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Depth and equivariant cohomology

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0. Introduction

Let G be a finite group and let X be a space on which G acts continuously. Choose a classifying bundle $PG \to BG$ for principal G-bundles. The group G acts freely on the contractible space PG, and there is a diagonal action of G on $PG \times X$. Let $PG \times^G X$ denote the orbit space of this diagonal action.

Let p be a prime integer. The mod-p equivariant cohomology ring of the G-space X is defined by the formula

$$H_G^*(X, Z/pZ) = H^*(PG \times^G X, Z/pZ).$$

The main result of this paper is:

THEOREM 1 (see Section 2). *The depth of $H_G^*(X, Z/pZ)$ is greater than or equal to the maximum rank of a central p-torus acting trivially on X.*

Here a *p-torus* is a direct product of cyclic groups of order p, and the *rank* of a p-torus A is the number of cyclic factors of A.

In a series of papers (Q1, Q2) D. Quillen investigated the algebraic structure of this ring. For example, suppose X has finite-dimensional mod-p cohomology. In this case Quillen proves the following

THEOREM (Theorem 7.7 of (Q1)). *The Krull dimension of the commutative ring*

$$H_G(X, Z/pZ) = \begin{cases} H_G^{\mathrm{ev}}(X, Z/pZ) & p \text{ odd} \\ H_G^*(X, Z/pZ) & p = 2 \end{cases}$$

is equal to the maximum rank of a p-torus A of G such than $X^A \neq \varnothing$.

In Section 4 of this paper, we use Theorem 1 and this dimension theorem of

Quillen's to show that certain cohomology modules are Cohen-Macaulay (Theorem 2).

Most of this work is contained in my thesis, written at M.I.T. under the guidance of Daniel Quillen. He suggested that I study these cohomology rings from a commutative-algebraic point of view; he also gave me most of my general insights into the subject and most of my specific methods of attack. I am very grateful to him for this guidance.

1. Preliminaries

The purpose of this preliminary section is to state some basic definitions and results, and to set notation.

Let G be a finite group. There is a classifying bundle $PG \to BG$ for principal G-bundles with paracompact base. The spaces PG and BG may be assumed to be (paracompact) CW-complexes.

Suppose that G acts continuously on a topological space X. Following Borel (B) we define

$$X_G = PG \times^G X$$

to be the orbit space of the diagonal action of G on $PG \times X$. We assume that the space X is such that X_G is a paracompact, locally contractible, Hausdorff space. For example, take X to be locally compact, paracompact, locally contractible and Hausdorff.

If R is a commutative ring, define the equivariant cohomology ring of the G-space X with coefficient in R to be

$$H_G^*(X, R) = H^*(X_G, R),$$

where the right hand side of this equation is ordinary singular cohomology with coefficients in R. The restrictions on X enable us to say that this definition of equivariant cohomology agrees with that of Borel (B) and Quillen (Q1) (these authors use sheaf cohomology); so we may use some results of their work.

We will make use of the following properties:

(a) (Q1, Section 1) $H_G^*(X, R)$ is independent of the choice of classifying bundle for G.

(b) (Q1, (1.5)) Functoriality: If $u: G \to G'$ is a homomorphism of finite groups and $f: X \to X'$ is u-equivariant, then there is a homomorphism $(u, f)^*: H_{G'}^*(X', R) \to H_G^*(X, R)$. If f and u are inclusions, this homomorphism will be denoted "res".

(c) If $X = \mathrm{pt}$ is a point, then $H_G^*(\mathrm{pt}, R) = H^*(PG \times^G X, R) = H^*(BG, R)$. In other words, $H_G^* = H_G^*(\mathrm{pt}, R)$ is classical group cohomology with coefficients in the trivial G-module R.

2. A regular sequence in $H_G^*(X, Z/pZ)$

In this section, the cohomology groups have coefficients in Z/pZ, where p is a fixed prime, unless otherwise indicated.

