Tranverse foliations of Seifert bundles and self homeomorphism of the circle.

Autor(en): **Eisenbud, D. / Hirsch, U. / Neumann, W.**

Objekttyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **56 (1981)**

PDF erstellt am: **22.07.2024**

Persistenter Link: https://doi.org/10.5169/seals-43263

Comment. Math. Helvetici **56** (1981) 638–660 0010–2571/81/004638-23$01.50+0.20/0

Transverse foliations of Seifert bundles and self homeomorphism of the circle

DAVID EISENBUD, ULRICH HIRSCH and WALTER NEUMANN*

1. Introduction

In this paper we give criteria for a Seifert circle bundle over a compact surface to admit foliations whose leaves are all transverse to the fibers, and we discuss which foliations may be deformed to foliations of this type.

Our criteria for transverse foliations, presented in Section 3, are simple numerical inequalities involving the Seifert pairs of the fibration and the euler number of the base (Theorems 3.1 to 3.4). They generalize criteria of Milnor [Mi] and Wood [W], who treat the case of locally trivial circle bundles with orientable total space (see also Sullivan [Su] for a higher dimensional generalization of Milnor and Wood). They are complete except for the case that the base is S^2, in which case we only have partial results (Theorems 3.3, 5.3). Our criteria are valid both for the case of C^0 foliations and, as we show in Section 4, for analytic foliations, hence also for any intermediate degree of smoothness.

We reduce the geometric question of the existence of foliations to algebra in a way similar to that of Milnor and Wood. We let $\mathcal{D}$ be the group of self-homeomorphisms $f:\mathbf{R}\to\mathbf{R}$ which are lifts of self-homeomorphisms of the circle. $\mathcal{D}$ contains the group $\mathcal{D}^+=\{f:\mathbf{R}\to\mathbf{R}\mid f$ monotonically increasing and $f(r+1)=f(r)+1$ for all $r\in\mathbf{R}\}$ as a subgroup of index 2 (the "flip" $t(r)=-r$ is a coset representative for the non-trivial coset of $\mathcal{D}^+$ in $\mathcal{D}$). For each real number γ we write sh $(\gamma)\in\mathcal{D}^+$ for the "shift" by γ, that is sh $(\gamma): r\mapsto r+\gamma$ for $r\in\mathbf{R}$.

It turns out that the problem of transversely foliating a Seifert fibered manifold M is equivalent to the problem of finding a homomorphism $\pi_1(M)\to\mathcal{D}$ which takes the class of a nonsingular fiber of M to the element $\mathrm{sh}(1)\in\mathcal{D}^+$ (Theorem 3.5). This in turn is equivalent to a problem of representating a certain product of conjugates of shifts as a product of a certain number of commutators in

*The authors wish to express their gratitude to the SFB 40 and the mathematical institute of the University of Bonn, under whose roof most of the work of this paper was done. The first and last authors are also grateful for the partial support of the NSF.

$\mathcal{D}^+$ or $\mathcal{D}$, or as a product of a certain number of squares of elements of $\mathcal{D}-\mathcal{D}^+$.

For this reason we begin, in Section 2, by characterizing the elements of $\mathcal{D}^+$ that can be written as products of g commutators, or of g squares of elements of $\mathcal{D}-\mathcal{D}^+$, etc., and by partially characterizing the elements that can be written as a product of conjugates of a given collection of shifts. Perhaps the most surprising result is that these classes of elements can be simply characterized in terms of the invariants

$$\underline{m}f=\min_{r\in\mathbf{R}}(f(r)-r), \qquad \bar{m}f=\max_{r\in\mathbf{R}}(f(r)-r),$$

of an element $f\in\mathcal{D}^+$ (this min and max exist, since $f(r)-r$ is a continuous perodic function on $\mathbf{R}$). In Section 4 we prove the same characterizations with $\mathcal{D}^+$ and $\mathcal{D}$ replaced by their subgroups $\widetilde{\mathrm{PSL}}(2,\mathbf{R})$ and $\widetilde{\mathrm{PGL}}(2,\mathbf{R})$ respectively, obtained by lifting the subgroups $\mathrm{PSL}(2,\mathbf{R})$ and $\mathrm{PGL}(2,\mathbf{R})$ of $\mathrm{Homeo}(S^1)$. A different characterization for products of commutators in $\widetilde{\mathrm{PSL}}(2,\mathbf{R})$ was given by Wood.

In Section 5 we describe a natural analytic family of examples due to Maria Carmen del Gazolas [dG]. We are grateful to the referee for bringing them to our attention. They are more general than the examples we originally had here and yield better results for base S^2.

The last section (Section 6) of our paper contains results on deforming foliations of Seifert fibered manifolds to make them transverse. Here we only consider transversely oriented C^2 foliations on Seifert manifolds whose base is not S^2 or P^2. Roughly speaking the theorem is that this can be done if the foliation has no compact leaves. The techniques are the same as for the case of locally trivial bundles, done by Thurston [T] and Levitt [L], so, at the referee's suggestion we omit proofs.

We fix the notations $\mathcal{D}, \mathcal{D}^+, \underline{m}, \bar{m}$ and, for $\gamma\in\mathbf{R}$, $\mathrm{sh}(\gamma)$ for use throughout this paper. In addition, we use Knuth's "floor and ceiling" notations $\lfloor\gamma\rfloor$ and $\lceil\gamma\rceil$ for $\gamma\in\mathbf{R}$ to denote respectively, the greatest integer $\leq\gamma$ and least integer $\geq\gamma$.

2. Self-homeomorphisms of the circle

We will actually need information on the "lifted" homeomorphisms in $\mathcal{D}$ and $\mathcal{D}^+$. To make the connection explicit, note that the center of $\mathcal{D}^+$ is

$$Z(\mathcal{D}^+)=\{\mathrm{sh}(n)\mid n\in\mathbf{Z}\}\cong\mathbf{Z},$$

and that $\mathcal{D}^+/Z(\mathcal{D}^+)$ is the group of orientation preserving self-homeomorphisms of the circle $S^1=\mathbf{R}/\mathbf{Z}$.

We write $a^b=b^{-1}ab$ for the *conjugate* of a by b. A *commutator* is an element of the form $a^{-1}b^{-1}ab$.

Before we describe our main results characterizing certain products of special elements of $\mathscr{D}^+$, we describe some elementary properties of $\mathscr{D}^+$ and of the functions $\underline{m}$ and $\bar{m}$.

LEMMA 2.1. *Let f, f_i, g be elements of $\mathscr{D}^+$. Then*

(1) *If $x, y\in\mathbf{R}$, then $x-y\in\mathbf{Z}$ implies $fx-fy=x-y$. If, on the contrary; $n<x-y<n+1$ for some $n\in\mathbf{Z}$, then $n<fx-fy<n+1$.*

(2) $0\le\bar{m}f-\underline{m}f<1$.

(3) $\underline{m}f^{-1}=-\bar{m}f$, *and dually*, $\bar{m}f^{-1}=-\underline{m}f$.

(4) $\sum_{i=1}^n \underline{m}f_i\le\underline{m}(\prod_{i=1}^n f_i)\le\underline{m}f_n+\sum_{i=1}^{n-1}\bar{m}f_i<(\sum_{i=1}^n\underline{m}f_i)+n-1,$ *and dually,*

$$\sum_{i=1}^n \bar{m}f_i\ge\bar{m}\Big(\prod_{i=1}^n f_i\Big)\ge\bar{m}f_n+\sum_{i=1}^{n-1}\underline{m}f_i>\Big(\sum_{i=1}^n \bar{m}f_i\Big)-n+1.$$

(5) $\lfloor\underline{m}f^g\rfloor=\lfloor\underline{m}f\rfloor$ *and* $\lceil\underline{m}f^g\rceil=\lceil\underline{m}f\rceil$ *(so, in particular, if $\underline{m}f\in\mathbf{Z}$, then $\underline{m}f^g=\underline{m}f$) and the same for $\bar{m}$.*

Proof. (1) Apply f to the equality $y+n=x$ or to the inequality $y+n<x<y+n+1$, for $n\in\mathbf{Z}$.

(2) Choose $x, y\in\mathbf{R}$ with $fx-x=\underline{m}f$ and $fy-y=\bar{m}f$ and observe that if $\bar{m}f-\underline{m}f\ge1$ then x and y contravene (1).

(3) is clear, as is the first inequality in (4). For the second inequality choose $x\in\mathbf{R}$ with $f_nx-x=\underline{m}f_n$ and observe that $(\prod f_i)x-x\le\underline{m}f_n+\sum_{i=1}^{n-1}\bar{m}f_i$. The third inequality now follows by (2), and the dual inequalities follow by (3).

(5) By (1) we have $\lfloor x-y\rfloor=\lfloor g^{-1}x-g^{-1}y\rfloor$ for $x, y\in\mathbf{R}$. Choosing $y=gz$ and $x=fgz$ gives $\lfloor fgz-gz\rfloor=\lfloor f^gz-z\rfloor$, and minimizing over $z\in\mathbf{R}$ gives $\lfloor\underline{m}f\rfloor=\lfloor\underline{m}f^g\rfloor$. Replacing $\lfloor\quad\rfloor$ by $\lceil\quad\rceil$ and/or min by max in this argument proves the rest of (5).

We shall also need a special subset of $\mathscr{D}^+$. We say $f\in\mathscr{D}^+$ has a *stable fixed point* at $r\in\mathbf{R}$ if for any $s\in\mathbf{R}$ sufficiently close to r, the iterates $f^n(s)$ converge to r as n goes to infinity. Equivalently, $f(r-\varepsilon)>r-\varepsilon$ and $f(r+\varepsilon)<r+\varepsilon$ for all ε sufficiently small. We say $f\in\mathscr{D}^+$ has an *unstable fixed point at* $r\in\mathbf{R}$ if r is a stable fixed point of f^{-1}. For $k>0$ we write

$$f\in\mathrm{SUF}(k)$$

(SUF stands for: Stable or Unstable Fixpoints) if f has exactly $2k$ fixed points on some (or equivalently every) half open unit interval $[r, r+1)$, and exactly k of them are stable and the other k are unstable. It is clear that the fixed points of f then alternate type (stable or unstable) along the real line. The significance of this definition is given by the following well known lemma.

