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Graph theoretic techniques in algebraic geometry II: construction of singular complex surfaces of the rational cohomology type of $\mathbf{CP}^2$

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Abstract. Methods of graph theory are used to obtain rational projective surfaces with only rational double points as singularities and with rational cohomology rings isomorphic to that of the complex projective plane. Uniqueness results for such cohomology $\mathbf{CP}^2$'s and for rational and integral homology $\mathbf{CP}^2$'s are given in terms of the types A_k, D_k, or E_k of singularities allowed by the construction.

This paper continues our discussion [4] of the use of graph theoretic methods in the construction of compact projective algebraic surfaces with rational singularities. In [4] we were concerned with complex spaces which compactify affine 2-space $\mathbf{C}^2$. The purpose of the present work is to apply similar techniques to provide examples of, and to classify according to singularity type, certain singular complex projective surfaces which have the same rational cohomology ring as the complex projective plane $\mathbf{CP}^2$. Our main result is:

THEOREM 1. *For each of the following twelve 8-point Dynkin diagrams Γ there exists a complete rational complex projective algebraic surface of the rational cohomology type of $\mathbf{CP}^2$ whose singularities are precisely the rational double points associated to the components of Γ: A_8, D_8, E_8, A_7+A_1, E_7+A_1, E_6+A_2, D_5+A_3, A_4+A_4, D_4+D_4, $A_5+A_2+A_1$, $D_6+A_1+A_1$, and $A_3+A_3+A_1+A_1$.*

Introduction

In an earlier paper ([2]), methods of the classical geometry of algebraic surfaces were applied to questions of existence and uniqueness of complex spaces of the homotopy or cohomology type of $\mathbf{CP}^2$. Although there is an error in the examples of [2] (corrected in [3]), the main result of [2] gives the uniqueness result below. The statement of the result requires a bit of terminology. A compact

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complex surface X is a *rational homology* $\mathbf{CP}^2$ if $\forall i$,

$$H^i(X, \mathbf{Q}) \cong H^i(\mathbf{CP}^2, \mathbf{Q}) = \begin{cases} \mathbf{Q} & \text{for } i = 0, 2, 4 \\ 0 & \text{otherwise} \end{cases}$$

X is a *rational cohomology* $\mathbf{CP}^2$ if there is a natural ring isomorphism $\hat{H}^*(X, \mathbf{Z}) \to \hat{H}^*(\mathbf{CP}^2, \mathbf{Z})$, where $\hat{H}^*$ means $H^*/(\text{Torsion subgroup})$. "Natural" means that the isomorphism preserves the duals of the homology class of a point and of the entire space, regarded respectively as oriented 0- and 4-cycles.

THEOREM 2 ([2], Theorem 6). *Let X be a compact two-dimensional complex analytic space whose singularities are rational double points. Suppose further that X is a rational homology $\mathbf{CP}^2$ and that $\hat{H}^2(X, \mathbf{Z})$ is generated by an effective analytic divisor. Then X is a rational projective algebraic surface. Indeed, one of the following holds*:

(a) *X is biholomorphic to $\mathbf{CP}^2$ ($\Leftrightarrow X$ is non-singular).*

(b) *X is biholomorphic to the singular complex quadratic cone* $\mathbf{Q}_0^2 = \overline{\{x^2 + y^2 + z^2 = 0\}} \subset \overline{\mathbf{C}^3} = \mathbf{CP}^3$.

(c) *For some integer n with $3 \leq n \leq 8$, X is derived from $\mathbf{CP}^2$ by the successive application of n monoidal point transformations, followed by the blowing down of precisely n non-singular rational curves having self-intersection -2. In this case, the cohomology ring structure is determined by the fact that $g^2 = 9 - n$, where g is a generator of $H^2(X, \mathbf{Z}) \cong \mathbf{Z}$. X is a rational cohomology $\mathbf{CP}^2$ exactly when $n = 8$.*

Note: It was mistakenly supposed in [2] that for $n = 8$ these spaces are in fact homotopy equivalent to $\mathbf{CP}^2$. This error is rectified in [3]. The necessity of the condition "$\hat{H}^2(X, \mathbf{Z})$ is generated by an effective divisor" was recently shown by Mumford in [9], where an example is presented of a rational cohomology $\mathbf{CP}^2$ which is a non-singular projective surface of general type.

Following the algorithm of (c), examples of cohomology $\mathbf{CP}^2$'s were constructed in [2]. The details of the method are as follows. For some integer $m \leq 8$, start with m projective lines L_i on $\mathbf{CP}^2$ and blow up 8 points, possibly including infinitely near points, on $C = \bigcup_{i=1}^m L_i$. Call the resulting surface $\tilde{X}$ and let $\rho : \tilde{X} \to \mathbf{CP}^2$ be the map inverse to the monoidal transformations. Suppose that among the $m + 8$ components C_j of the curve $\rho^{-1}(C) \subset \tilde{X}$ there are precisely 8 that together comprise the exceptional set for the minimal resolution of one or more of the classical rational double points (that is, each of the 8 C_j in question satisfies $C_j^2 = -2$ and the dual intersection graph is the disjoint union of one or more of the Dynkin diagrams A_k, D_k, or E_k), while the remaining C_j are exceptional of the first kind (non-singular rational with $C_j^2 = -1$). Then the unique normal analytic space X obtained from $\tilde{X}$ by collapsing each of the connected components of the union of these 8 C_j separately to a point will satisfy the conditions of the theorem and will have the rational cohomology type of $\mathbf{CP}^2$.