Let

$$H = \begin{cases} H_G^{ev} = \bigoplus_{1 \geq 0} H_G^{2i}(\mathrm{pt}) & p \text{ odd} \\ H_G^* = \bigoplus_{i \geq 0} H_G^i(\mathrm{pt}) & p = 2. \end{cases}$$

The ring H is a commutative ring. The graded group $H_G^*(X)$ may be considered as an H-module via the map $X \to \mathrm{pt}$. An *H-sequence* on $M = H_G^*(X)$ (or on any H-module M) may be defined in the following way (K):

A sequence of elements $x_1, x_2, \ldots, x_N, \ldots$ of positive degree in H is said to be an *H-sequence* on M (or a *regular* sequence on M) if x_1 is not a zero divisor on M, and if for each $i > 1$, x_i is not a zero divisor on $M/(x_1, \ldots, x_{i-1})M$.

Let $n > 0$. For each i such that $1 \leq i < n$, the sequence of elements $x_1, \ldots, x_n$ of H is an H-sequence on $H_G^*(X)$ if and only if $x_1, \ldots, x_i$ is an H-sequence on $H_G^*(X)$ and $x_{i+1}, \ldots, x_n$ is an H-sequence on $H_G^*(X)/(x_1, \ldots, x_i)H_G^*(X)$.

Theorems of Evens (E) and Venkov (V, see also (Q1)) show that H is Noetherian and that $H_G^*(X)$ is a finitely generated H-module if $H^*(X)$ is finite-dimensional over Z/pZ. In this case, any two maximal H-sequences have the same (finite) length (e.g. see (K); in Theorem 121 of (K), take the ideal I to be the positive degree elements of H). This common length we call the *depth* of $H_G^*(X)$.

Here is the main theorem.

THEOREM 1. *Let A be a p-torus that contained in the center of G. Suppose also that A acts trivially on X. Then there is a regular sequence on $H_G^*(X)$ of length equal to rank (A). Thus, if $H^*(X)$ is finite dimensional over Z/pZ, then*

$$\textit{depth } H_G^*(X) \geq \textit{rank } (A).$$

This theorem will be proved by induction on the rank of A. Before we give the proof of Theorem 1, we need some preliminary results.

Let A be a cyclic group of order p contained in the center of G, such that $X^A = X$. ($X^A = \{x \in X \mid ax = x\ \forall a \in A\}$). Consider the representation $\rho: A \to \mathbf{C}^*$ given by $\rho(a) = \exp(2\pi i/p)$, where a is a fixed generator for A. The representation ρ of A gives an A-action on $\mathbf{C}$. Using this action we have a line bundle, also called ρ, over the classifying space BA for A (as in (A)):

$$PA \times^A \mathbf{C} \underset{\rho}{\rightarrow} BA.$$

The first Chern class for this bundle is $c_1(\rho) \in H^2_A(\text{pt}, Z) = H^2(BA, Z)$; via the homomorphism $Z \to Z/pZ$, we have a mod-p Chern class for ρ, $c_1(\rho) \in H^2(BA) = H^2_A$.

Since A acts trivially on X, there is an "equivariant" Lyndon–Hochschild–Serre spectral sequence (see Section 3):

$$E_2^{**} = H^*(X_{G/A}, \{H^*_A\}) \Rightarrow H^*(X_G).$$

Since A is central in G, and coefficients are in a field,

$$E_2^{p,q} = H^p(X_{G/A}, H^q_A) \simeq H^p(X_{G/A}) \otimes_{Z/pZ} H^q_A \quad \text{(see Section 3).}$$

It is well known that

$$H^*_A = \begin{cases} Z/pZ(c_1(\rho)) \otimes_{Z/pZ} \Lambda(x) & p \text{ odd} \\ & \text{(a polynomial algebra on } c_1(\rho) \text{ tensored with} \\ & \text{an exterior algebra on } x \text{ where the Bockstein} \\ & \text{of } x \text{ equals } c_1(\rho).) \\ Z/2Z(y) & p = 2 \\ & \text{(a polynomial algebra on } y, \text{ where } y \cdot y = c_1(\rho).) \end{cases}$$

Following Evens (E) we have:

LEMMA 1. *Let* $\alpha \in H^{2M}_G(X)$ *be any cohomology class such that* $H^{2M}_G(X) \underset{\text{res}}{\longrightarrow} H^{2M}_A = E_2^{0,2M}$ *takes* α *to* $c_1(\rho)^M$, *where* $M > 0$. *Then*

(a) $c_1(\rho) \in E_r^{0,2M}$ *for every* $r \geq 2$,

(b) *Multiplication by* $c_1(\rho)$ *induces an isomorphism* $E_r^{*,j} \xrightarrow{\simeq} E_r^{*,j+2M}$ *for every* $r \geq 2$ *and* $j \geq 0$ ($E_r^{*,j} = \bigoplus_{i \geq 0} E_r^{i,j}$), *and*

(c) $E^{**}_{2M+2} = E^{**}_\infty$

Proof.