LEMMA 2.2. *Any two elements of* SUF(k) *are conjugate in* $\mathscr{D}^+$.

Proof. We prove this for $k=1$, the only case we ever use. The general proof will then be clear. Thus let $f, g \in \mathrm{SUF}(1)$. Let x be an unstable fixed point of f and let y be the unique fixed point of f in the interval $(x, x+1)$, so y is necessarily stable. Let x' and y' be chosen similarly for g. We shall construct an $h \in \mathscr{D}^+$ with $hx = x'$, $hy = y'$ and $hgh^{-1} = f$.

It is enough to construct h on each of the intervals $[x, y]$ and $[y, x+1]$ so that the equation $hgh^{-1} = f$ holds where it is defined. We do this for $[x, y]$; the construction for $[y, x+1]$ is similar. Choose z and z' with $x < z < y$ and $x' < z' < y'$ and let $h_0 : [z, f(z)] \to [z', g(z')]$ be any monotonic homeomorphism. For $n \in \mathbf{Z}$ define $h_n = g^n h_0 f^{-n} : f^n[z, f(z)] \to g^n[z', g(z')]$. Since the intervals $f^n[z, f(z)]$ partition the interval (x, y) and the intervals $g^n[z', g(z')]$ partition (x', y'), these maps h_n fit together to give the desired map $h : [x, y] \to [x', y']$.

We now come to the main results of this section. The first result needed for our geometric application is:

THEOREM 2.3. *Let f be an element of $\mathscr{D}^+$.*

(1) *f can be written as a product of $g \geq 1$ commutators of elements of $\mathscr{D}^+$ if and only if $\underline{m}f < 2g-1$ and $\bar{m}f > 1-2g$.*

(2) *f can be written as a product of $g \geq 2$ squares of elements of $\mathscr{D} - \mathscr{D}^+$ if and only if $\underline{m}f < g-1$ and $\bar{m}f > 1-g$.*

Remark. (2) remains valid if "squares of elements of $\mathscr{D} - \mathscr{D}^+$" is replaced either by "elements of SUF(1)" or "elements of $\mathscr{D}^+$ having fixed points".

The situation for commutators in $\mathscr{D}$, which we need to handle the non-orientable case of our geometric problem, is much simpler:

THEOREM 2.4. *Every element of $\mathscr{D}^+$ is a commutator of an element of $\mathscr{D} - \mathscr{D}^+$ with an element of $\mathscr{D}^+$.*

We shall also need a partial characterization of certain products of conjugates in $\mathscr{D}^+$:

THEOREM 2.5. *Let $f_1, \ldots, f_n \in \mathscr{D}^+$ and $r \in \mathbf{R}$ be given with*

$$\sum \lfloor \underline{m}f_i \rfloor < r < \sum \lceil \bar{m}f_i \rceil.$$

Then there exist elements $e_i \in \mathscr{D}^+$ such that $d = \prod (f_i^{e_i})$ satisfies

$$\underline{m}d \leq r \leq \bar{m}d.$$

To deal with Seifert fibrations over the sphere we need to know when d can be the identity in the above theorem, when the f_i are shifts. We only have a partial answer to this question.

THEOREM 2.6. *Let $\gamma_1, \ldots, \gamma_k$ be real numbers.*

(1) *There exist $e_i \in \mathcal{D}^+$ such that $\prod \mathrm{sh}(\gamma_i)^{e_i} \in \mathrm{SUF}(1)$ if and only if $\sum \lfloor \gamma_i \rfloor \leq -1$ and $\sum \lceil \gamma_i \rceil \geq 1$.*

(2) *If $\sum \gamma_i = 0$ or $\sum \lfloor \gamma_i \rfloor \leq -2$ and $\sum \lceil \gamma_i \rceil \geq 2$, then there exist $e_i \in \mathcal{D}^+$ such that $\prod \mathrm{sh}(\gamma_i)^{e_i} = 1$.*

(3) *Conversely, if e_i exist as in* (2), *the either $\sum \gamma_i = 0$, or $\sum \lfloor \gamma_i \rfloor \leq -1$ and $\sum \lceil \gamma_i \rceil \geq 1$ with at least one of these inequalities strict.*

The rest of this section gives the proofs of Theorems 2.3 to 2.6. We start with the proof of 2.3, which is an induction, starting at the case $g = 1$ in part (1) and $g = 2$ in part (2). Since we shall often need this special case, we state it as a lemma, for easy reference.

LEMMA 2.7. *The following conditions on an element $f \in \mathcal{D}^+$ are equivalent:*

(1) *$\underline{m}f < 1$ and $\bar{m}f > -1$.*

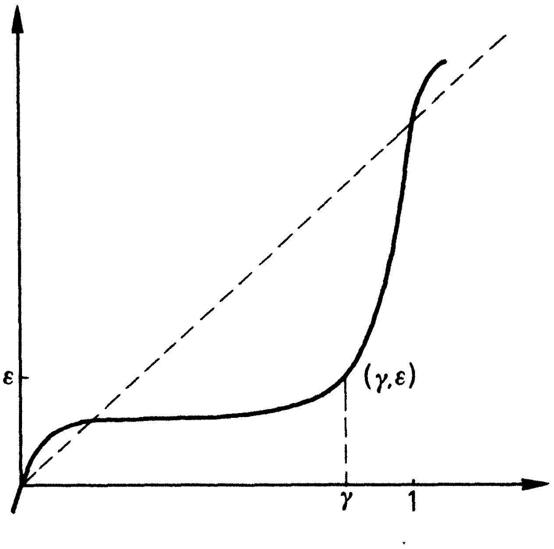
(2) *f is the product of two elements of* SUF(1).

(3) *f is a commutator in $\mathcal{D}^+$.*

(4) *f is the product of two squares of elements of $\mathcal{D} - \mathcal{D}^+$.*

Proof. We begin with the most delicate point, the constructive statement (1)$\Rightarrow$(2).

Let $\varepsilon, \gamma \in \mathbf{R}$ with $0 < \varepsilon < \gamma < 1$, to be chosen later, and let $c \in \mathcal{D}^+$ be a function whose graph looks like



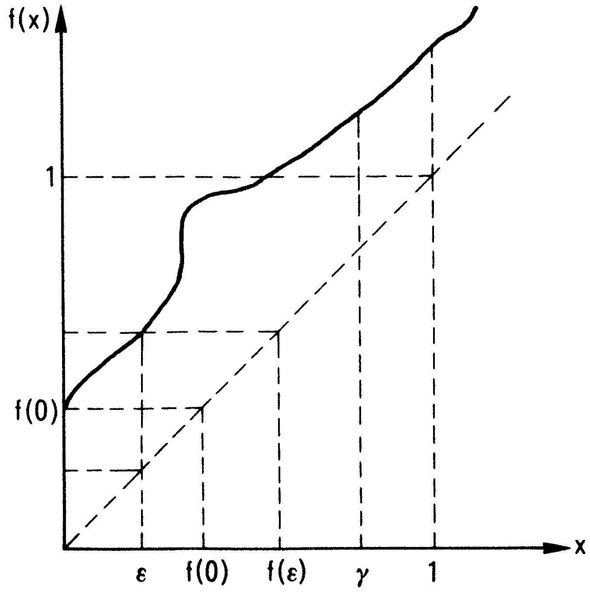
That is, $c(0)=0$, $c(\gamma)=\varepsilon$, and $c\in \mathrm{SUF}(1)$, with an unstable fixed point at 0 and a stable one in the interval $(0, \varepsilon)$.

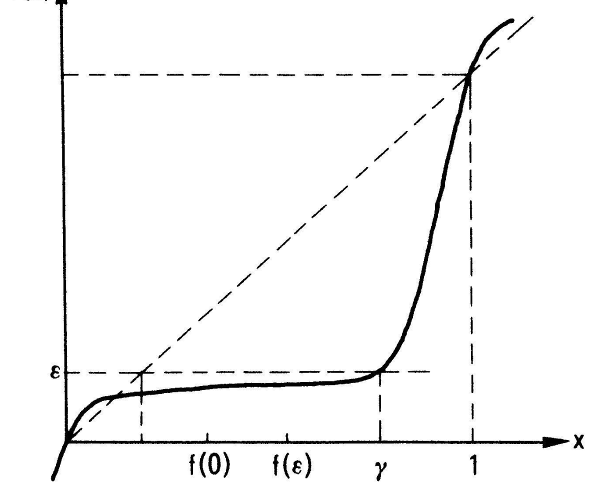
Given that f satisfies (1) and is not the identity, we may suppose, by inverting f if necessary, that $\bar{m}f>0$. Conjugating f by a shift if necessary, we can then further suppose that $0<f(0)<1$. Chosen ε close enough to zero and γ close enough to 1 that

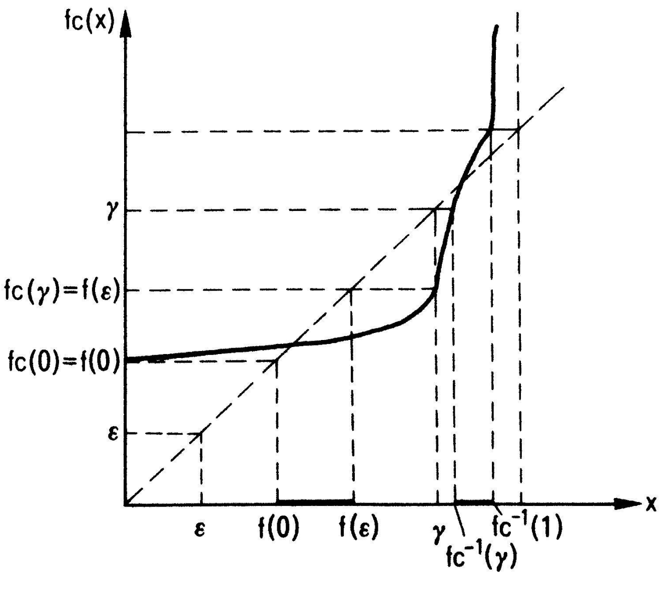
$$0<\varepsilon<f(0)<f(\varepsilon)<\gamma<1<f(\gamma).$$

We may write $f=(fc)c^{-1}$; we will see that this is nearly the required factorization.

