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On the quantitative boundary behavior of conformal maps

CH. POMMERENKE* and S. E. WARSCHAWSKI*

1. Introduction

Let Γ be a closed Jordan curve in $\mathbf{C}$ and let f map the unit disk $\mathbf{D}$ conformally onto the inner domain of Γ. For $\omega_1, \omega_2 \in \Gamma$, let $\Gamma(\omega_1, \omega_2)$ denote the arc (of smaller diameter) of Γ between ω_1 and ω_2. We shall study the relation between the geometric quantity

$$\eta(\delta) = \sup_{|\omega_1-\omega_2|\leq\delta} \sup_{\omega\in\Gamma(\omega_1,\omega_2)} \left(\frac{|\omega_2-\omega|+|\omega-\omega_1|}{|\omega_2-\omega_1|}-1\right)^{1/2} \tag{1.1}$$

and the analytic quantity

$$\beta(\delta) = \sup_{1-\delta\leqq|\zeta|<1} (1-|\zeta|)\left|\frac{f''(\zeta)}{f'(\zeta)}\right| \qquad (0<\delta\leqq 1). \tag{1.2}$$

The relation between $\eta(\delta)$ and other properties of f has been investigated in two papers by F. D. Lesley and the second author [4][5], and our main theorem is based in part on these results.

The curve Γ (which need not be rectifiable) is called *asymptotically conformal* if $\eta(\delta) \to 0$ as $\delta \to 0$; this holds [7, Th. 1] if and only if $\beta(\delta) \to 0$ as $\delta \to 0$. The connection with quasiconformal mappings was studied in a paper with J. Becker [2].

THEOREM 1. *Let f map $\mathbf{D}$ conformally onto the inner domain of the asymptotically conformal curve Γ. Then, for $0<\varepsilon<1/2$, there exists $\delta_0(\varepsilon)>0$ such that*

$$c\eta(\delta^{1+\varepsilon})<\beta(\delta)<M[\eta(\delta^{1-\varepsilon})+\delta^{\varepsilon/6}] \qquad (0<\delta<\delta_0(\varepsilon)) \tag{1.3}$$

where $c>0$ and M depend only on f.

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This theorem gives the best result if Γ is "not too smooth." For instance, if $c_1(\log 1/\delta)^{-a} < \eta(\delta) < M_1(\log 1/\delta)^{-a}$ for $0<\delta<\delta_1$ and some $a>0$ then Theorem 1 (with $\varepsilon = 1/4$) shows that

$$c_2\eta(\delta) < \beta(\delta) < M_2\eta(\delta) \qquad (0<\delta<\delta_2). \tag{1.4}$$

We shall study $\beta(\delta)$ and $\eta(\delta)$ in Section 2 and prove the lower estimate (1.3). The much more difficult proof of the upper estimate (1.3) will be given in Section 3.

In the last section, we derive some consequences of Theorem 1 and construct examples (using lacunary series):

(a) The curve Γ is smooth if [4, Prop. 3]

$$\int_0^1 \frac{\eta(t)}{t}\,dt < \infty, \tag{1.5}$$

and we shall see that this condition is best possible and that it does not imply that Γ is Dini-smooth. It follows from (1.5) that

$$c_3\eta(\delta) < \beta(\delta) < M_3 \int_0^\delta \frac{\eta(t)}{t}\,dt + M_3\delta \int_\delta^1 \frac{\eta(t)}{t^2}\,dt \qquad (0<\delta<\delta_3), \tag{1.6}$$

and this estimate is better than (1.3) if $\eta(\delta)$ behaves like δ^a. It also follows [4, Th. 3] from (1.5) that $\log f'$ is continuous in $\bar{\mathbf{D}}$, and we shall improve the estimate for the modulus of continuity.

(b) The curve Γ is rectifiable and even asymptotically smooth if

$$\int_0^1 \frac{\eta(t)^2}{t}\,dt < \infty, \tag{1.7}$$

and this condition is again best possible. Hence $\log f' \in \mathrm{VMOA}$ [7, Th. 2], and we shall show that

$$\log f' \in \mathrm{BMO}_{\partial\mathbf{D}}(\rho)$$

for a certain $\rho(\delta)$; see Sarason's lecture notes [11, Chapter 5] for a discussion of these function classes.