At the time that [2] was written it was not known how many different constructions of this type were possible nor what combinations of singular points the resulting surfaces could have. A priori there are 39 different graphs with 8 vertices and with components of the form A_k, D_k, or E_k, but not all 39 occur. To discover which of the 39 occur and which do not is the "thankless task" mentioned in [2], page 429. This question has assumed new interest with recent work of Ronald Fintushel on rational cohomology $\mathbf{CP}^2$'s which are singular 4-manifolds, each singularity being the cone on a rational homology 3-sphere. Considered as topological spaces, the surfaces X of type (c) constructed as above are certainly such objects – indeed, they are singular 4-manifolds with singularities of the required type which in addition support a complex analytic structure.

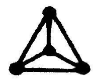
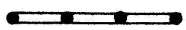
In this paper we will give the complete list of all rational cohomology $\mathbf{CP}^2$'s that can arise by this construction.

THEOREM 3. *Let X be a rational cohomology $\mathbf{CP}^2$ constructed by the above technique. Then X has at most 4 singular points x_i. Let $\Gamma = \bigcup_i \Gamma_i$ be the disjoint union of the Dynkin diagrams associated to the x_i and let* $\det(\Gamma)$ *be the determinant of the Cartan matrix associated to Γ. Then* $\det(\Gamma)$ *is the square of an integer less than or equal to* 8. *Conversely, for each* 8*-point graph $\Gamma \neq D_4 + 2A_2$ with* 4 *or fewer components, each a Dynkin diagram of type A_k, D_k, or E_k, and with* $\det(\Gamma) = j^2$ *for some integer $j \leq 8$, at least one such space X exists. Explicitly, the graphs Γ satisfying these conditions are the* 12 *listed in Theorem* 1 *above.*

The method of proof is purely graph theoretical, but at several important points the graph theory sheds light on matters of topological and geometric interest as well. We thank Paul Catlin, Daniel Frohardt, Peter Malcolmson, and the other participants of the Wayne State University Graph Theory Seminar (April, 1979) for valuable conversations about these ideas.

Preliminaries

By a *hypergraph* on a set V of m distinct vertices $v_1, \ldots, v_m$ we mean a system (V, Γ) where Γ is a set of non-empty subsets (called *edges*) of V. A *graph* is a hypergraph in which each edge has cardinality 2. A singleton edge $\{v_i\}$ is pictured as a loop at v_i. In this paper however, we will deal only with hypergraphs that have no singleton edges. Edges of cardinality $k > 2$ will be indicated by the symbol  . If a hypergraph Γ has no singleton edges and in addition satisfies the condition that for each pair of distinct vertices v_i, v_j there exists a unique edge $S \in \Gamma$ with $v_i, v_j \in S$, then Γ will be called *minimally complete.* For example, the minimally complete hypergraphs on 4 vertices are

 ,  , and 

A *weighted hypergraph* is a hypergraph in which each vertex v_i is assigned an integer "weight" n_i.

If Γ is a weighted hypergraph on m vertices v_i, with weight n_i on v_i, the *intersection matrix of* Γ is the m by m square symmetric matrix (a_{ij}) where a_{ij} equals n_i if $i=j$, -1 if $i \neq j$ and some element of Γ contains $\{v_i, v_j\}$, and 0 otherwise. By the *determinant* (respectively, *trace*) of Γ (abbreviated $\det(\Gamma)$, $\operatorname{tr}(\Gamma)$), we shall mean the determinant (trace) of the intersection matrix. Note that the trace is just the sum $\sum_{i=1}^{m} n_i$ of the weights.

Now let Y be an algebraic surface and let $C = \bigcup_{i=1}^{m} C_i$ be a curve on Y whose components C_i meet transversally with C_i meeting C_j in at most one point whenever $i \neq j$. By the *dual intersection hypergraph* associated to $C \subset Y$ we mean the weighted hypergraph on m vertices $v_1, \ldots, v_m$ defined by $\{v_{i_1}, \ldots, v_{i_k}\} \in \Gamma \Leftrightarrow C_{i_1} \cap \cdots \cap C_{i_k} \neq \emptyset$, with weight $-C_i^2$ on v_i. (We use the negatives of the self-intersection numbers to avoid having to alternate the signs of the determinants with successive monoidal transformations – see below.) A hypergraph Γ is called (*complex*) *projective planar* if it is dual to a collection of projective lines on $\mathbf{CP}^2$. A projective planar hypergraph is necessarily minimally complete, reflecting the fact that any two lines on the projective plane meet in exactly one point.

Let Γ be a weighted hypergraph on vertices $v_1, \ldots, v_m$ with weight n_i on v_i, and for $S = \{v_{i_1}, \ldots, v_{i_k}\}$ an edge of Γ define a new hypergraph $\sigma_S(\Gamma)$ on $m+1$ vertices $v_1, \ldots, v_{m+1}$ by

$$\sigma_S(\Gamma) = \Gamma - \{S\} \cup \{\{v_{i_1}, v_{m+1}\}, \ldots, \{v_{i_k}, v_{m+1}\}\}$$

with weights 1 on v_{m+1}, $n_i + 1$ on v_i if $v_i \in S$, and n_i on v_i if $v_i \notin S$, $i \leq m$. Similarly, if v_i is a vertex of Γ, denote by $\sigma_{v_i}(\Gamma)$ the hypergraph on $v_1, \ldots, v_{m+1}$ defined by

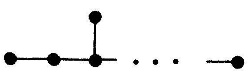
$$\sigma_{v_i}(\Gamma) = \Gamma \cup \{v_i, v_{m+1}\}$$

with weights 1 on v_{m+1}, $n_i + 1$ on v_i, and n_j on v_j for $j \neq i, m+1$. These operations are dual to the monoidal transformation for non-singular curves meeting transversally on an algebraic surface. The operation σ_S (respectively, σ_{v_i}) is called the