First, since $fc(0)>0$, $fc(\gamma)<\gamma$, and $fc(1)>1$, the function fc must have at least one fixed point on each of $(0, \gamma)$ and $(\gamma, 1)$. On $(0, \gamma)$, the fixed points must of course occur on $fc(0, \gamma)=f(0, \varepsilon)$, while on $(\gamma, 1)$ they must occur on $(fc)^{-1}(\gamma, 1)\subset(\gamma, 1)$.







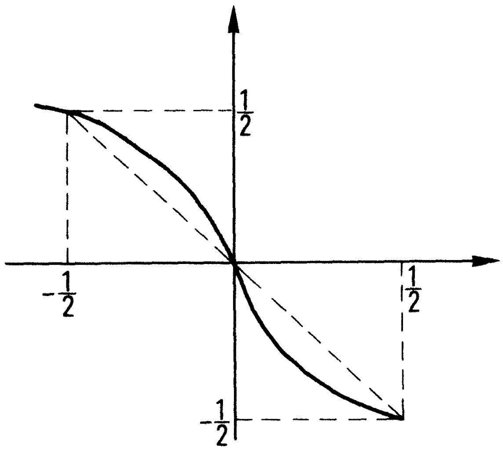
If f is nicely behaved, say f and f^{-1} Lipschitz, then by making c sufficiently flat on $f(0, \varepsilon)$ and sufficiently steep on $(\gamma, 1)$, we can assure that fc has exactly one (stable respectively unstable) fixed point on each of these intervals, so the factorization $f = (fc)c^{-1}$ is the desired one.

In general, we proceed as follows. Let g be the unique element of $\mathcal{D}^+$ which agrees with fc on $[0, 1] - f(0, \varepsilon) - (fc)^{-1}(\gamma, 1)$, and which is linear on $f[0, \varepsilon]$ and on $(fc)^{-1}[\gamma, 1]$. Clearly g is in SUF(1) (with a stable fixed point on $f(0, \varepsilon)$ and an unstable one on $(fc)^{-1}(\gamma, 1)$). We shall replace the factorization $f = (fc)c^{-1}$ by a factorization $f = g(c')^{-1}$, so it suffices to show that $c' = f^{-1}g$ is in SUF(1).

Now $c' = c(fc)^{-1}g$, so $c' = c$ except on $f(0, \varepsilon)$ and $(fc)^{-1}(\gamma, 1)$. Since $c'f(\varepsilon) = cf(\varepsilon) < c(\gamma) = \varepsilon < f(0)$, we see that $c'f(0, \varepsilon)$ is disjoint from $f(0, \varepsilon)$, so c' has no fixed point on $f(0, \varepsilon)$. Similarly $c'(fc)^{-1}(1) = c(fc)^{-1}(1) = f^{-1}(1) < \gamma < (fc)^{-1}(\gamma)$, so $c'((fc)^{-1}(\gamma, 1))$ is disjoint from $(fc)^{-1}(\gamma, 1)$, and so c' has no fixed points there either. Thus $c' \in \text{SUF}(1)$, and the proof of (1)$\Rightarrow$(2) is concluded.

(2)$\Rightarrow$(3). Write $f = dc$, with $c, d \in \text{SUF}(1)$. By Lemma 2.2, d^{-1} is a conjugate of c in $\mathcal{D}^+$, say $d = b^{-1}c^{-1}b$, so $f = b^{-1}c^{-1}bc$, as required.

(2)$\Rightarrow$(4). It suffices to show that any $c \in \text{SUF}(1)$ is a square of an element of $\mathcal{D} - \mathcal{D}^+$. By Lemma 2.2 it suffices to show this for just one such c. Let $a \in \mathcal{D} - \mathcal{D}^+$ have the following graph on $[-\frac{1}{2}, \frac{1}{2}]$ that is $a(0) = 0$, $a(\frac{1}{2}) = -\frac{1}{2}$, $a(-\frac{1}{2}) = \frac{1}{2}$, and



$a(x) < -x$ for $x \in (0, \frac{1}{2})$ and $a(x) > -x$ for $x \in (-\frac{1}{2}, 0)$. Clearly $c = a^2$ has a stable fixed point at $\frac{1}{2}$, an unstable one at 0, and no others in the interval $[0, 1)$.

(4)$\Rightarrow$(1). If $c \in \mathcal{D}^+$ has a fixed point then $-1 < \underline{m}c \leq 0$ and $0 \leq \bar{m}c < 1$. Thus if both c and d have fixed points, then $\underline{m}(cd) \leq \underline{m}c + \bar{m}d < 0 + 1 = 1$ and similarly $\bar{m}(cd) > -1$.

(3)$\Rightarrow$(1). Say $f = c^{-1}d^{-1}cd$. Using Lemma 2.1 we see

$$\begin{aligned}\underline{m}f = \underline{m}(c^{-1}d^{-1}cd) &\leq \underline{m}(c^{-1}) + \bar{m}(d^{-1}cd) \leq -\bar{m}c + \lceil \bar{m}(d^{-1}cd) \rceil \\ &= -\bar{m}c + \lceil \bar{m}c \rceil < 1.\end{aligned}$$

Similarly $\bar{m}f > -1$.

The inductive step in the proof of Theorem 2.3 will use the following consequence of Lemma 2.7.

PROPOSITION 2.8. *Let $f \in \mathscr{D}^+$ and $0<\varepsilon<1$ be given. Then*
(1) *$f=kh$ for some $h, k \in \mathscr{D}^+$ with*

$$0<\underline{m}h<1,$$
$$\underline{m}k<\underline{m}f-2+\varepsilon<\bar{m}k.$$

(2) *$f=kh$ for some $h, k \in \mathscr{D}^+$ with $h \in \mathrm{SUF}(1)$ and $\underline{m}k<\underline{m}f-1+\varepsilon<\bar{m}k$.*

Proof. By Lemma 2.7, $\mathrm{sh}(-\underline{m}f+1-\varepsilon)f=k'h'$, where each of k' and h' is in SUF (1). The decomposition $f=(\mathrm{sh}(\underline{m}f-1+\varepsilon)k')h'$ satisfies (2), while $f=(\mathrm{sh}(\underline{m}f-2+\varepsilon)k')(\mathrm{sh}(1)h')$ satisfies (1).

Proof of Theorem 2.3. The "only if" statements follow at once from Lemma 2.7 and Lemma 2.1(4). We prove the "if" statements by induction on g, the cases $g=1$ of (1) and $g=2$ of (2) being Lemma 2.7. Inverting f if necessary, we may assume $\bar{m}f>0$.

(1) If $\underline{m}f<2g-1$, then we use Proposition 2.8(1) to write $f=kh$ with $0<\underline{m}h<1$ and $\underline{m}k<\underline{m}f-2+\varepsilon<\bar{m}k$. By Lemma 2.7, h can be written as a commutator, and if ε is small enough, then k will be a product of $g-1$ commutators by induction hypothesis.

(2) If $\underline{m}f<g-1$, then we apply Proposition 2.8(2). The element h will, as noted in the proof of Lemma 2.7, be the square of an element of $\mathscr{D}-\mathscr{D}^+$, while k will be a product of $g-1$ such squares by induction hypothesis.

Proof of Theorem 2.4. We first note two simple equations. If $t \in \mathscr{D}$ is given by $t(x)=-x$, and if $a \in \mathscr{D}-\mathscr{D}^+$ is arbitrary, then

$$t^{-1}\cdot \mathrm{sh}(\gamma)\cdot t=\mathrm{sh}(-\gamma) \quad \text{for all} \quad \gamma \in \mathbf{R},$$
$$a^{-1}\cdot \mathrm{sh}(n)\cdot a=\mathrm{sh}(-n) \quad \text{for all} \quad n \in \mathbf{Z}.$$

Given $f \in \mathscr{D}^+$, we wish to write it as $f=a^{-1}b^{-1}ab$ with $a \in \mathscr{D}-\mathscr{D}^+$ and $b \in \mathscr{D}^+$. If f is a shift we use the equation $t^{-1}\cdot(\mathrm{sh}(\gamma))^{-1}\cdot t\cdot \mathrm{sh}(\gamma)=\mathrm{sh}(2\gamma)$, so we may assume f is not a shift. We can then find $n \in \mathbf{Z}$ such that $\underline{m}f-2n<1$ and $\bar{m}f-2n>-1$ (namely let $2n$ be the one of $\lfloor \underline{m}f \rfloor$ and $\lfloor \underline{m}f \rfloor+1$ which is even). Put $f'=\mathrm{sh}(-2n)f$.

Then f' satisfies condition (1) of Lemma 2.7, so $f'=cd$ with $c, d \in \mathrm{SUF}(1)$. Now $tc^{-1}t^{-1}$ is still in SUF (1), so by Lemma 2.2, $tc^{-1}t^{-1}=e^{-1}de$ for some $e \in \mathscr{D}^+$. Putting $a=et$, we get $c=a^{-1}d^{-1}a$, so $f'=a^{-1}d^{-1}ad$. Finally $f=\mathrm{sh}(2n)f'=$

$a^{-1}\cdot \mathrm{sh}(-n)\cdot a\cdot \mathrm{sh}(n)\cdot a^{-1}d^{-1}ad=a^{-1}(\mathrm{sh}(n)d)^{-1}a(\mathrm{sh}(n)d)$ is in the desired form.