(a) Referring to the fibration (see Section 3) giving rise to the spectral sequence

$$BA \to PG \times^G X \to P(G/A) \times^{G/A} X$$

it is enough to note that $c_1(\rho)$ is the restriction to the cohomology of the fibre BA of the class α in the cohomology of the total space $PG \times^G X$. Thus, for every $r \geq 2$, $d_r(c_1(\rho)^M) = 0$, when d_r is the appropriate r-th differential of the spectral sequence.

(b) Multiplication by $c_1(\rho)$ is an isomorphism

$$H^*(X_{G/A}) \otimes_{Z/pZ} H^j_A \xrightarrow[\simeq]{} H^*(X_{G/A}) \otimes_{Z/pZ} H^{j+2M}_A$$

for every $j \geq 0$ since $c_1(\rho)$ is a polynomial generator of H^*_A. So, for $r = 2$, (b) is true.

Suppose (b) has been proven for $r \geq 2$. Suppose $j \geq 0$. Consider the following diagram:

$$\begin{array}{ccccc}
E_r^{i-r,j+r-1+2M} & \xrightarrow{d_r} & E_r^{i,j+2M} & \xrightarrow{d_r} & E_r^{i+r,j-r+1+2M} \\
\uparrow{\scriptstyle \cdot c_1(\rho)^M} & & \uparrow{\scriptstyle \cdot c_1(\rho)^M} & & \uparrow{\scriptstyle \cdot c_1(\rho)^M} \\
E_r^{i-r,j+r-1} & \xrightarrow{d_r} & E_r^{i,j} & \xrightarrow{d_r} & E_r^{i+r,j-r+1}
\end{array}$$

d_r is the differential of the spectral sequence. The diagram is commutative since

$$d_r(c_1(\rho)^M \cdot x) = c_1(\rho)^M \cdot d_r(x) \pm d_r(c_1(\rho)^M) \cdot x$$

by the multiplicative property of d_r, and since $d_r(c_1(\rho)^M) = 0$.

If $j - r + 1 \geq 0$, then by induction all the vertical maps are isomorphisms, so the induced map on homology,

$$E_{r+1}^{i,j} \xrightarrow[\cdot c_1(\rho)^M]{} E_{r+1}^{i,j+2M},$$

is an isomorphism.

If $j - r + 1 < 0$, $E_r^{i+r,j-r+1} = 0$, and a diagram case shows that

$$E_{r+1}^{i,j} \xrightarrow[\cdot c_1(\rho)^M]{} E_{r+1}^{i,j+2M}$$

is an isomorphism in this case also.

(c) $d_{2m+2}: E_{2M+2}^{i,j} \to E_{2M+2}^{i+2M+2,j-2M+1}$. Suppose that $j \leq 2M$; then

$d_{2M+2}(E^{*,j}_{2M+2})=0$ since the spectral sequence is first quadrant. If $j>2M$, there is an integer $m>0$ such that $j=2mM+k$, where $k\leq 2M$. Let $y\in E^{i,j}_{2M+2}=E^{i,2mM+k}_{2M+2}$; then (b) and induction on m show that $y=(c_1(\rho)^M)^m\cdot x$ for some $x\in E^{i,k}_{2M+2}$. So

$$d_{2M+2}(y)=c_1(\rho)^{mM}\cdot d_{2M+2}(x)\pm d_{2M+2}(c_1(\rho)^{mM}\cdot x=0\pm 0=0.$$

So $d_{2M+2}\equiv 0$, and $E^{**}_{2M+2}=E^{**}_{\infty}$. QED

Thus, we have

COROLLARY 1. *The cohomology class α of Lemma 1 is not a zero divisor on $H^*_G(X)$.*

Proof. Lemma 1 shows that multiplication by $c_1(\rho)^M$ is injective on E^{**}_{∞}. So multiplication by α, which restricts to $c_1(\rho)^M$, must be injective on $H^*_G(X)$. QED

Let $l=N/p$ be the index of A in G, where N is the order of G. Corresponding to the one-dimensional representation ρ of A, there is the l-dimensional induced representation ind(ρ) of G. There is an l-dimensional vector bundle, also called ind(ρ), over the classifying space BG for G:

$$PG\times^G \mathbf{C}^l\to BG.$$

The mod-p Chern classes for this vector bundle are $c_i(\text{ind}(\rho))\in H^{2i}(BG)=H^{2i}_G$ for $0\leq i\leq l$.