σ-process centered at S (resp., v_i), and applying σ_S (resp., σ_{v_i}) to Γ is called "blowing up" the edge S (resp., the vertex v_i). Any hypergraph can be transformed into a graph by blowing up all of its edges of cardinality $k>2$. If $S=\{v_{i_1},\dots,v_{i_k}\}$ is an edge of Γ of cardinality k, then $\operatorname{tr}(\sigma_S(\Gamma))=\operatorname{tr}(\Gamma)+k+1$, since each of the k weights n_{i_j}, $j=1,\dots,k$, is raised by 1 and a new vertex v_{m+1} of weight 1 is inserted, while the other weights remain the same. Similarly, $\operatorname{tr}(\sigma_{v_i}(\Gamma))=\operatorname{tr}(\Gamma)+2$. Finally, we remind the reader of the Dynkin diagrams (Coxeter graphs)

A_k:  , $k=1,2,\dots$

D_k:  , $k=4,5,\dots$, and

E_k:  , $k=6,7,$ or 8

(all weights are 2). These are the graphs associated to the fundamental root systems of the simple complex Lie algebras $\mathfrak{sl}(k+1,\mathbf{C})$, $\mathfrak{o}(2k,\mathbf{C})$, $\mathfrak{e}_k$ (cf., eg., [8], §§1, 11). They are also the weighted dual intersection graphs of the exceptional curves appearing in the minimal resolutions of the double points

$$A_k\colon\ x^2=x^{k+1}+y^2,\qquad D_k\colon\ z^2=x^{k-1}+xy^2,\qquad E_6\colon\ z^2=x^4+y^3,$$
$$E_7\colon\ z^2=x^3y+y^3,\qquad E_8\colon\ z^2=x^5+y^3.$$

These are the only two-dimensional hypersurface singularities $x\in X$ which are *rational* ($R^1\pi_*\mathcal{O}_{\tilde X}$ vanishes at x for $\pi:\tilde X\to X$ a resolution – see Artin [1]). They are precisely the singularities of the form $\mathbf{C}^2/G$ where G is a finite subgroup of $\mathrm{SL}(2,\mathbf{C})$ (namely, a cyclic group or a binary dihedral, tetrahedral, octahedral, or icosahedral group). Indeed, much attention has been directed toward understanding the relations among the various settings in which these graphs occur (see especially [10], [5], and [7]).

The determinants of these graphs are $\det(A_k)=k+1$, $\det(D_k)=4$, and $\det(E_k)=9-k$. This can be verified directly, or by the method of [6], or by computing the quotients G/G' for $G\subset\mathrm{SL}(2,\mathbf{C})$ the appropriate Kleinian group, where G' is the commutator subgroup of G. Since G/G' is also the first homology group of a spherical neighborhood of the associated rational double point, these determinants have topological significance in the construction of the complex surfaces X.

The main result

The problem of constructing rational homology and cohomology $\mathbf{CP}^2$'s by the technique described in the introduction reduces to the following question. Let n be an integer ≤ 8, and let Γ_0 be a projective planar hypergraph on m vertices, $m \leq n$, with each vertex of weight -1. Perform n successive σ-processes on Γ_0 in such a way that the final hypergraph $\tilde{\Gamma} = \sigma^n(\Gamma_0)$ has m vertices of weight 1 and n vertices – including the m original ones – of weight 2. Require further that the sub-hypergraph Γ of $\tilde{\Gamma}$ obtained by deleting all the vertices of weight 1 and all the edges adjoining them be the disjoint union of Dynkin diagrams of the form A_k, D_k, or E_k. In how many ways can this be done?

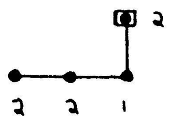
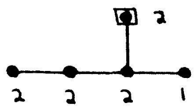
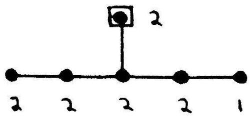
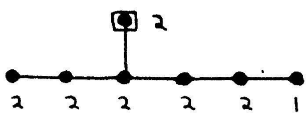
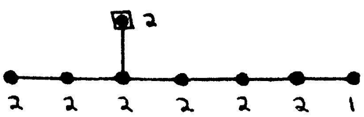
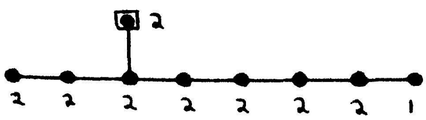
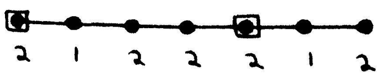
The complete solution is contained in the first 51 rows of the following table. The last 3 rows show the only 3 examples which satisfy every condition except complex projective planarity of the initial hypergraph. That is, these represent solutions to the dual graph theoretic problem for minimally complete hypergraphs, but do not translate into the geometric construction. In the last column the rational cohomology type of the resulting space X is given. By $\bar{S}_j$, $j = 3, 4, \ldots$, we mean the singular complex rational surface obtained by collapsing to a point the zero section of the $\mathbf{CP}^1$-bundle on $\mathbf{CP}^1$ with Chern class $-j$ ($\bar{S}_2 =$ the singular quadric surface $\mathbf{Q}_0^2$, $\bar{S}_1 = \mathbf{CP}^2$). These are the prototypical homology $\mathbf{CP}^2$'s with cohomology ring structure given by $g^2 = j$, for g a generator of $H^*(X, \mathbf{Z})$. In the final graphs the (proper transforms of) the original vertices are denoted by the symbol ◙.