Proof of Theorem 2.5. First note that given $f\in\mathcal{D}^+$ and $0<\varepsilon<1$, we can find a conjugate f' of f with $\underline{m}f'\le\lfloor \underline{m}f\rfloor+\varepsilon$. Indeed, if $\underline{m}f=\lfloor\underline{m}f\rfloor$ this is trivial; otherwise choose $x\in\mathbf{R}$ with $fx=x+\lfloor\underline{m}f\rfloor+t$ and $0<t<1$, and choose $a\in\mathcal{D}^+$ with $a(x)=x$ and $a(x+\varepsilon)=x+t$, and then $a^{-1}fa(x)=x+\lfloor\underline{m}f\rfloor+\varepsilon$, so $\underline{m}(a^{-1}fa)\le\lfloor\underline{m}f\rfloor+\varepsilon$.

We can thus replace each f_i in Theorem 2.5 by a conjugate f_i' with $\underline{m}f_i'\le\lfloor\underline{m}f_i\rfloor+\varepsilon$ for some small ε. We next observe that by conjugating each f_i' by a suitable shift we can move the points at which they attain their $\underline{m}$'s so as to achieve also: $\underline{m}(\prod f_i')=\sum\underline{m}f_i'$.

Thus if $r\in\mathbf{R}$ is as in the theorem, we can find $b_i\in\mathcal{D}^+$ such that $\underline{m}(\prod f_i^{b_i})<r$. Dually, one finds $c_i\in\mathcal{D}^+$ such that $r<\bar{m}(\prod f_i^{c_i})$.

Now let $e_i:[0,1]\to\mathcal{D}^+$ be a continuous path with $e_i(0)=b_i$ and $e_i(1)=c_i$ for each i ($\mathcal{D}^+$ is a convex subset of $\mathbf{R}^{\mathbf{R}}$, hence pathwise connected). If $d_t=\prod f_i^{e_i(t)}$ for $0\le t\le 1$ then, since $\underline{m}$ and $\bar{m}$ are continuous, the intermediate value theorem implies $\underline{m}(d_t)\le r\le\bar{m}(d_t)$ for some t, proving the theorem.

Proof of Theorem 2.6. The proof uses the following lemma.

LEMMA 2.9. (1) *If* $-1<\gamma_1<0<\gamma_2<1$, *then there exists* $e\in\mathcal{D}^+$ *such that* $\mathrm{sh}(\gamma_1)(\mathrm{sh}(\gamma_2))^e\in\mathrm{SUF}(1)$.

(2) *If* $-1<\gamma<1$ *and* $f\in\mathrm{SUF}(1)$, *then there exists* $e\in\mathcal{D}^+$ *such that* $f(\mathrm{sh}(\gamma))^e\in\mathrm{SUF}(1)$.

This lemma is easy to prove directly, but since it follows immediately from a stronger result, Lemma 4.2, to be proved later, we postpone its proof for now.

Returning to the proof of Theorem 2.6(1), note that the "only if" is immediate from Lemma 2.1, so we must prove the "if". Assume therefore $\sum\lfloor\gamma_i\rfloor\le -1$ and $\sum\lceil\gamma_i\rceil\ge 1$. Our first step is to "normalize" the γ_i's.

Observe that inserting or deleting a $\gamma_i=0$ does not change the problem. Also, since $\mathrm{sh}(n)$ is in the center of $\mathcal{D}^+$ for $n\in\mathbf{Z}$, if we replace each γ_i by γ_i+n_i with $n_i\in\mathbf{Z}$ and $\sum n_i=0$, then we also do not change the problem. We can thus normalize and reindex γ_i so that they become:

$$\gamma_0,\gamma_1,\ldots,\gamma_l;\qquad \gamma_0\in\mathbf{Z},\quad 0<\gamma_i<1\quad\text{for}\quad i=1,\ldots,l.$$

The conditions $\sum\lfloor\gamma_i\rfloor\le -1$ and $\sum\lceil\gamma_i\rceil\ge 1$ then become: $\gamma_0=-n$ with $1\le n\le l-1$. By eliminating γ_0 by subtracting 1 from n of the γ_i, we can renormalize once more to obtain:

$$\gamma_1,\ldots,\gamma_l;\qquad -1<\gamma_i<1\quad\text{for}\quad i=1,\ldots,l.\tag{$*$}$$

Since exactly n of these γ_i are negative, and $1 \le n \le l-1$, we can assume $-1<\gamma_1<0$ and $0<\gamma_2<1$.

We now apply part (1) of Lemma 2.9 to find $e \in \mathfrak{D}^+$ with $\operatorname{sh}(\gamma_1)\operatorname{sh}(\gamma_2)^e \in \mathrm{SUF}(1)$. Applying part (2) of the lemma iteratively, with $\gamma = \gamma_3, \ldots, \gamma_l$, then completes the proof.

To prove 2.6(2), note that if $\sum \gamma_i = 0$ we can take all $e_i = 1$. If $\sum \lfloor \gamma_i \rfloor \le -2$ and $\sum \lceil \gamma_i \rceil \ge 2$, then we can renormalize as above, to get the γ_i in the form (*) with $-1<\gamma_1, \gamma_3<0$ and $0<\gamma_2, \gamma_4<1$. Then $\{\gamma_1, \gamma_2\}$ and $\{\gamma_3, \ldots, \gamma_l\}$ each satisfy the inequalities of part (1) of the theorem, so (2) follows from (1) and Lemma 2.2.

(3) Assume some product $\prod \operatorname{sh}(\gamma_i)^{e_i}$ equals 1. If all the γ_i are integers then we must have $\sum \gamma_i = 0$. Otherwise, for each i we have $\lfloor \gamma_i \rfloor \le \underline{m}(\operatorname{sh}(\gamma_i)^{e_i}) \le \bar{m}(\operatorname{sh}(\gamma_i)^{e_i}) \le \lceil \gamma_i \rceil$, with both the first and third inequality strict for some i, so the inequalities $\sum \lfloor \gamma_i \rfloor < 0$ and $\sum \lceil \gamma_i \rceil > 0$ follow from Lemma 2.1(4).

Now suppose $\sum \lfloor \gamma_i \rfloor = -1$ and $\sum \lceil \gamma_i \rceil = +1$. Then the normalization procedure used in the proof of part (1) implies that we may assume $k=2$ and $-1<\gamma_1<0<\gamma_2<1$. The equation $\operatorname{sh}(\gamma_1)^{e_1}\operatorname{sh}(\gamma_2)^{e_2} = 1$ implies $\operatorname{sh}(\gamma_2)$ is conjugate to $\operatorname{sh}(-\gamma_1)$. The following lemma thus shows $\gamma_1 + \gamma_2 = 0$, completing the proof of (3).

LEMMA 2.10. *If $a \in \mathfrak{D}^+$ is conjugate to* $\operatorname{sh}(\gamma)$ *then $\underline{m}a \le \gamma \le \bar{m}a$.*

Proof. By Lemma 2.1(5), $\lim_{n\to\infty} \underline{m}(a^n)/n = \gamma$. But by Lemma 2.1(4), $\lim \underline{m}(a^n)/n \ge \underline{m}(a)$. Thus $\underline{m}a \le \gamma$, and similarly $\bar{m}a \ge \gamma$.

Remark. For *any* $a \in \mathfrak{D}^+$ the number $s(a) = \lim \underline{m}(a^n)/n = \lim \bar{m}(a^n)/n$ is a well defined (and well known) conjugacy invariant, and satisfies $\underline{m}a \le s(a) \le \bar{m}(a)$.

3. Seifert manifolds with transverse foliations

We consider a Seifert bundle $p: M \to F$ over a closed surface F, the fiber being S^1. The total space M may be orientable or not. Such a bundle may be described as follows: There exists some finite non-empty collection $D_1, \ldots, D_k$ of disjoint closed discs in F so that $p^{-1}(F - \cup \operatorname{int} D_i) \to F - \cup \operatorname{int} D_i$ is a locally trivial fibration admitting a section $s: F - \cup \operatorname{int} D_i \to M$, while $p^{-1}D_i \cong D^2 \times S^1$ is a solid torus. With each D_i is associated a coprime "Seifert" pair (α_i, β_i) of integers, with $\alpha_i \ge 1$, so that the class of $s(\partial D_i)$ in $\pi_1(p^{-1}D_i) = \mathbf{Z}$ is $-\beta_i$, and over D_i the map p is given in suitable coordinates by

$$p^{-1}D_i = D^2 \times S^1 \ni (re^{i\theta}, e^{i\psi}) \mapsto re^{i(\alpha_i\theta - \nu_i\psi)} \in D_i.$$

Here ν_i is an inverse of β_i modulo α_i and we are identifying D^2 and D_i with the unit disc in $\mathbf{C}$. The integer α_i is thus the class in $\pi_1(p^{-1}D_i)=\mathbf{Z}$ of the "general fiber" $p^{-1}(u)$, $u \in D_i - \{0\}$.

We shall write the Seifert pairs as rational numbers β_i/α_i. The collection $\{\beta_1/\alpha_1, \ldots, \beta_k/\alpha_k\}$ of Seifert pairs associated to $p: M \to F$ is not unique, but depends on the choice of the D_i's and of s. By changing these choices, $\{\beta_i/\alpha_i\}$ can be changed in the following ways:

(a) permute the indices;

(b) add or delete a Seifert pair $0/1$;

(c) alter each β_i/α_i by an integer, but keeping $\sum \beta_i/\alpha_i$ fixed;

(d) (only if M is non-orientable:) replace any β_i/α_i by $-\beta_i/\alpha_i$:

(e) (only if M is non-orientable:) replace any β_i/α_i by $(\beta_i \pm 2\alpha_i)/\alpha_i = (\beta_i/\alpha_i) \pm 2$.

If M is orientable, the Seifert fibration is completely classified (up to orientation preserving homeomorphisms) by the *Seifert invariant*

$$(g; \beta_1/\alpha_1, \ldots, \beta_k/\alpha_k),$$

where $g = g(F)$ is the genus of the base surface F (with the convention that g is negative for F non-orientable). The fact that the Seifert invariant is well determined up to (a), (b), and (c) above can also be formulated: M is classified by g, by the unordered set of β_i/α_i modulo 1 (omitting those that are zero modulo 1), and by $e(M \to F) = -\sum \beta_i/\alpha_i$. This number $e(M \to F)$ is called the *euler number* of the fibration, see [N–R].