LEMMA 2. *The cohomology class $e=c_l(\text{ind}(\rho))$ is a non-zero divisor on $H^*_G(X)$.*

Proof. Since A is central, the Mackey induction formula (e.g., see (Se)) shows that

$$\text{res ind}(\rho))=l\rho=\underbrace{\rho\oplus\cdots\oplus\rho}_{l\text{ times}}.$$

Thus $\text{res}(c_l(\text{ind}(\rho)))=c_l(\text{res}(\text{ind}(\rho))=c_l(l\rho)=c_1(\rho)^l$ by various properties of Chern classes. (Here "res" is ambiguously used to denote either the restriction of the reprêsentation ind(ρ) of G to the subgroup A, or the restriction map in cohomology from H^*_G to H^*_A.)

We may now proceed with the proof of Theorem 1.

Proof of Theorem 1. We begin the induction by noting that Lemma 2 proves the Theorem in case rank$(A)=1$.

Now, suppose $\operatorname{rank}(A)=n>1$. Thus, A is central, and $X^A=X$. Let $l=N/p^n$ be the index of A in G. Let A_1 be a subgroup of rank 1 in A, and write $A=A_1\times B$ where B is a p-torus of rank $n-1$.

There is a one-dimensional representation $\hat{\rho}: A_1\times B\to \mathbf{C}$ of A given by ρ (the same ρ as in Lemma 1) on A_1 and the trivial representation on B. Let $e=c_l(\operatorname{ind}_{A\to G}(\hat{\rho}))\in H_G^{2l}\leq H$ be the top Chern class of the l-dimensional representation $\operatorname{ind}_{A\to G}(\hat{\rho})$ of G. If $\operatorname{res}_{A_1\to G}: H_G^*\to H_{A_1}^*$, then we have $\operatorname{res}_{A_1\to G}(e)=c_1(\rho)^l$. This follows again from the Mackey induction formula, which implies that $\operatorname{res}_{A_1\to G}(\operatorname{ind}_{A\to G}(\hat{\rho}))=\operatorname{res}_{A_1\to A}(\operatorname{res}_{A\to G}(\operatorname{ind}_{A\to G}(\hat{\rho})))=\operatorname{res}_{A_1\to A}(l\hat{\rho})=l\rho$; and standard properties of Chern classes.

Thus, Lemma 2 says that e is not a zerodivisor on $H_G^*(X)$.

The finite group G acts on $\mathbf{C}^{2l}$ via $\operatorname{ind}_{A\to G}(\hat{\rho})$ and therefore on $\mathbf{C}^{2l}\times X$ diagonally. So there is a vector bundle

$$\begin{array}{c} PG\times^G(\mathbf{C}^{2l}\times X)\xrightarrow[\xi]{} PG\times^G X,\\ \Vert \\ (PG\times X)\times^G\mathbf{C}^{2l}\end{array}$$

and the associated (orientable) sphere bundle is

$$PG\times^G(S^{2l-1}\times X)\xrightarrow[\xi']{} PG\times^G X;$$

recall that $\operatorname{ind}_{A\to G}(\hat{\rho})$ is unitary, since $\hat{\rho}$ is unitary. Associated to this sphere bundle ξ' is a mod-p Euler class; it is the top Chern class of the vector bundle ξ. Therefore this Euler class is equal to e.

There is the exact Gysin sequence for ξ' (S):

$$\cdots\to H^{i-2l}(X_G)\xrightarrow[\cdot e]{} H^i(X_G)\xrightarrow{\theta} H^i((S^{2l-1}\times X)_G)\to\cdots.$$

The map $H^{i-2l}(X_G)\to H^i(X_G)$ is multiplication by e as indicated, and since e is a non-zero-divisor, this map in injective.