The next section will be devoted to justifying the table. Note that Theorems 1 and 3 follow from the table. That is, in the 31 rows of the table which represent cohomology $\mathbf{CP}^2$'s, the distinct Dynkin diagrams that appear are exactly the 12 listed in Theorem 1. To check the characterization in terms of determinants (Theorem 3) it is sufficient to calculate the determinants of the twelve 8-point Dynkin diagrams that appear in the table and the 27 which do not. (Recall that in all there are thirty-nine 8-point graphs with components of type A_k, D_k, or E_k.) Note that, except for $A_7 + A_1$, each of the disconnected diagrams which appear consists of two disjoint graphs Γ' and Γ'' (not necessarily connected) with $\det(\Gamma') = \det(\Gamma'')$. Thus $\det(\Gamma)$ is the square $(\det(\Gamma'))^2$. (The "missing" graph $D_4 + 2A_2$ does not have this property, although its determinant is the square of an integer ≤ 8 and it has 4 or fewer components.)

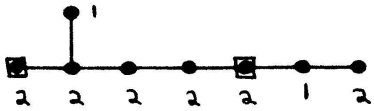
Proof of the main result. To justify the table, we must determine all initial hypergraphs Γ_0, and for each Γ_0, all sequences of $n \leq 8$ blow-ups permitted by the conditions on the final hypergraph $\tilde{\Gamma}$, as given in the first paragraph of the previous section.

First we note that since each of the original vertices of Γ_0 becomes part of the

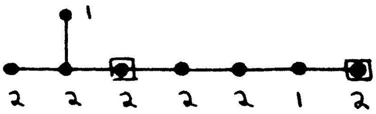
Table for the construction of rational homology $\mathbb{CP}^2$'s

Final graph	Singularities	Determinant	Cohomology type
Initial hypergraph •			
	$A_2 + A_1$	6	$\bar{S}_6$
	A_4	5	$\bar{S}_5$
	D_5	4	$\bar{S}_4$
	E_6	3	$\bar{S}_3$
	E_7	2	Q_0^2
	E_8	1	$\mathbb{P}^2$
Initial hypergraph •—•			
	$A_3 + 2A_1$	16	$\bar{S}_4$

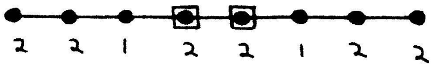
Initial hypergraph •—• (continued)



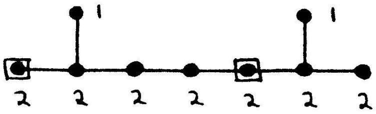
$A_5 + A_1$ 12 $\bar{S}_3$



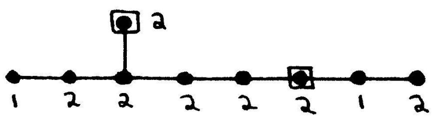
$A_5 + A_1$ 12 $\bar{S}_3$



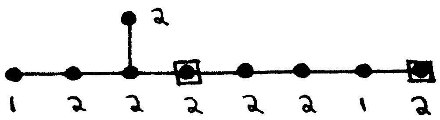
$3A_2$ 27 $\bar{S}_3$



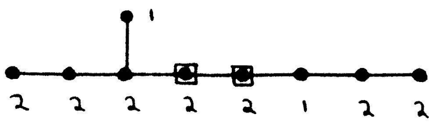
A_7 8 Q_0^2



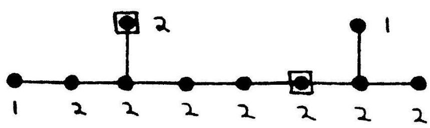
$D_6 + A_1$ 8 Q_0^2



$D_6 + A_1$ 8 Q_0^2

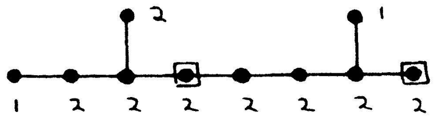
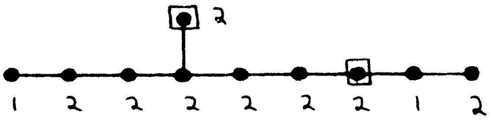
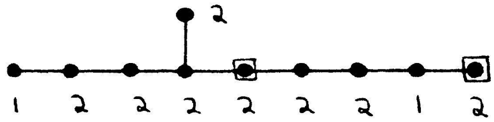
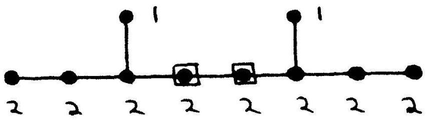
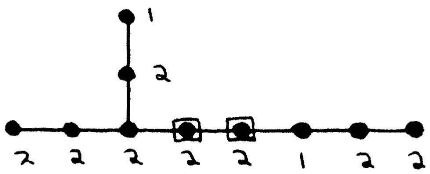


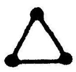
$A_5 + A_2$ 18 Q_0^2

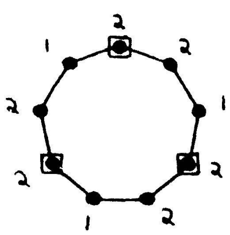
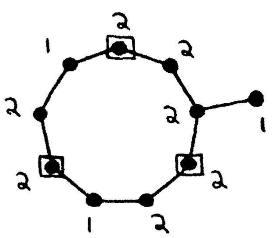