Staying with the case M orientable, note that the Seifert invariant can always be put in the form

$$(g; \beta_0/1, \beta_1/\alpha_1, \ldots, \beta_k/\alpha_k),$$
$$0 < \beta_i/\alpha_i < 1 \quad \text{for} \quad i = 1, \ldots, k,$$

and this form, called *normal form*, is unique up to permutation of the indices $i = 1, \ldots, k$.

If M is non-orientable then one must include in the classifying data the information as to whether the fibers can be given mutually consistent orientations or not, and then the resulting "Seifert invariant" (well defined up to (a), (b), (c), (d), and (e)) classifies. We will not need a notation for this, so we do not introduce one. The euler number of the fibration is of course not defined in this case.

The above discussion is a modified presentation of Seifert's original classification [S], where the invariants are given in normalized form. The unnormalized version was introduced (in the oriented case) in [N] and [N–R].

We will say that M admits a *transverse foliation* if M has a codimension 1 foliation whose leaves are transverse to the fibers of $p: M \to F$. Foliations here are always C^0 foliations, however, when we give conditions below for existence of foliations, these foliations can actually be chosen to be analytic, by the results of the next section.

We can now state the main geometric results of this paper. Let $p: M \to F$ be a Seifert fibration over the closed surface F with Seifert pairs $\beta_1/\alpha_1, \ldots, \beta_k/\alpha_k$. Let χ be the euler characteristic of F, so $\chi = 2-2g$ if F is orientable (that is, $g \geq 0$) and $\chi = 2+g$ if F is non-orientable (that is $g<0$).

THEOREM 3.1. *If M is non-orientable, then it admits a transverse foliation. This foliation can be chosen with all leaves compact.*

THEOREM 3.2. *If M is orientable and F is not the sphere* (*that is* $\chi \neq 2$), *then M admits a transverse foliation if and only if*

$$\sum \lfloor \beta_i/\alpha_i \rfloor \leq -\chi \quad \textit{and} \quad \sum \lceil \beta_i/\alpha_i \rceil \geq \chi$$

or $M \to F$ has normal form Seifert invariant $(-1; 0/1)$. *For a normal form invariant* $(g; \beta_0/1, \beta_1/\alpha_1, \ldots, \beta_k/\alpha_k)$ *these inequalities may be rewritten* $\chi - k \leq \beta_0 \leq -\chi$.

THEOREM 3.3. *If* $F = S^2$ (*that is* $\chi = 2$) *and* $\sum \beta_i/\alpha_i = 0$ *or* $\sum \lfloor \beta_i/\alpha_i \rfloor \leq -\chi = -2$ *and* $\sum \lceil \beta_i/\alpha_i \rceil \geq \chi = 2$, *then M admits a tranverse foliation.*

Conversely, if $M \to S^2$ admits a transverse foliation then either $\sum \beta_i/\alpha_i = 0$ *or* $\sum \lfloor \beta_i/\alpha_i \rfloor \leq -1$ *and* $\sum \lceil \beta_i/\alpha_i \rceil \geq 1$ *and at least one of these inequalities is strict.*

Before proving 3.1 to 3.3, we pause to note the corresponding results for foliations with only compact leaves in case M is orientable. Recall that in this case the euler number of the fibration, $e(M \to F) = -\sum \beta_i/\alpha_i$, is defined.

THEOREM 3.4. *Suppose that M is orientable. The following statements are equivalent*

(i) $e(M \to F) = 0$;

(ii) *M has a transverse foliation with a compact leaf*;

(iii) *M has a smooth transverse foliation with all leaves compact.*

Proof. The euler number $e(M \to F)$ is the obstruction to finding a "rational section" to the Seifert fibration, that is, a closed compact surface $\bar{F} \rightsquigarrow M$ immersed transverse to all fibers. This is implicit in [N–R], but can be seen explicitly as follows using the naturality properties of e of [N–R, Theorem 1.2] (which contained a misprint: "homeomorphism" should have been "continuous map"). By taking a double cover if necessary one can reduce to the case $g \geq 0$.

The Seifert fibration is then given by an S^1-action, and by factoring a $(\mathbf{Z}/a)\subset S^1$ one can reduce to the case of a genuine S^1-bundle. Finally, for a genuine S^1-bundle the existence of a rational section is equivalent, by standard obstruction theory, to the rational euler class in $H^2(F:\mathbf{Q})$ being trivial.

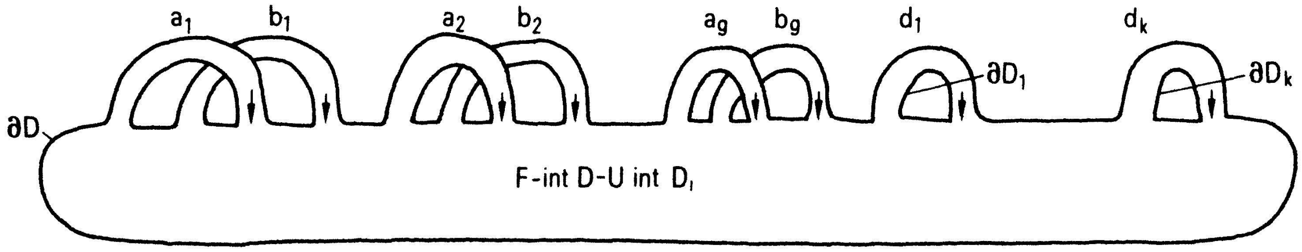
Thus (ii)$\Rightarrow$(i). Clearly (iii)$\Rightarrow$(ii).

Finally (i)$\Rightarrow$(iii) by Conner and Raymond [C–R]. Various approaches to this are also discussed in [N–R]. In fact, a complete set of models for the foliations given by (iii) is as follows. Those with orientable base are given by fiber products $M=Z\times_{\mathbf{Z}/a}S^I$, where $(X,\mathbf{Z}/a)$ is an oriented $(\mathbf{Z}/a)$-action on a closed surface and $\mathbf{Z}/a$ acts standardly on S^1. The foliation is by fibers of $M\rightarrow S^1/(\mathbf{Z}/a)\cong S^1$. Those with non-orientable base are given by fiber products $M=X\times_{D_{2a}}S^1$, where D_{2a} is the dihedral group $\langle t,g\mid t^2=g^a=1, t^{-1}gt=g^{-1}\rangle$ acting standardly on S^1 and acting on the closed oriented surface X in such a way that g is orientation preserving and t is free and orientation reversing on $X/\langle g\rangle$. The foliation is by fibers of $M\rightarrow S^1/D_{2a}\cong[0,1]$.

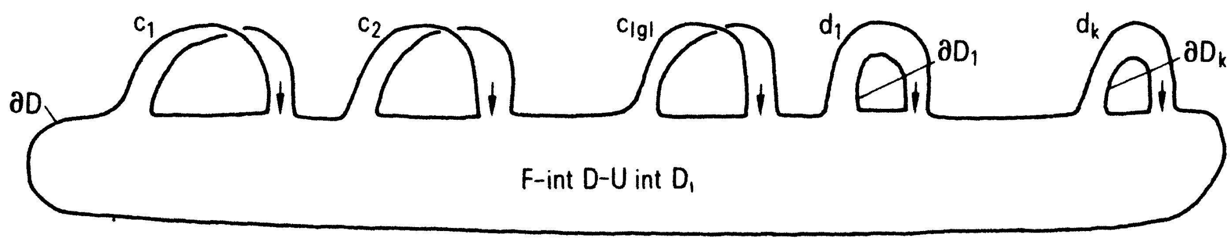
Generalizing the ideas of Milnor and Wood, we reduce 3.1–3 to algebraic theorems as follows.

Suppose $M\rightarrow F$ has a transverse foliation. We will utilize the notions of the introduction to this section, and in particular consider disks $D_1,\dots,D_k$ in F so that $M\rightarrow F$ admits a section s over $F-U\operatorname{int}D_i$.

Cut F open along some additional disk, D, and open out $F-\operatorname{int}D-U\operatorname{int}D_i$ to represent it as a disk with oriented handles a_j, b_j, d_i:



in case F is orientable; or as a disc with handles c_i, d_i:



in case F is non-orientable.

Using the section s, we may trivialize the fibration over $F-\operatorname{int}D-\bigcup\operatorname{int}D_i$. Moreover we may assume this trivialization of the fibration agrees with the foliation over the disk $F-\operatorname{int}D-\bigcup\operatorname{int}D_i-\{\text{Handles}\}$. Then by following the leaves of the foliation around a handle we may associate to each handle an element of $\mathcal{D}^+$, if the fiber orientation is preserved over a path going around that handle, or of $\mathcal{D}-\mathcal{D}^+$ if the fiber orientation is reversed over a path around the handle. The reason why the element corresponding to a handle is in $\mathcal{D}$ rather than Homeo (S^1) is that our section s on the handle tells us how to lift to $\mathcal{D}$.

Of course the fiber orientation is preserved around each handle d_i, and we see that the element of $\mathcal{D}^+$ associated to d_i is conjugate to the shift $\mathrm{sh}\,(-\beta_i/\alpha_i)$. Indeed, in the coordinates in $p^{-1}D_i = D^2 \times S^1$ introduced at the beginning of this section, let m denote a meridean $S^1 \times \{1\} \subset \partial(D^2 \times S^1)$, l denote a longitude $\{1\} \times S^1$, h denote a typical fiber $p^{-1}(u)$ with $u \in \partial D_i$, and q denote $s(\partial D_i)$. Then our explicit description of p implies the following homology relations in $H_1(\partial(D^2 \times S^1))$: $h = \nu_i m + \alpha_i l$, $q = -(\mu_i m + \beta_i l)$, where μ_i is defined by $\nu_i\beta_i - \mu_i\alpha_i = 1$. This implies the homology relation $m = \beta_i h + \alpha_i q$. But m represents the homology class of the intersection of a leaf of our foliation with $\partial(D^2 \times S^1)$, so this homology relation says that as the leaf winds α_i times around the handle it also winds β_i times around the fiber in our given trivialization of the bundle structure on $\partial(D^2 \times S^1)$. Thus in one circuit of the handle in the q-direction we get a conjugate of $\mathrm{sh}\,(\beta_i/\alpha_i)$, so d_i is conjugate to $\mathrm{sh}\,(-\beta_i/\alpha_i)$, as claimed.