Throughout the paper, we denote by $\delta_0, \delta_1, \dots$, by $c, c_1, \dots$ and by $M, M_1, \dots$ positive constants that depend only on the function f and possibly on displayed parameters, while $K, K_1, \dots$ will denote absolute constants.

2. The lower estimate

2.1. *Some properties of β.* let $\beta(\delta)$ be defined by (1.2). The maximum principle shows that

$$(1-|\zeta|)\left|\frac{f''(\zeta)}{f'(\zeta)}\right| \leqq (t+\delta)\frac{\beta(\delta)}{\delta} \quad \text{for} \quad 1-t-\delta \leqq |\zeta| \leqq 1-\delta.$$

Hence $\delta\beta(t+\delta) \leqq (t+\delta)\beta(\delta)$ and similarly $t\beta(t+\delta) \leqq (t+\delta)\beta(t)$, and it follows that

$$\beta(t+\delta) \leqq \beta(t)+\beta(\delta) \quad \text{for} \quad t, \delta>0, \qquad t+\delta \leqq 1. \tag{2.1}$$

Thus β is increasing and subadditive.

It follows from (1.2) by integration that

$$\left|\log\frac{f'(\rho e^{it})}{f'((1-\delta)e^{it})}\right| \leqq \beta(\delta)\log\frac{\delta}{1-\rho} \quad \text{for} \quad 1-\delta \leqq \rho < 1.$$

If $\beta(\delta) \to 0$ as $\delta \to 0$ we conclude that, for $\varepsilon > 0$,

$$(1-|\zeta|)^{\varepsilon} \leqq |f'(\zeta)| \leqq (1-|\zeta|)^{-\varepsilon} \qquad (1-\delta_1(\varepsilon) \leqq |\zeta| < 1). \tag{2.2}$$

THEOREM 2.1. *Let f map $\mathbf{D}$ conformally onto the inner domain of an asymptotically conformal curve. If $|\zeta| = 1-\delta < 1$ and $a \geqq 1$ then*

$$\frac{\beta(\delta)}{2} \leqq \max_{\substack{z\in\bar{\mathbf{D}} \\ |z-\zeta|\leqq a\delta}} \left|\frac{f(z)-f(\zeta)}{(z-\zeta)f'(\zeta)}-1\right| \leqq 60a^3\beta(\delta) \tag{2.3}$$

for $0<\delta<\delta_2(a)$.

Proof. (a) Since

$$\frac{f(z)-f(\zeta)}{(z-\zeta)f'(\zeta)}-1 = \frac{f''(\zeta)}{2f'(\zeta)}(z-\zeta)+O(|z-\zeta|^2) \qquad (z \to \zeta),$$

the left-hand inequality (2.3) follows from a well-known coefficient estimate applied to $\{|z-\zeta| = \delta\}$.

(b) Let $z \in \mathbf{D}$ and $|z-\zeta| \leqq a\delta$. We see from (2.1) that $\beta((a+1)\delta) \leqq$

$(a+2)\beta(\delta) \leqq 3\beta(\delta)$. Hence, by definition (1.2),

$$\left|\log \frac{f'(z)}{f'(\zeta)}\right| = \left|\int_\zeta^z \frac{f''(t)}{f'(t)}\,dt\right| \leqq \int_\zeta^z \frac{6a\beta(\delta)}{1-|t|^2}\,|dt|$$
$$= 3a\beta(\delta)\log\frac{1+|s|}{1-|s|} \leqq 6a\beta(\delta)\log\frac{1}{1-|s|}$$

where we integrated along the non-euclidean segment from ζ to z and where $s=(z-\zeta)/(1-\bar{\zeta}z)$. Writing $\beta=\beta(\delta)$ we deduce that

$$\left|\frac{f'(z)}{f'(\zeta)}-1\right| \leqq \exp\left|\log\frac{f'(z)}{f'(\zeta)}\right| - 1 \leqq (1-|s|)^{-6a\beta}-1.$$