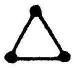
D_8 4 $\mathbb{P}^2$

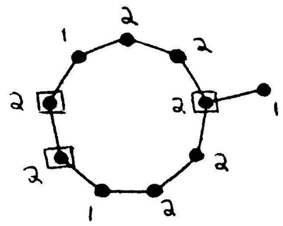
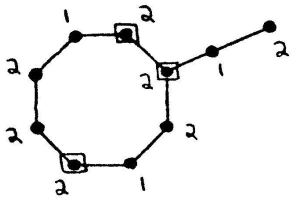
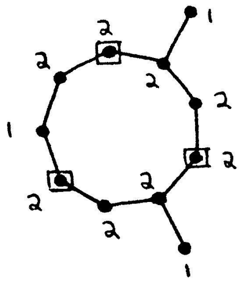
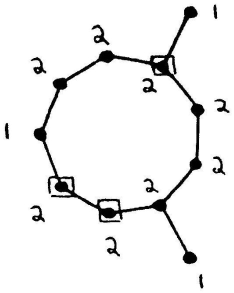
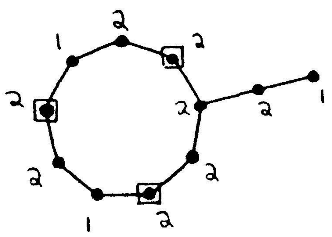
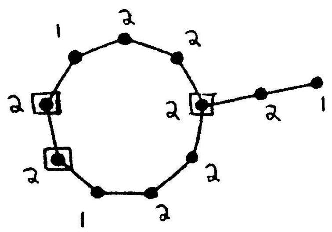
Initial hypergraph  (continued)

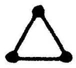
	D_8	4	$\mathbb{P}^2$
	$E_7 + A_1$	4	$\mathbb{P}^2$
	$E_7 + A_1$	4	$\mathbb{P}^2$
	A_8	9	$\mathbb{P}^2$
	$E_6 + A_2$	9	$\mathbb{P}^2$

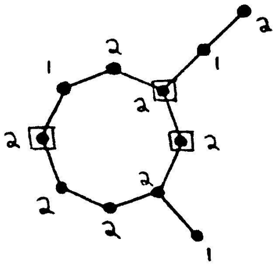
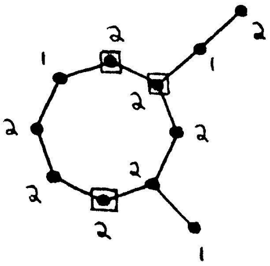
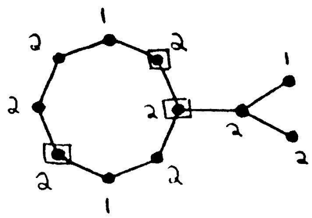
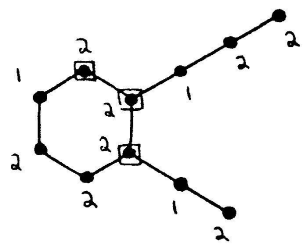
Initial hypergraph 

	$3A_2$	27	$\bar{S}_3$
	$A_5 + A_2$	18	Q_0^2

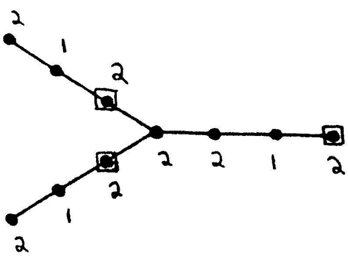
Initial hypergraph  (continued)

	$A_5 + A_2$	18	Q_0^2
	$2A_3 + A_1$	32	Q_0^2
	A_8	9	$\mathbb{P}^2$
	A_8	9	$\mathbb{P}^2$
	$E_6 + A_2$	9	$\mathbb{P}^2$
	$E_6 + A_2$	9	$\mathbb{P}^2$

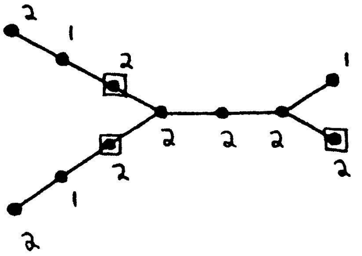
Initial hypergraph  (continued)

	$A_7 + A_1$	16	$\mathbb{P}^2$
	$A_7 + A_1$	16	$\mathbb{P}^2$
	$D_5 + A_3$	16	$\mathbb{P}^2$
	$A_5 + A_2 + A_1$	36	$\mathbb{P}^2$