So, there is a short exact sequence of H-modules

$$0\to H^*(X_G)\xrightarrow[\cdot e]{} H^*(X_G)\xrightarrow[\theta]{} H^*((S^{2l-1}\times X)_G)\to 0,$$

where $H^*(X_G)=\bigoplus_{i\geq 0}H^i(X_G)$. Multiplication by e is an H-module map since e has even degree.

This short exact sequence shows that there is an H-module isomorphism

$$H^*((S^{2l-1}\times X)_G) \xleftarrow[\bar{\bar{\theta}}]{\cong} H^*(X_G)/(e)H^*(X_G)$$

induced by θ.

The isomorphism $\bar{\bar{\theta}}$ provides the inductive step. For, how does B act on $S^{2l-1}\times X$? In fact, B acts trivially. To see this it is enough to note that

(1) $\mathrm{res}_{B\to G}(\mathrm{ind}_{A\to G}(\hat{\rho}))=l$ (the l-dimensional trivial representation).

(*Proof*: B is central since A is. So $\mathrm{res}_{B\to G}(\mathrm{ind}_{A\to G}(\hat{\rho}))=\mathrm{res}_{B\to A}(\mathrm{res}_{A\to G}(\mathrm{ind}_{A\to G}(\hat{\rho})))=\mathrm{res}_{B\to A}(l\hat{\rho})=l\cdot \mathrm{res}_{B\to A}(\hat{\rho})=l$.) and

(2) $X^B=X$; this follows because $X^A=X$ and $B\le A$.

Since $\mathrm{rank}(B)<\mathrm{rank}(A)$, B is central and $(S^{2l-1}\times x)^B=S^{2l-1}\times X$, we may use induction to obtain an H-sequence $e_1, e_2, \ldots, e_{n-1}$ of length $n-1$ on $H^*((S^{2l-1}\times X)_G)$. Using the isomorphism $\bar{\bar{\theta}}$; $e, e_1, e_2, \ldots, e_{n-1}$ is an H-sequence of length n on $H^*(X_G)$. QED

We notice that it is possible to explicitly write down an H-sequence on $H^*(X_G)$ in the following manner. Write $A=A_1\times A_2\times\cdots\times A_n$ as a direct product of cyclic groups of order p. For $1\le i\le n$, let $\rho_i: A\to \mathbf{C}^*$ be the one dimensional representation of A given by the trivial representation of A on all but the i-th factor of A and by (our usual) ρ on A_i. If $e_i=c_l(\mathrm{ind}_{A\to G}(\rho_i))$ is the l-th Chern class of ρ_i $(l=N/p^n)$ the proof of Theorem 1 shows that $e_1, e_2, \ldots, e_n$ is an H-sequence on $H^*(X_G)$.

Also,

$$H^*_G(X)/(e_1,\ldots,e_i)H^*_G(X)\cong H^*\Big(\Big(\Big(\underbrace{S^{2l-1}\times\cdots\times S^{2l-1}}_{i\ \text{factors}}\Big)\times X\Big)_G\Big),$$

where G acts on $(S^{2l-1})^i$ via $\mathrm{ind}(\rho_j)$ on the j-th factor (for $1\le j\le i$) and on $(S^{2l-1})^i\times X$ diagonally.

3. A spectral sequence

If G and X are as in Section 2, let N be a normal subgroup of G acting *trivially* on X.

In this section we point out that there is a fibration

(*) $BN\to X_G\to X_{G/N}$

giving rise to a Serre spectral sequence

$$E_2^{**} = H^*(X_{G/N}, \{H_N^*\}) \Rightarrow H^*(X_G).$$

($\{\cdot\}$ denotes local coefficients.)

To get the fibration (*) we start with a classifying bundle $P(G/N) \to B(G/N)$ for principal G/N bundles. Then, there is a classifying bundle $PG \to BG$ for principal G-bundles and a commutative diagram

$$\begin{array}{ccc} PG & \to & P(G/N) \\ \downarrow & & \downarrow{\scriptstyle \xi} \\ BG & \underset{f}{\to} & B(G/N) \end{array}$$

with f a *fibration.*

Since N acts trivially on $P(G/N)$ there is a commutative diagram

$$\begin{array}{cccc} PG & & & \\ & \searrow & \overset{\tilde{f}}{\searrow} & \\ & PG/N & \to & P(G/N) \\ & {\scriptstyle \xi'}\downarrow & \square & \downarrow{\scriptstyle \xi} \\ & BG & \underset{f}{\to} & B(G/N) \end{array}$$

and the big square $\square$ is cartesian.