Since $|dz/ds| = |1-\bar{\zeta}z|^2/(1-|\zeta|^2) \leqq 5a^2\delta$ we obtain by another integration that

$$\left|\frac{f(z)-f(\zeta)}{(z-\zeta)f'(\zeta)}-1\right| \leqq \frac{5a^2\delta}{|z-\zeta|}\int_0^{|s|}[(1-\sigma)^{-6a\beta}-1]\,d\sigma$$
$$= \frac{5a^2\delta}{|1-\bar{\zeta}z|}\left(\frac{1-(1-|s|)^{1-6a\beta}}{(1-6a\beta)|s|}-1\right) \leqq 60a^3\beta$$

for $0<\delta\leqq\delta_0$ if δ_0 is chosen so small that $6a\beta(\delta)<1/2$. This proves the right-hand inequality (2.3).

2.2. *Geometric properties of* η. By elementary geometry, the definition (1.1) of η means that $\Gamma(\omega_1, \omega_2)$ lies in an ellipse with loci ω_1 and ω_2 and with minor half axis $(2+\eta(\delta)^2)^{1/2}\eta(\delta)\delta/2$; this is $<\eta(\delta)\delta$ for small δ. We need a somewhat different description in terms of the width of a strip; this result was independently proved by C. FitzGerald.

LEMMA 2.1. *If* $\eta(\delta)\to 0$ $(\delta\to 0)$ *then, for* $0<\delta<\delta_0$,

$$\frac{\eta(\delta)}{3} < \sup_{|\omega_1-\omega_2|\leqq\delta}\ \sup_{\omega\in\Gamma(\omega_1,\omega_2)}\left|\operatorname{Im}\frac{\omega-\omega_1}{\omega_2-\omega_1}\right| < \eta(\delta). \tag{2.5}$$

Proof. The right-hand inequality follows at once from the remark about the enclosing ellipse. We prove now that $\eta(\delta)<3\eta^*(\delta)$ where $\eta^*(\delta)$ denotes the middle term in (2.5).

Let $\omega_1, \omega_2 \in \Gamma$ with $|\omega_1-\omega_2|\leqq\delta$ and let $\omega\in\Gamma(\omega_1,\omega_2)$. We may assume that

$$\omega_1=0, \qquad \omega_2=\delta; \qquad \omega=re^{i\theta}, \qquad 0\leqq\theta<\pi, \qquad r\cos\theta\leqq\delta/2. \tag{2.6}$$

If δ is sufficiently small then $\eta^*(\delta)<\eta(\delta)<1/2$. Hence $r\sin\theta\leqq\delta\eta^*(\delta)<\delta/2$ which, together with $r\cos\theta\leqq\delta/2$, shows that $r<\delta/\sqrt{2}$. Since $\sqrt{x+y}-\sqrt{x}\leqq y/(2\sqrt{x})$ for $x>0$, $y>0$, we conclude that

$$\frac{|\omega_2-\omega|+|\omega-\omega_1|}{|\omega_2-\omega_1|}-1=\frac{[(\delta-r)^2+4r\delta\sin^2(\theta/2)]^{1/2}-(\delta-r)}{\delta}$$
$$\leqq\frac{2r\delta\sin^2(\theta/2)}{\delta-r}\leqq\frac{2\delta\eta^*(\delta)^2}{\delta(1-1/\sqrt{2})},$$

and thus that $\eta^*(\delta)^2<8\eta(\delta)^2$ for small δ.