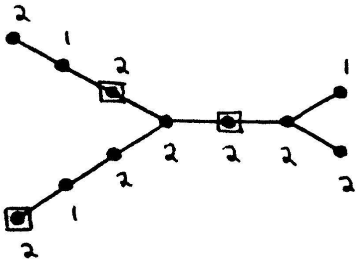
Initial hypergraph

	$D_4 + 3A_1$	32	Q_0^2

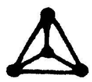
Initial hypergraph  **(continued)**

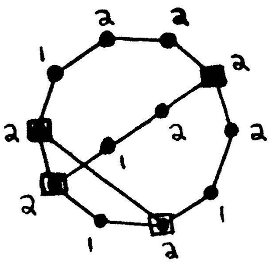


$D_6 + 2A_1$ 16 $\mathbb{P}^2$

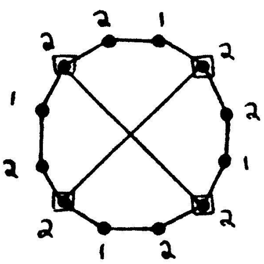


$D_6 + 2A_1$ 16 $\mathbb{P}^2$

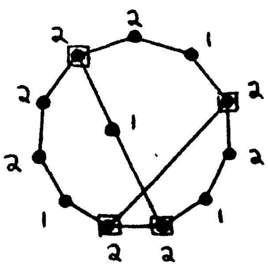
Initial hypergraph 



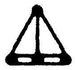
$D_5 + A_3$ 16 $\mathbb{P}^2$

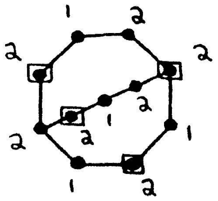
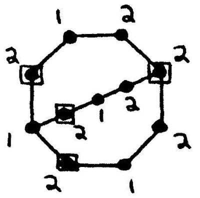
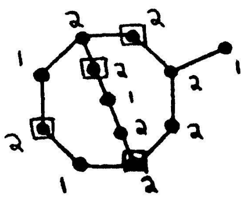
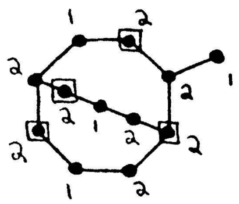
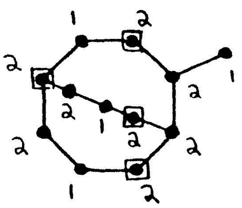
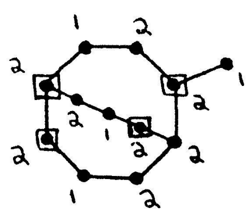


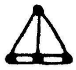
$2A_4$ 25 $\mathbb{P}^2$

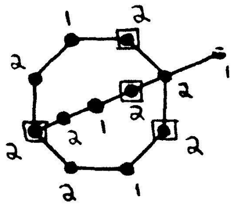
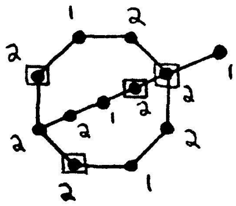
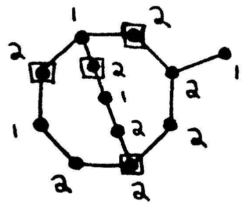


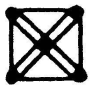
$2A_4$ 25 $\mathbb{P}^2$

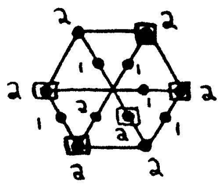
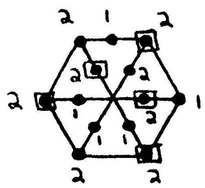
Initial hypergraph 

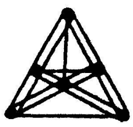
	$2A_3 + A_1$	32	Q_0^2
	$D_4 + 3A_1$	32	Q_0^2
	$A_7 + A_1$	16	$\mathbb{P}^2$
	$D_5 + A_3$	16	$\mathbb{P}^2$
	$D_5 + A_3$	16	$\mathbb{P}^2$
	$D_5 + A_3$	16	$\mathbb{P}^2$

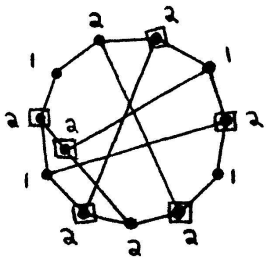
<u>Initial hypergraph</u>  (continued)

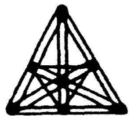
	$2D_4$	16	$\mathbb{P}^2$
	$2D_4$	16	$\mathbb{P}^2$
	$D_6 + 2A_1$	16	$\mathbb{P}^2$

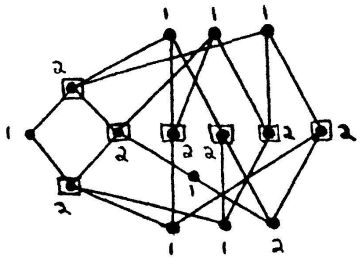
<u>Initial hypergraph</u> 

	$2A_4$	25	$\mathbb{P}^2$
	$A_5 + A_2 + A_1$	36	$\mathbb{P}^2$

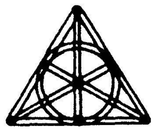
<u>Initial hypergraph</u> 

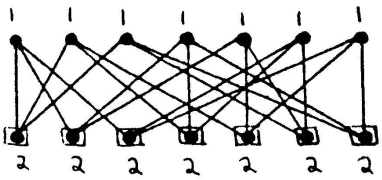
	$A_5 + A_2 + A_1$	36	$\mathbb{P}^2$

Initial hypergraph 

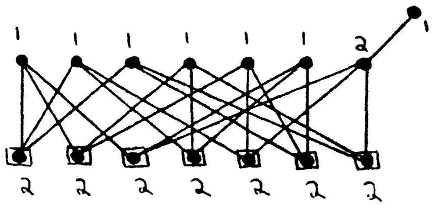


$2A_3 + 2A_1$ 64 $\mathbb{P}^2$

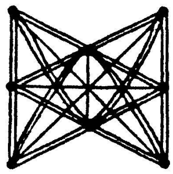
Initial hypergraph 

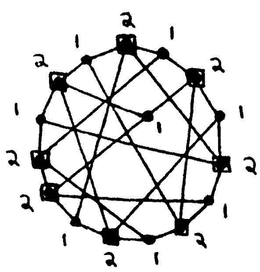


$7A_1$ 128 Does not occur over $\mathbb{C}$



$D_4 + 3A_1$ 64 Does not occur over $\mathbb{C}$

Initial hypergraph 



$4A_2$ 81 Does not occur over $\mathbb{C}$

graph $\Gamma \subset \tilde{\Gamma}$, all of the multiple edges of Γ_0 must be blown up. Furthermore, Γ_0 can have no edges of cardinality greater than 3. For if $S = \{v_{i_1}, \ldots, v_{i_k}\}$ is an edge of Γ_0 with $k \geq 4$, then $\sigma_S(\Gamma_0)$ contains k vertices v_{i_j} of weight 0 with no two in an edge of $\sigma_S(\Gamma_0)$, and these k vertices must have weight 2 in the final graph $\tilde{\Gamma}$. Thus each v_{i_j} must be at the center of 2 more blow-ups, neither of which can involve v_{i_l} for $l \neq j$. This makes a total of at least $1 + 2k > 8$ blow-ups, violating the restriction $n \leq 8$.