Replacing the fibres of the principal G/N bundles ξ and ξ' by the G/N-space X and noting that $PG/N \times^{G/N} X$ is homeomorphic to $PG \times^G X$, we have a commutative diagram of fibrations with the indicated square cartesian.

$$\begin{array}{ccccc} BN & \to & X_G & \to & X_{G/N} \\ \| & & \downarrow & \square & \downarrow \\ BN & \to & BG & \to & B(G/N) \end{array}$$

Since the fibration $X_G \to X_{G/N}$ is induced from the map $BG \to B(G/N)$, if the local coefficient system is trivial for the latter fibration, it is trivial for the former fibration (S).

Now, if N is central in G, it is a fact that the local coefficient system $\{H_N^*\}$ is trivial for the fibration $BG \to B(G/N)$. Therefore, in this case, the local coefficient system $\{H_N^*\}$ is trivial for the fibration $X_G \to X_{G/N}$. Since we are using field coefficients we see that

$$E_2^{**} = H^*(X_{G/N}, H_N^*) \cong H^*(X_{G/N}) \otimes_{Z/pZ} H_N^*$$

if N is central.

(Note: The results in this section are of course also true if G is a compact Lie group, and N is a closed normal subgroup.)

4. Certain H-modules are Cohen-Macaulay

Suppose H is a (graded) commutative, Noetherian ring and M is a (graded) commutative Noetherian H-algebra that is also a finitely generated (graded) H-module. We may define (as is usual) the *H-depth* of M as the maximal rank of an H-sequence (defined as in Section 2) on M and the *H-dimension* of M as the Krull dimension of $H/\mathrm{ann}_H(M)$. Standard results in commutative algebra ensure that the H-depth of M equals the M-depth of M; and the H-dimension of M equals the M-dimension of M (M, Se2). Thus we may refer without ambiguity to the *depth* of M and the *dimension* of M without the "H" or "M" prefixes. We also have the standard result that depth $M \geq \dim M$. We say that M is *Cohen-Macaulay* if depth $M = \dim M$.

(The reader who is used to seeing these results for *local* rings is reminded of the accurate (at least for these results) analogy between local and graded rings. The ideal of positive degree elements plays the part that the maximal ideal does in local algebra.)

Fix a prime p. Let G be a finite group and let X be a G-space with finite-dimensional (over Z/pZ) mod-p cohomology. Define H as in Section 2. Let

$$M = H_G(X, Z/pZ) = \begin{cases} H_G^{\mathrm{ev}}(X, Z/pZ) & p \text{ odd} \\ H_G^*(X, Z/pZ) & p \text{ even.} \end{cases}$$

As in Section 2, M is a finitely generated H-module via the map $X \to \mathrm{pt}$.

In this section we show that H and M are Cohen-Macaulay for certain groups G and certain G-spaces X. Namely, we have:

THEOREM 2. *Let G be a finite group with a unique p-torus A of maximal rank. Assume A is also central. Let X be a G-space such that $X^A = X$. Then M and H are Cohen-Macaulay.*

Proof. According to Quillen (Q1: also, see Section 0), $\dim M = \operatorname{rank}(A) = \dim H$. Theorem 1 shows that depth $H \geq \operatorname{rank}(A)$, and that depth $M \geq \operatorname{rank}(A)$. Now, since dimension always dominates depth, we obtain the theorem. QED

If G is a finite group let $C_G(A) = \{g \in G \mid ga = ag \; \forall a \in A\}$ be the centralizer of A in G.

COROLLARY 2. *If G is a finite group and X is a G-space with finite dimensional (mod-p) cohomology, then for any maximal p-torus A in G, $H_{C_G(A)}(X^A)$ is Cohen-Macaulay.*

Proof. If A is maximal, then A is the unique maximal p-torus of $C_G(A)$. (For, if B is another p-torus in $C_G(A)$, then for every a in A and b in B, $ab = ba$. Thus, the subgroup generated by A and B, $\langle A, B\rangle$, is a p-torus in G containing A. Since A is maximal, $\langle A, B\rangle = A$ and $B \leq A$.) It is clear that A is central in $C_G(A)$ and acts trivially on X^A. QED

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