Remark 2.1. The last result implies that $\eta(2\delta)\leqq K\eta(\delta)$. We only indicate the proof. With the convention (2.6), choose $\omega_1', \omega_2'\in\Gamma(0,\delta)$ such that $|\omega_1'-\omega_2'|=\delta/2$, $\omega\in\Gamma(\omega_1',\omega_2')$ and $[\omega_1',\omega_2']$ is parallel to $[0,\delta]$. Let $\omega\in\Gamma(\omega_1',\omega_2')$ be a point on the perpendicular bisector of $[\omega_1',\omega_2']$. We consider now the pairs $\{0,\omega\}$, $\{\omega,\delta\}$, $\{\omega_1',\omega_2'\}$ and see by elementary geometry that

$$\delta\eta^*(\delta)=\max|\mathrm{Im}\,\omega|<3\eta^*(\delta/\sqrt{3})$$

for small δ. Applying this twice we obtain $\eta^*(\delta)<9\eta^*(\delta/3)<9\eta^*(\delta/2)$.

2.3. *Proof of the lower estimate* (1.3). Let $z_1, z_2\in\partial\mathbf{D}$, $|z_1-z_2|=\delta$ and choose $\zeta\in\mathbf{D}$ on the perpendicular bisector of $[z_1,z_2]$ such that $|\zeta|=1-\delta$. It follows from Theorem 2.1 with $a=2$ that, for z on $\partial\mathbf{D}$ between z_1 and z_2,

$$\frac{f(z)-f(\zeta)}{f'(\zeta)}=(z-\zeta)+b\quad\text{with}\quad|b|\leqq K_1\delta\beta(\delta).\tag{2.7}$$

Writing b_j instead of b for the cases $z=z_j$, we thus see that

$$\frac{f(z)-f(z_1)}{f(z_2)-f(z_1)}=\frac{(z-z_1)+(b-b_1)}{(z_2-z_1)+(b-b_2)}.\tag{2.8}$$

Since $|\mathrm{Im}[(z-z_1)(\bar z_2-\bar z_1)]|\leqq\delta^3$ we deduce from (2.7) and (2.8) that

$$\left|\mathrm{Im}\frac{f(z)-f(z_1)}{f(z_2)-f(z_1)}\right|\leqq\frac{\delta^3+K_2\delta^2\beta(\delta)}{[\delta-2K_1\delta\beta(\delta)]^2}\leqq M_2\beta(\delta)\tag{2.9}$$

for sufficiently small δ because $\delta\leqq M_1\beta(\delta)$ by (2.1).

Since Γ is a quasiconformal curve it follows [6, p. 315] that

$$\frac{|f(z_1)-f(z_2)|}{(1-|\zeta|)|f'(\zeta)|} \geqq c_1 \frac{\operatorname{diam} \Gamma(f(z_1), f(z_2))}{(1-|\zeta|)|f'(\zeta)|} \geqq c_2. \tag{2.10}$$

Hence (2.2) with $\varepsilon/2$ instead of ε shows that

$$|f(z_1)-f(z_2)| \geqq c_2(1-|\zeta|)^{1+\varepsilon/2} = c_2\delta^{1+\varepsilon/2} \geqq \delta^{1+\varepsilon} \tag{2.11}$$

if $0<\delta<\delta_0(\varepsilon)$, and the lower estimate (1.3) follows from (2.9) and Lemma 2.1.

3. The upper estimate

3.1. *Connection with conformal mapping of strips.* To obtain an upper bound for $\beta(\delta)$ we map $\mathbf{D}$ and the inner domain of Γ conformally onto infinite strips. Let, as above, f denote a univalent function in $\mathbf{D}$, as well as its continuous extension to $\partial\mathbf{D}$, which maps $\mathbf{D}$ onto Ω, and let $f(\zeta_0) = \omega_0(|\zeta_0| = 1)$. The functions

$$z = x+iy = h(\zeta) = \operatorname{Log} \frac{\zeta_0+\zeta}{\zeta_0-\zeta} \quad \text{and} \quad w = u+iv = H(\omega) = -\log(\omega-\omega_0), \tag{3.1}$$