Using the fact that in a minimally complete hypergraph each pair of triple edges shares at most 1 vertex, the following lemma is quite easy to check case by case.

SUBLEMMA. *The maximum number e of triple edges in a minimally complete hypergraph on m points, $1 \leq m \leq 8$, is as follows*:

$m = 1, 2$	$e = 0$	$m = 6$	$e = 4$
$m = 3, 4$	$e = 1$	$m = 7$	$e = 7$
$m = 5$	$e = 2$	$m = 8$	$e = 8$

Furthermore, the minimally complete hypergraphs realizing these maxima are unique up to isomorphism.

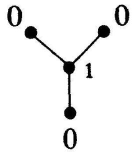
Let x, y, and z denote the number of 3-edges, 2-edges, and vertices blown up in passing from Γ_0 to $\tilde{\Gamma}$. Then $x + y + z = n \leq 8$, and the traces satisfy $\operatorname{tr}(\tilde{\Gamma}) = \operatorname{tr}(\Gamma_0) + 4x + 3y + 2z = \operatorname{tr}(\Gamma_0) + 2n + 2x + y$. Since $\operatorname{tr}(\Gamma_0) = -m$ (m vertices, each with weight -1) and $\operatorname{tr}(\tilde{\Gamma}) = 2n + m$ (n vertices of weight 2, m vertices of weight 1) we obtain $2n + m = -m + 2n + 2x + y$, or

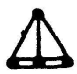
$$(*) \qquad m = (y/2) + x \leq 4 + (x/2).$$

We want to determine all possibilities for Γ_0 and the triple (x, y, z).

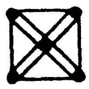
Case 1. $x = 0$, $m = y/2 \leq 4$. Γ_0, having no triple edges, is the complete graph K_m on m points ($1 \leq m \leq 4$), and $y = 2m$. If $m = 1$, of course z cannot be zero, for K_1 has no edge to be blown up. Likewise if $m = 2$ ($\Leftrightarrow y = 4$) then $z \neq 0$ since it is impossible to insert 4 new vertices by σ-processes into the lone edge of K_2 without increasing the weight of one of the vertices to more than 2. Thus the possible initial hypergraphs and their triples (x, y, z) are in this case:

Γ_0	(x, y, z)
K_1	$(0, 2, z), 1 \leq z \leq 6$
K_2	$(0, 4, z), 1 \leq z \leq 4$
K_3	$(0, 6, z), 0 \leq z \leq 2$
K_4	$(0, 8, 0)$.

Case 2. $x = 1$, $m = (y/2) + 1 \leq 4$. Since Γ_0 has a triple edge, m must be at least 3, so the only choices for Γ_0 are  and . If $m=3$, $y=4$. After the triple edge of  is blown up to obtain  , at most one of the three 2-edges can be blown up, and no weight can be added to the other two original vertices except by blowing them up. Thus $z \geq 2$ in this case. The possibilities are

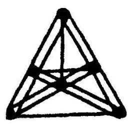
Γ_0	(x, y, z)
	$(1, 4, 2)$
	$(1, 4, 3)$
	$(1, 6, 0)$
	$(1, 6, 1)$

Case 3. $x=2$, $m=(y/2)+2 \leq 5$. By the sublemma, $m>4$. Thus $m=5$, $y=6$, and Γ_0 is the unique minimally complete hypergraph on 5 points with 2 triple edges.

Γ_0	(x, y, z)
	$(2, 6, 0)$

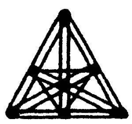
Case 4. $x = 3$, $m=(y/2)+3 \leq 5$. By the sublemma, this cannot occur.

Case 5. $x = 5$, $m=(y/2)+4 \leq 6$. The sublemma gives $m=6$, whence $y=4$.

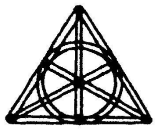
Γ_0	(x, y, z)
	$(4, 4, 0)$

Case 6. $x=5$, $m=(y/2)+4 \leq 6$. This, too, is impossible by the sublemma.