We write a_j, b_j, c_j, and d_i again for the element of $\mathcal{D}$ associated to the corresponding handle. Since ∂D is the boundary of a disk, over which the fibration and foliation will be trivial, the element of $\mathcal{D}$ induced over ∂D must be 1. Thus, in the case F orientable, we have

$$\prod_{j=1}^{g} a_j b_j^{-1} a_j^{-1} b_j \prod_{i=1}^{k} d_i = 1, \tag{$*$}$$

while if F is non-orientable we get

$$\prod_{j=1}^{|g|} c_j^2 \prod_{i=1}^{k} d_i = 1. \tag{$**$}$$

Conversely, given elements a_j, b_j, d_i and $\mathcal{D}$ satisfying $(*)$ or c_j, d_i satisfying $(**)$, with d_i conjugate to $\mathrm{sh}\,(-\beta_i/\alpha_i)$, the above discussion yields a construction of a transverse foliation for the corresponding Seifert fibration. Thus the problem of the existence of transverse foliations becomes the problem of finding appropriate factorizations $(*)$ or $(**)$.

Proof of Theorem 3.1. In this case, either F is orientable, and some a_j and/or b_j is in $\mathcal{D}-\mathcal{D}^+$, or F is non-orientable and some c_j lies in $\mathcal{D}^+$.

In the former case suppose it is a_l which is required to lie in $\mathcal{D}-\mathcal{D}^+$. We use Theorem 2.4 to find $a_l \in \mathcal{D}-\mathcal{D}^+$ and $b_l' \in \mathcal{D}^+$ such that $a_l(b_l')^{-1}a_l^{-1}b_l'$ is the shift by

$\sum \beta_i/\alpha_i$. If b_l is required to lie in $\mathcal{D}^+$ take $b_l = b_l'$; if b_l is required to lie in $\mathcal{D}-\mathcal{D}^+$ take $b_l = a_l b_l'$. If we choose the remaining a_j's and b_j's so their commutators are trivial we have satisfied equation $(*)$, with $d_i = \operatorname{sh}(-\beta_i/\alpha_i)$.

To see the statement about compact leaves, observe that we do not really need Theorem 2.4, we can work more explicitly. Namely choose a_l in the above argument to be $a_l = t$ with $tx = -x$, and choose $b_l' = \operatorname{sh}(\gamma)$ with $\gamma = (\sum \beta_i/\alpha_i)/2$. Since $t \operatorname{sh}(-\gamma)t^{-1} \operatorname{sh}(\gamma) = \operatorname{sh}(2\gamma)$, this does what is required, and clearly leads to a smooth foliation with compact leaves.

In the case that F is non-orientable it is enough to note that any shift has a square root in $\mathcal{D}^+$ to see that $(**)$ is solvable, and to choose this square root as a rational shift to get a smooth foliation with compact leaves.

Proof of Theorem 3.2. We first consider the case that F is orientable. Then our discussion shows that a transverse foliation exists if and only if we can represent some product of conjugates of the shifts $\operatorname{sh}(\beta_i/\alpha_i)$,

$$d = \prod_{i=1}^{k} \operatorname{sh}(\beta_i/\alpha_i)^{c_i},$$

as a product of g commutators. By Theorem 2.3 we can do this if and only if we can find d as above with

$$\underline{m}d < 2g-1 = 1-\chi$$
$$\bar{m}d > 1-2g = \chi-1.$$

But by Lemma 2.1, any d as above satisfies $\sum \lfloor \beta_i/\alpha_i \rfloor \leq \underline{m}d$ and $\sum \lceil \beta_i/\alpha_i \rceil \geq \bar{m}d$, so $\sum \lfloor \beta_i/\alpha_i \rfloor < 2g-1$ and $\sum \lceil \beta_i/\alpha_i \rceil > 1-2g$. Since these are inequalities between integers, the necessity of the condition in Theorem 3.2 is shown. Conversely, suppose the inequalities of Theorem 3.2 are satisfied. If the β_i/α_i are all integral then $d = \operatorname{sh}(\sum \beta_i/\alpha_i)$ does what is required. Otherwise $\sum \lfloor \beta_i/\alpha_i \rfloor < \sum \lceil \beta_i/\alpha_i \rceil$ and the inequalities $\sum \lfloor \beta_i/\alpha_i \rfloor \leq -\chi$ and $\sum \lceil \beta_i/\alpha_i \rceil \geq \chi$ imply that we can find $\sum \lfloor \beta_i/\alpha_i \rfloor < r < \sum \lceil \beta_i/\alpha_i \rceil$ with $\chi - 1 < r < 1-\chi$. Theorem 2.5 then implies the existence of the desired element d.

If F is non-orientable of genus $g \leq -2$, then the proof is just the same, using $(**)$ in place of $(*)$. We may thus assume that $g = -1$, so $\chi = 1$ and $F = \mathbf{R}P^2$. In this case condition $(**)$ becomes that the element d above can be chosen to equal the square of an element of $\mathcal{D}-\mathcal{D}^+$.

If we eliminate the trivial case that the β_i/α_i are all integral and $\sum \beta_i/\alpha_i = 0$, which is the exceptional case of Theorem 3.2, then the sufficiency of the conditions $\sum \lfloor \beta_i/\alpha_i \rfloor \leq -\chi = -1$ and $\sum \lceil \beta_i/\alpha_i \rceil \geq \chi = 1$ is just part (1) of Theorem 2.6

together with the observation, made in the proof of 2.7, that any element of SUF (1) is a square of an element of $\mathcal{D}-\mathcal{D}^+$.

To see the converse, observe first that if $c\in\mathcal{D}-\mathcal{D}^+$ with $c^2\neq 1$, then $\underline{m}c^2<0<\bar{m}c^2$ (since $c^2x<x\Leftrightarrow c^2x>cx$, since c reverses orientation). Thus if $d=c^2\neq 1$ the inequalities $\sum\lfloor\beta_i/\alpha_i\rfloor\leq -1$ and $\sum\lceil\beta_i/\alpha_i\rceil\geq 1$ follow from Lemma 2.1, while if $d=1$ then these inequalities follow from Theorem 2.6(3), unless $\sum\beta_i/\alpha_i=0$. But in the latter case the inequalities still hold, unless the β_i/α_i are all integral, which is the exceptional case of Theorem 3.2.

Proof of 3.3. Here $(*)$ becomes $d=1$, with d as in the previous proof, so Theorem 3.3 becomes parts (2) and (3) of Theorem 2.6.

There is an interesting corollary of the above proofs. Namely if F is orientable, then the fundamental group $\pi_1(M)$ can be presented as follows: it is generated by elements a_j, b_j, d_i, and z subject to the relation $(*)$ above and the additional relations

$$d_i^{\alpha_i}z^{\beta_i}=1,\qquad d_iz=zd_i,$$
$$a_jz=za_j^{\pm 1},\qquad b_jz=zb_j^{\pm 1},$$

where the exponents are $+1$ or -1 according as the corresponding element a_j or b_j preserves or reverses fiber orientation in M. For F non-orientable the corresponding statement holds using elements c_j, d_i, z, and replacing $(*)$ by $(**)$. These are by an easy Van Kampen argument, see Seifert [5]. The element z is represented by a generic fiber of M; The other elements have their obvious geometric meanings. Our proof thus showed the following.

THEOREM 3.5. *The Seifert manifold M admits a transverse foliation if and only if there exists a homomorphism $\varphi:\pi_1(M)\to\mathcal{D}$ with $\varphi(z)=\mathrm{sh}\,(1)$, where $z\in\pi_1(M)$ is the class of a generic fiber of M.*

Theorems 3.1 to 3.3 thus give numerical conditions for existence of such a φ.

Given φ as in the above theorem, one can reconstruct M with its transverse foliation as follows. $G=\pi_1(M)/\langle z\rangle$ can be represented as a group of isometries of a geometry X, where X is S^2, euclidean space $\mathbf{R}^2$, or the hyperbolic plane $\mathbf{H}$. Thus $\pi_1(M)$ acts (non-effectively) on X. Via φ it also acts on $\mathbf{R}$, so it acts diagonally on $X\times\mathbf{R}$ and we can form the quotient space $X\times_{\pi_1(M)}\mathbf{R}=M$. The Seifert fibration is given by $X\times_{\pi_1(M)}\mathbf{R}\to X/\pi_1(M)=X/G=F$, and the foliation is induced from the foliation of $X\times\mathbf{R}$ by fibers of $X\times\mathbf{R}\to\mathbf{R}$.

Remark. The foliation will have all leaves compact if and only if the image of φ acts discretely on $\mathbf{R}$. One can check that φ can always be so chosen if $X = S^2$ or $\mathbf{R}^2$.

To close this section we discuss briefly the case of Seifert manifolds with boundary. In this case M always admits transverse foliations, but a specified transverse foliation on ∂M may not extend to one on M. The condition for such an extension to exist can be derived just as in the closed case. For simplicity we just discuss the case of orientable M.

Let the boundary components of M be denoted $T_1, \ldots, T_r$. We make the following simplifying assumption: $r \geq 1$ *and the foliation restricted to each* T_i *has non-compact leaves.* This is no loss of generality, since, if the foliation had only compact leaves on some T_i, then we could eliminate this boundary component by pasting in a solid torus over which the foliation extends.