where Log denotes the principal value for $\zeta \in \mathbf{D}$ and log is a determination of the logarithm for $\omega \in \Omega$ obtained by fixing a branch at a point of $\bar{\Omega}-\{\omega_0\}$, map $\mathbf{D}$ onto the strip $\Sigma = \{z \mid -\infty < x < +\infty,\ |y| < \pi/2\}$ and Ω onto a striplike domain S, depending on ω_0. Its boundary is a closed Jordan curve C with a point, w_∞, at $w=\infty$. Then $F = h \circ f^{-1} \circ H^{-1}$ is a conformal map of S onto Σ. Let $f(-\zeta_0) = \omega_0'$ and $w_0' = H(\omega_0')$; then $\lim_{w\to w_\infty} \operatorname{Re} F(w) = \infty$ and $\lim_{w \to w_0'} \operatorname{Re} F(w) = -\infty$. The points w_0' and w_∞ decompose C into two subarcs C_+ and C_-, where the notation is so chosen that, under the mapping F, C_+ corresponds to $\{y = \pi/2\}$ and C_- to $\{y = -\pi/2)$.

A simple calculation leads to the equation $(w = -\log(f(\zeta)-f(\zeta_0)))$

$$\frac{F''(w)}{[F'(w)]^2} + \frac{1}{F'(w)} - 1 + \frac{\zeta_0-\zeta}{\zeta_0} = -\frac{\zeta_0^2-\zeta^2}{2\zeta_0} \frac{f''(\zeta)}{f'(\zeta)} \quad (|\zeta|<1). \tag{3.2}$$

3.2. *A comparison strip.* Let $0<\varepsilon<1/10$. We assume in the following that Γ is an asymptotically conformal curve and use the notations of Section 3.1; $K, K_1, K_2, \ldots$ denote absolute constants, $M, M_1, M_2, \ldots$ depend only on f and parameters. We write

$$\tilde{\eta}(u) = \eta(2e^{-u}) + 2e^{-u\varepsilon}. \tag{3.3}$$

LEMMA 3.1. *There exists a constant* a_1 *which depends only on* f *(but not on* ω_0*) and a strip*

$$S_1 = \{w = u + iv \mid v > a_1, \varphi_-(u) < v < \varphi_+(u)\} \subset S$$

where φ_- *and* φ_+ *are continuous, piecewise linear functions in* $[a_1, \infty)$ *with the following properties:*

(i) *The corners of both curves* $\{v = \varphi_\pm(u)\}$ *occur at most at points* $u = u_n$ $(n = 1, 2, \dots)$ *with* $u_{n+1} - u_n = 1/2$.

(ii) *If for* $u \geqq a_1$

$$\varepsilon(u) = \sup_{t \geqq u} \{|\varphi_+'(t)|, |\varphi_-'(t)|\}$$

then

$$\varepsilon_1(u) \equiv \varepsilon(u) + 2e^{-u\varepsilon} \leqq K_1 \tilde{\eta}(u-1) \qquad (u \geqq a_1 + 1). \tag{3.4}$$

(iii) *For* $u > a_1$, *let* θ_u *denote the crosscut* $\{\mathrm{Re}\, w = u, \varphi_-(u) < v < \varphi_+(u)\}$ *of* S_1 *and* $\theta(u) = \varphi_+(u) - \varphi_-(u)$ *its length. Then there exists exactly one crosscut* Θ_u *of* S *which contains* θ_u *and joins a point of* C_+ *to one on* C_-. *If* $\Theta(u)$ *is the length of* Θ_u *then*

$$\Theta(u) - \theta(u) + 2e^{-u\varepsilon} \leqq K_2 \tilde{\eta}(u-1) \qquad (u \geqq a_1 + 1) \tag{3.5}$$

and

$$|\theta(u) - \pi| \leqq K_3 \tilde{\eta}(u-1) \qquad (u \geqq a_1 + 1). \tag{3.6}$$

It should be noted that, while S and S_0 change with $\omega_0 \in \Gamma$, a_1 is *independent* of ω_0.

The *proof* of this lemma is contained in Section 2.2 of [4]. Note the difference in the definition of $\tilde{\eta}(u)$ here in (3.3) and in [4]. The strip S_0 constructed there is denoted here by S_1. The fact that for $u > a_1$ the region S has one and only one crosscut Θ_u is stated in Section 1.2 of [4] which is referred to in 2.2.