Cases 7, 8, *and* 9 similarly give the following possibilities

Γ_0	(x, y, z)
	$(6, 2, 0)$

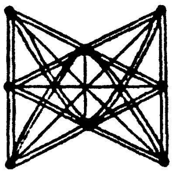
(the unique minimally complete hypergraph on 7 points with 6 triple edges)



$(7, 0, 0)$

$(7, 0, 1)$

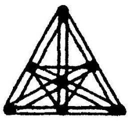
(the Fano projective plane $\mathbf{P}^2(\mathbf{Z}_2)$)



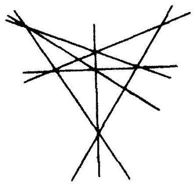
$(8, 0, 0)$

$m = x = 8$

It is easy to verify directly that the first 9 candidates for Γ_0 on this list are complex projective planar and that the last 2 are not. For instance, the hypergraph

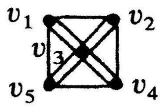


is dual to the collection of projective lines



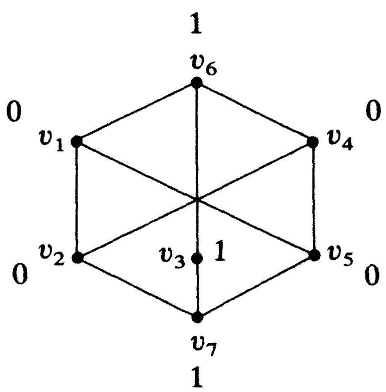
(The 7-point graph of Fano and the 8-point graph do, however, occur over fields of characteristic 2 and 3 respectively. This phenomenon will be explored in part III of this series of papers.)

From here, an easy but tedious exhaustion of cases yields the possible final graphs $\tilde{\Gamma}$ for each pair $(\Gamma_0, (x, y, z))$. We will give one example to illustrate the technique.



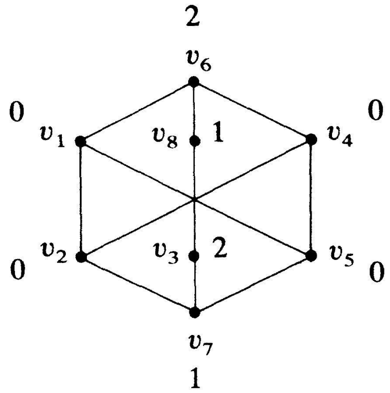
$(2, 6, 0)$

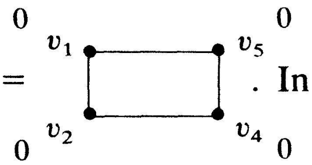
After blowing up the 2 triple edges we obtain the weighted graph



where v_6 and v_7 are the new vertices inserted by blowing up the triples $\{v_1, v_3, v_4\}$ and $\{v_2, v_3, v_5\}$ respectively. Since no vertices can be blown up ($z = 0$), the only way to raise the weight of v_3 is by blowing up one of the two edges to which it belongs. By symmetry we may suppose that $\{v_1, v_6\}$ is blown up.

$\sigma^3(\Gamma_0)$:

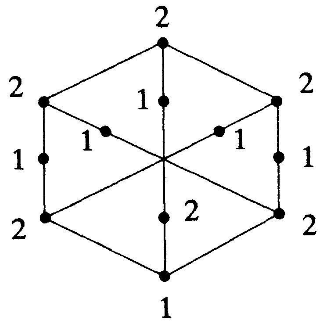


Consider the subgraph $\Gamma' =$ 

. In the 5 remaining σ-processes suppose that y' edges of Γ' are blown up, together with $z' \leq 5 - y'$ "vertices" of Γ' (i.e., edges of one of the graphs $\sigma^k(\Gamma_0)$, $3 \leq k \leq 3 + y'$ which contain only 1 of the vertices v_1, v_2, v_4, and v_5). Since the weight of each vertex of Γ' must be raised by 2, $2y' + z' = 8$. The only solutions are

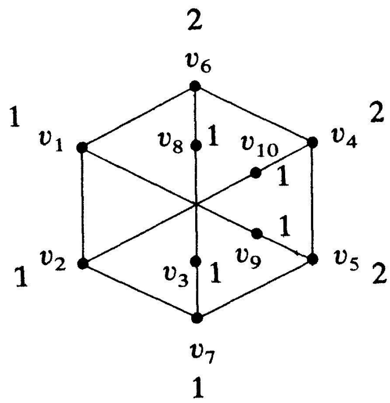
(A) $y' = 3, z' = 2$, and (B) $y' = 4, z' = 0$.

Case (B) cannot occur, for after blowing up each edge of Γ' we obtain the dead end

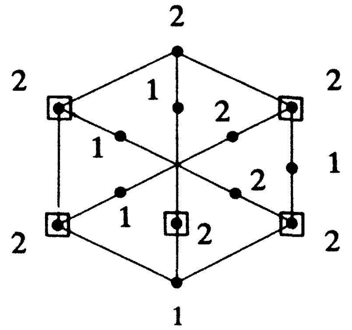


in which no further blow-ups of edges are possible. In case (A), by symmetry we may suppose that $\{v_1, v_5\}$, $\{v_2, v_4\}$, and $\{v_4, v_5\}$ are blown up.

$\sigma^6(\Gamma_0)$:

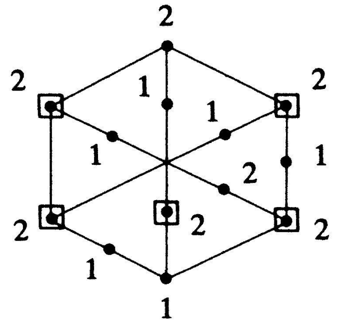


The remaining edge $\{v_1, v_2\}$ of Γ' cannot be blown up, no edge of $\sigma^6(\Gamma_0)$ can be subdivided twice, and v_2 can be involved in only one more blow-up, so it is clear that the remaining 2 σ-processes must blow up $\{v_1, v_9\}$ and either $\{v_2, v_{10}\}$ or $\{v_2, v_7\}$. The resulting final graphs are



$(A_5+A_2+A_1)$

and



(A_4+A_4)

These are lines 49 and 48 of the table, and, up to the order in which the 8 σ-processes are performed, these are the only permissible constructions with an initial hypergraph on 5 points.

The other 52 lines of the table are derived similarly. This completes the proof.

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