Choose a section to the Seifert fibration on each boundary component T_j. Then a transverse foliation on T_j determines an element $h_j \in \mathscr{D}^+$, by taking holonomy in $\text{Homeo}^+(S^1)$ and using the section to lift to $\mathscr{D}^+$. This element is well defined up to conjugacy, so $\lfloor \underline{m} h_j \rfloor$ and $\lceil \bar{m} h_j \rceil$ are well defined. Moreover, our choice of sections on ∂M lets us define the Seifert invariant of M,

$$(g, r; \beta_1/\alpha_1, \ldots, \beta_k/\alpha_k)$$

(we now include the number of boundary components in our notation), well defined up to the usual indeterminacy, namely any $\beta_i/\alpha_i = 0/1$ can be added or deleted and each β_i/α_i can be altered by an integer so long as $\sum \beta_i/\alpha_i$ remains constant.

We put $\chi = 2-2g$ if $g \geq 0$ and $\chi = 2+g$ if $g < 0$, so χ is the euler characteristic of F with its boundary components capped by discs.

THEOREM 3.6. *If* $g \neq 0$ *or* -1 *then the inequalities* $\sum \lfloor \beta_i/\alpha_i \rfloor + \sum \lfloor \underline{m} h_j \rfloor \leq -\chi$ *and* $\sum \lceil \beta_i/\alpha_i \rceil + \sum \lceil \bar{m} h_j \rceil \geq \chi$ *are necessary and sufficient for the given foliation on* ∂M *to extend to a transverse foliation on* M. *This is true also for* $g = -1$ *unless the* β_i/α_i *are all integral and* $r = 1$. *If* $g = 0$ *these inequalities, with* χ *replaced by* 1, *are necessary, but not sufficient in general.*

Note that the inequalities are independent of the choice of sections on ∂M, since altering the section on T_j, say, alters $\lfloor \underline{m} h_j \rfloor$ and $\lceil \bar{m} h_j \rceil$ by an integer, and alters $\sum \lfloor \beta_i/\alpha_i \rfloor$ and $\sum \lceil \beta_i/\alpha_i \rceil$ by the negative of this integer.

The proof of this theorem is just like the closed case, so we omit it. We do not have complete results for $g = 0$. However, in the exceptional case with $g = -1$, omitted in the above theorem, we can assume, by changing the section on ∂M if necessary, that the Seifert invariant is $(-1, 1; 0/1)$, and a necessary and sufficient condition for extension is that h_1 be the square of an element of $\mathscr{D}-\mathscr{D}^+$.

4. The projective linear group

In this section we show that the results of Section 2 also hold in the universal cover $\widetilde{\mathrm{PGL}}(2, \mathbf{R})$ of the projective linear group $\mathrm{PGL}(2, \mathbf{R})$. In particular this gives a rather pleasanter characterization of products of g commutators in $\widetilde{\mathrm{PSL}}(2, \mathbf{R})$ than that given by Wood [W]. It also shows that the transverse foliations whose existence is given by the theorems of Section 3 can always be chosen with "structure group" $\mathrm{PGL}(2, \mathbf{R})$, and in particular they can be chosen as analytic foliations.

We denote $\widetilde{\mathrm{PGL}}(2, \mathbf{R})$ by G. since $\mathrm{PGL}(2, \mathbf{R})$ acts on $\mathbf{R}P^1 = S^1$, we have an inclusion $\mathrm{PGL}(2, \mathbf{R}) \subset \mathrm{Homeo}(S^1)$, which induces an inclusion $G \subset \mathscr{D}$. Denote $G^+ = G \cap \mathscr{D}^+ = \widetilde{\mathrm{PSL}}(2, \mathbf{R})$. G^+ has index 2 in G with the nontrivial coset represented by the "flip" $tx = -x$.

Let $A = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \in \mathrm{PSL}(2, \mathbf{R})$. Then A has two fixed points on $\mathbf{R}P^1 = S^1$, so it has a unique lift $\tilde{A} \in G^+$ with fixed points. Let $K \subset G^+$ be the conjugacy class of $\tilde{A}$ (this is a slight modification of Wood's notation).

THEOREM 4.1. *Theorems* 2.3, 2.4, 2.5, 2.6 *and the remark after* 2.3 *are all valid with* $\mathscr{D}, \mathscr{D}^+$ *and* SUF(1) *replaced by* G, G^+ *and* K.

This can be deduced without too much trouble from a comparison of Wood's results with ours, but we give a self-contained version, based on the following improvement of Wood's Proposition 5.1.

LEMMA 4.2. *Given* $f, g \in G^+$ *with* $-1 < \bar{m}f$, $\underline{m}f < 0$, $0 < \bar{m}g$, $\underline{m}g < 1$, *there exist conjugates* f', g', *of* f, g *in* G^+ *with* $f'g' = \tilde{A}$. *In particular, by replacing either* f *or* g *by a conjugate one can achieve* $fg \in K$ *and* $gf \in K$ (*note* $gf = g(fg)g^{-1} \in K \Leftrightarrow fg \in K$).

Proof. Since $\mathrm{SL}(2, \mathbf{R})$ is a connected 2-fold cover of $\mathrm{PSL}(2, \mathbf{R})$, the group G^+ is also the universal cover of $\mathrm{SL}(2, \mathbf{R})$. Let F, $G \in \mathrm{SL}(2, \mathbf{R})$ be the images of f and g. We first show we can solve the relevant conjugacy problem in $\mathrm{SL}(2, \mathbf{R})$.

Elements $H \in \mathrm{SL}(2, \mathbf{R}) - \{\pm 1\}$ are classified up to conjugacy as follows. If $|\mathrm{tr}\, H| > 2$ then $\mathrm{tr}\, H$ classifies H up to conjugacy. For each value of $\mathrm{tr}\, H$ with $|\mathrm{tr}\, H| \leq 2$ there are exactly two conjugacy classes, distinguished as follows: there exists $v \in \mathbf{R}^2$ such that v, Hv forms a basis of $\mathbf{R}^2$, and this basis will be an oriented basis for one conjugacy class and non-oriented for the other. The corresponding element $[H] \in \mathrm{PSL}(2, \mathbf{R})$ has either 2 fixed points on $\mathbf{R}P^1 = S^1$, one fixed point, or is conjugate to a rotation, according as $|\mathrm{tr}\, H| > 2$, $|\mathrm{tr}\, H| = 2$, or $|\mathrm{tr}\, H| < 2$.

Note also that, since $\mathrm{SL}(2, \mathbf{R})$ is the 2-fold cover of $\mathrm{PSL}(2, \mathbf{R})$, the element $1 \in \mathrm{SL}(2, \mathbf{R})$ lifts to $\mathrm{sh}(2n) \in G^+$ with $n \in \mathbf{Z}$ and $-1 \in \mathrm{SL}(2, \mathbf{R})$ lifts to $\mathrm{sh}(2n+1) \in G^+$ with $n \in \mathbf{Z}$.

We claim that our element F above is conjugate to a unique $F_0 \in \mathrm{SL}(2, \mathbf{R})$ of the form

$$F_0 = \begin{pmatrix} 0 & 1 \\ -1 & a \end{pmatrix}, \qquad a \in \mathbf{R},$$

and every such F_0 occurs. Of course, a is given by $a = \mathrm{tr}\, F$. Indeed, our condition on f is equivalent to saying $-1 < fr - r < 0$ for some $r \in \mathbf{R}$, in other words, there exists $v \in \mathbf{R}^2$ such that v, Fv is an unoriented basis of $\mathbf{R}^2$. Thus $F \in \mathrm{SL}(2, \mathbf{R}) - \{\pm 1\}$, and if $|\mathrm{tr}\, F| \le 2$ then just the "negative" conjugacy class is permitted. Since F_0 is in this conjugacy class if $|a| \le 2$, our claim follows.

Similarly G is conjugate to a unique G_0 of the form

$$G_0 = \begin{pmatrix} b & -\frac{1}{2} \\ 2 & 0 \end{pmatrix}.$$

Since $F_0 G_0 = \begin{pmatrix} 2 & 0 \\ * & \frac{1}{2} \end{pmatrix}$ is conjugate to $A = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$, by a further conjugation if necessary we can find F', G' conjugate to F, G such that $F'G' = A$.

Now let $f', g' \in G^+$ be the lifts of F', G' which satisfy $-1 < \bar{m}f'$, $\underline{m}f' < 0$, $0 < \bar{m}g'$, $\underline{m}g < 1$. They are unique, since lifts of elements of $\mathrm{SL}(2, \mathbf{R})$ are determined up to even integral shifts. Then f', g' are conjugate to f, g, and $f'g' = \mathrm{sh}(2n)\tilde{A}$ for some $n \in \mathbf{Z}$, since $f'g'$ is a lift of A. If F', G' are each conjugate to rotations, that is $|a| < 2$ and $|b| < 2$, then $-1 < \underline{m}f' \le \bar{m}f' < 0$ and $0 < \underline{m}g' \le \bar{m}g' < 1$, so $-1 < \underline{m}(f'g') \le \bar{m}(f'g') < 1$. Thus in this case $n = 0$ and $f'g' = \tilde{A}$. Thus by continuity this holds for any value of a and b, completing the proof.

We now return to the proof of 4.1. We must first prove the analog of Lemma 2.7, with $\mathscr{D}, \mathscr{D}^+$, SUF (1) replaced by G, G^+, K. For the implication (1)$\Rightarrow$(2), observe that if $\underline{m}f < 1$ and $\bar{m}f > -1$ then, by replacing f by f^{-1} if necessary we can assume f is as in Lemma 4.2. Then choosing $g \in K$, so $g^{-1} \in K$, Lemma 4.2 implies that f is the product of two elements of K. For the proof of (2)$\Rightarrow$(4) we observe that $\tilde{A}$ is the square of a lift to G of the element $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \in \mathrm{PGL}(2, \mathbf{R})$, so any element of K is the square of an element of $G - G^+$. The rest of the proof of the analogues of 2.7, 2.8, 2.3, 2.4 now are exactly as before. For the proof of the analogue of 2.5, note that the conjugating element a used in the proof of 2.5 can be chosen in G^+, so that proof also goes through with no change. Finally Lemma 2.9 is a special case of Lemma 4.2, so the proof of the analogue of 2.6 goes through with no change. This completes the proof of 4.1. Using Theorem 3.5 and the discussion after it, we get:

COROLLARY 4.3. *The necessary and/or sufficient conditions of Section* 3 *for the existence of a transverse foliation on a Seifert manifold M are also necessary and/or sufficient for the existence of an analytic such foliation, and for the existence of a homomorphism* $\varphi : \pi_1(M) \to G = \widetilde{\mathrm{PGL}}(2, \mathbf{R})$ *taking the class h of a non-singular fiber to* $sh\ (1) \in G$.