LEMMA 3.2. *There exists an* $a > a_1 + 1$ *which depends only on* f *which the following properties. Let* $S_0 = \{w = u + iv \mid u > a,\ \varphi_-(u) < v < \varphi_+(u)\}$ *where* φ_+ *and* φ_- *are the functions in the definition of* S_1 *of Lemma* 1; *thus* $S_0 \subset S$. *Let* $F_0 = X_0 + iY_0 : S_0 \to \Sigma$ *denote the one-to-one conformal map of* S_0 *onto* Σ *such that, for* $w \in S_0$, $\lim_{u \to +\infty} \operatorname{Re} F_0(w) = +\infty$ *and* $F_0(a + i\varphi_\pm(a)) = \pm i\pi/2$. *Then for* $Y(w) = \operatorname{Im} F(w)$

$$|Y(w) - Y_0(w)| < M_1 \tilde{\eta}\left(\frac{u}{1+3\varepsilon}\right) \quad \textit{for} \quad w \in S_0,\ u < M_2(\varepsilon). \tag{3.7}$$

Again we note that our constants are independent of ω_0.

Proof. We refer to the proof of Lemma 2 in Section 2.6 of [4] up to and including equation (2.6.14). There an a is determined such that S_0 satisfies the hypotheses (a) and (b) of Theorem 2 of [5] with $L = 2\pi$, $l = 1/8$, c replaced by c_0, an absolute constant defined in ([4], (2.6.13)), $\mu = 1/2$, $\alpha_+(u) = \alpha_-(u) = 2\sqrt{2}\ \tilde{\eta}(u)/\pi$, and, by ([4], (2.6.14)),

$$\lambda(u) = \left[1 + \frac{2\sqrt{2}}{\pi}\tilde{\eta}(u)\right]^{-1}.$$

Note that a depends only on η and thus on f, but not on ε. Furthermore, in the notation of this theorem, $\varepsilon(u) \leqq \varepsilon_1(u) \equiv \varepsilon(u) + 2e^{-u\varepsilon} \leqq K_1 \tilde{\eta}(u-1)$ by (3.4) and $\delta(u) \leqq \delta_1(u) \equiv \delta(u) + 2e^{-u\varepsilon} \leqq K_2 \tilde{\eta}(u-1)$ by (3.5). (Theorem 2 of [5] assumes that $\varepsilon(u) \geqq 2e^{-pu}$ and $\delta(u) \geqq e^{-pu}$ for some $p > 0$. However, if this condition is not satisfied for any $p > 0$, $\varepsilon(u)$ and $\delta(u)$ may be replaced by $\varepsilon_1(u) \equiv \varepsilon(u) + 2e^{-pu}$ and $\delta_1(u) \equiv \delta(u) + e^{-pu}$ for some $p > 0$.) Hence we can apply the result of Part (i) of the proof of Theorem 2 in [5], namely, the inequality (4.5). Here we take $p = \varepsilon$, $p_1 = 5\varepsilon/4$, $\nu_1 = 1 + 2 \cdot 5\varepsilon/4 = 1 + 5\varepsilon/2$ and we obtain for $w \in S_0$

$$|Y(w) - Y_0(w)| \leqq M_3[\tilde{\eta}(u/\nu_1 - 1)]^{\lambda(u/\nu_1 - 1)}$$

for $u \geqq q_3 \nu_1$ (see [5], (4.5)). We now determine $M_2 > q_3 \nu_1$ such that

$$\frac{u}{1+\frac{5}{2}\varepsilon} - 1 > \frac{u}{1+3\varepsilon} \quad \text{and} \quad \tilde{\eta}\left(\frac{u}{1+3\varepsilon}\right) < e^{-1} \quad \text{for} \quad u > M_2(\varepsilon).$$