Remark. The homomorphism $\varphi : \pi_1(M) \to G$ of this corollary can practically never be found injective, with discrete image in G. Indeed this will be so if and only if the foliation is an Anosov foliation, discussed in the next section, which considerably restricts the possibilities for M.

Problem. Are the results of Section 2 valid also for groups between G and $\mathscr{D}$? This is particularly interesting for $C^r\mathscr{D} = \{f \in \mathscr{D} \mid f \text{ is } C^r\text{-smooth}\}$. All one needs is a suitable substitute for (1)$\Rightarrow$(2) in Lemma 2.7, the rest of the proof then runs itself.

5. Examples

We describe briefly examples of Maria Carmen del Gazolas [dG]. They generalize the well known Anosov foliation of a quotient $M = \Gamma\backslash\mathrm{PSL}(2, \mathbf{R})$ of $\mathrm{PSL}(2, \mathbf{R})$ by a discrete subgroup, induced by the foliation of $\mathrm{PSL}(2, \mathbf{R})$ by fibers of $\mathrm{PSL}(2, \mathbf{R}) \to \mathrm{PSL}(2, \mathbf{R})/U$, where $U \subset \mathrm{PSL}(2, \mathbf{R})$ is the subgroup of upper triangular matrices.

Let S be a compact surface with a riemannian metric with a finite number k of conical metric singularities with cone angles $2\pi\delta_i$, $i = 1, \ldots, k$. A neighborhood of such a point is obtained from a solid angle of measure $2\pi\delta_i$ by identifying its sides. Define the Euler characteristic of S by

$$\chi(S) = \chi(S_0) - \sum_{i=1}^{k} (1 - \delta_i)$$

where $\chi(S_0)$ is Euler characteristic of the underlying topological surface. The Gauss–Bonnet formula is $2\pi\chi(S) = \int_S K\,dv$, so if we assume S has constant curvature K, we see that K has the same sign as $\chi(S)$.

Now suppose each δ_i is a rational number γ_i/α_i. Then the unit tangent bundle $M=T^1S$ of S is well defined and is a Seifert bundle with an α_i-fold fiber over the i-th cone point. It is not hard to compute the Seifert invariant explicitly using the definition at the start of Section 3. Namely choose the section s to be a vector field which is radially outward at each of the cone points of S and has a unique singularity of index $\chi(S_0)-k$ otherwise. A simple calculation shows this leads to Seifert invariant

$$(g;(k-\chi(S_0))/1, -\gamma_1/\alpha_1, \ldots, -\gamma_k/\alpha_k)$$

which has normal form

$$(g;\beta_0/1, \beta_1/\alpha_1, \ldots, \beta_k/\alpha_k)$$

with

$$\beta_i = b_i\alpha_i - \gamma_i \qquad b_i = \lceil \gamma_i/\alpha_i \rceil > 0$$

$$\beta_0 = -b+k-\chi(S_0), \qquad b=\sum_1^k b_i.$$

Notice that $\beta_0 \le -\chi(S_0)$. Moreover $e(M \to S)=\chi(S)$, so if S has non-positive curvature we get $e(M \to S) \le 0$, or equivalently $-\sum_{i=1}^k (\beta_i/\alpha_i) \le \beta_0$.

Now suppose S is as above with constant curvature ≤ 0, that is, it is hyperbolic or euclidean. Then the usual parallelism on $M=T^1S$ gives a foliation on M transverse to the fibers. Precisely, two vectors in euclidean or hyperbolic space are parallel if their directed geodesics stay a bounded distance apart in positive time. The unit tangent bundle of a euclidean or hyperbolic surface element is thus canonically foliated by families of parallel vectors. This foliates M away from its singular fibers, and this foliation is easily seen to extend across the singular fibers.

We have already observed that a necessary condition for an M with normal form invariant $(g;\beta_0/1, \beta_1/\alpha_1, \ldots, \beta_k/\alpha_k)$ to arise as one of these examples is

$$-\sum_{i=1}^k (\beta_i/\alpha_i) \le \beta_0 \le -\chi(S_0). \tag{5.1}$$

Since $-M$ has normal form invariant $(g;-k-\beta_0/1, (\alpha_1-\beta_1)/\alpha_1, \ldots, (\alpha_k-\beta_k)/\alpha_k)$, the same condition for $-M$ becomes, after trivial simplification,

$$\chi(S_0)-k \le \beta_0 \le -\sum_{i=1}^k (\beta_i/\alpha_i). \tag{5.2}$$

If one shows that these necessary conditions are also sufficient, one has an alternate proof of sufficiency in Theorems 3.2 and 3.3. This presumably can be done in general; in particular, if the desired cone angles $2\pi\delta_i$ of S are all at most 2π, it is not hard to show the existence of S. This already gives sufficient examples to show that the sufficient condition of Theorem 3.3 for the $g=0$ case can be weakened as follows:

PROPOSITION 5.3. *If M is orientable and Seifert fibered over S^2, then a sufficient condition for M to admit a transverse foliation is that either* (5.1) *or* (5.2) *is valid. Here* $\chi(S_0)=2$.

It seems a reasonable conjecture that this condition is also necessary. The results of Greenberg [G] prove this for transverse foliations with structure group $\mathrm{PSL}(2,\mathbf{R})$ when $k=3$. We have some slight improvements of the necessary condition of Theorem 3.3 in general, but they do not come close to this conjecture, so we omit them.

The Anosov foliation mentioned at the beginning of this section are the special case of Maria Carmen del Gazolas' examples for cone angles of the form $2\pi/\alpha_i$.

6. Making foliations tranverse

Very many foliations of Seifert manifolds are isotopic to transverse foliations. We will state two results in this direction which generalize the results of Thurston [T] and Levitt [L] on S^1-bundles. The proofs rely on the results and techniques of [L] and [T], and we omit them. The possibility of making such a generalization was also observed by Johannson in discussions with one of the authors in 1976.

We consider a closed 3-manifold M with Seifert fibration $p:M\to F$ and a foliation $\mathcal{F}$ of M which is transversely orientable and C^2. We will suppose that no leaf of $\mathcal{F}$ is a torus or a Klein-bottle.

THEOREM 6.1. *If $\chi(F)<-1$ then $\mathcal{F}$ is homotopic to a foliation transverse to every fiber of p. The same is true for $\chi(F)\geq -1$ provided that*

(1) *$F\neq S^2$ or $\mathbf{RP}^2$,*

(2) *If $F=T^2$ then p has exceptional fibers, and*

(3) *either* (*a*) *The orientation of each fiber of p is preserved along each curve in F, or* (b) *The leaves of $\mathcal{F}$ are orientable.*

In proving Theorem 6.1, it is convenient to isotope $\mathcal{F}$ to make one leaf transverse and then apply the following somewhat more general result:

THEOREM 6.2. *If $\mathcal{F}$ is transverse to one fiber of p then $\mathcal{F}$ is isotopic to a foliation transverse to every fiber of p.*

REFERENCES

[C–R] P. E. CONNER and F. RAYMOND, *Injective actions of toral groups*, Topology *10* (1971), 283–296.

[dG] MARIA CARMEN DEL GAZOLAS, *Foliations on Seifert bundles*, to appear.

[G] L. GREENBERG, *Homomorphisms of triangle groups into* PSL(2, **C**), *Riemann surfaces and Related Topics*, Proceedings of the 1978 Stony Brook conference (Princeton Univ. Press 1980), 167–181.

[L] G. LEVITT, *Feuilletages des variétés de dimension* 3 *qui sont des fibrés en circles.* Comment Math. Helv. *53* (1978), 572–594.

[Mi] J. MILNOR, *On the existence of a connection with curvature zero*, Comment. Math. Helv. *32* (1957–58), 215–223.

[Ne] W. D. NEUMANN, *S^1-actions and the α-invariant of their involutions*, Bonner Math. Schriften *44* (Bonn 1970).

[N–R] W. D. NEUMANN and F. RAYMOND, Seifert manifolds, plumbing, μ-invariant and orientation reversing maps, Alg. and Geom. Topology, Springer Lecture Notes in Math. *664* (1978), 163–196.

[No] S. P. Novikov, Topology of foliations, Trans. of the Moscow Math. Soc. for the year 1965, *14* (1967), 268–304.

[O] P. ORLIK, Seifert fibre spaces, Springer Lecture Notes in Math. *291* (1972).

[R] R. ROUSSARIE, *Plongements dans les variétés feuilletées et classification de feuilletages sans holonomie*, Publ. math. I.H.E.S. *43* (1974), 101–141.

[Se] H. SEIFERT, *Topologie dreidimensionaler gefaserter Raüme*, Acta. Math. *60* (1933), 147–238.

[Sch] A. J. SCHWARTZ, *A generalization of a Poincaré–Bendixson theorem to closed two dimensional manifolds*, Amer. J. Math. *85* (1963), 453–458.

[Su] D. SULLIVAN, *A generalization of Milnor's inequalities concerning affine foliations and affine manifolds*, Comment. Math. Helv. *51* (1976), 183–184.

[T] W. P. THURSTON, Foliations of 3-manifolds which are circle bundles, thesis, Berkeley 1972.

[W] J. W. WOOD, *Bundles with totally disconnected structure group*, Comment. Math. Helv. *46* (1971), 257–273.

Department of Mathematics
University of Maryland
College Park, MD 20742 USA

Received May 19, 1980, November 1981