Since $[\tilde{\eta}(u)]^{\lambda(u)}$ increases with decreasing u, the factor of M_3 is

$$\leqq \left[\tilde{\eta}\left(\frac{u}{1+3\varepsilon}\right)\right]^{\lambda(u/(1+3\varepsilon))} = \tilde{\eta} \cdot \left[\frac{1}{\tilde{\eta}}\right]^{K\tilde{\eta}/(1+K\tilde{\eta})} \leqq \eta \left[\frac{1}{\tilde{\eta}}\right]^{K\tilde{\eta}} \left(K = \frac{2\sqrt{2}}{\pi}\right)$$

For $u>M_2(\varepsilon)$ we have $\tilde{\eta}(u/(1+3\varepsilon))<e^{-1}$ and, therefore $[1/\tilde{\eta}]^{K\tilde{\eta}}\leqq\exp(2\sqrt{2}/\pi e)$. Hence we obtain (3.7) with $M_1=M_3\exp(2\sqrt{2}/\pi e)$.

3.3. *Estimates for $F_0'(w)$ and $F_0''(w)$.* The following Lemma is in part a *quantative* version of a known result on L-strips [13, Theorem X] adapted to our special situation.

We choose an absolute constant α with $3/4<\alpha^3<1$, say

$$\alpha=(4/5)^{1/3}$$

and use the notation of Lemma 3.3. Let $\psi(u)=\frac{1}{2}[\varphi_+(u)+\varphi_-(u)]$ and $\Lambda=\{u\geqq a, v=\psi(u)\}$.

LEMMA 3.3. *There exists a (ε) and $x_0(\varepsilon)$ depending only on ε and f such that, with $S(\alpha)=\{u\geqq a(\varepsilon), |v-\psi(u)|\leqq\alpha^2\pi/2\}$,*

$$\{|w-w^*|\leqq\alpha\pi/2\}\subset S_0 \quad \text{for} \quad w^*\in\Lambda, \qquad \operatorname{Re} w^*\geqq a(\varepsilon), \tag{3.8}$$

$$|F_0'(w)-1|\leqq K\tilde{\eta}\left(\frac{u}{1+4\varepsilon}\right) \quad \text{for} \quad w\in S(\alpha), \tag{3.9}$$

$$\left|\frac{F_0''(w)}{F_0'(w)}\right|\leqq K\tilde{\eta}\left(\frac{u}{1+4\varepsilon}\right) \quad \text{for} \quad w\in S(\alpha), \tag{3.10}$$

$$F^{-1}([x_0(\varepsilon),+\infty))\subset S(\alpha). \tag{3.11}$$

Proof. Let $\{u_n\}$ be the sequence of points, $u_{n+1}-u_n=1/2$, $u_n\geqq a$, at which possible corners of the graphs represented by φ_+ and φ_- occur. By considering the module of the quadrilateral formed by the crosscuts θ_{u_n} and $\theta_{u_{n+1}}$ and the arcs $\{u_n\leqq u\leqq u_{n-1}, v=\varphi_\pm(u)\}$ with respect to the family of curves joining these arcs, we obtain by a known argument (see e.g. [8, pp. 598–599]) that, for $w_n=u_n+i\psi(u_n)$, $X_0(w)\equiv X_0(u,v)$,

$$X_0(w_{n+1})-X_0(w_n)\leqq\pi\int_{u_n}^{u_{n+1}}\frac{du}{\theta(u)}+\frac{\pi}{2}\int_{u_n}^{u_{n+1}}\frac{\varphi_+'^2+\varphi_-'^2}{\theta(u)}\,du+\sigma(u_n)+\sigma(u_{n+1}) \tag{3.11}$$

and

$$X_0(w_{n+1})-X_0(w_n)\geqq\pi\int_{u_n}^{u_{n+1}}\frac{du}{\theta(u)}-[\sigma(u_n)+\sigma(u_{n+1})]. \tag{3.12}$$

Here

$$\sigma(u)=\operatorname*{Max}_{w_j\in\theta_u}\operatorname{Re}(F_0(w_2)-F_0(w_1)),$$

the oscillation of $\operatorname{Re} F_0(w)$ on θ_u. We note that (using $\theta(u)\leqq 2\pi$)

$$\pi\int_{u_n}^{u_{n+1}}\frac{du}{\theta(u)}\geqq\pi\frac{u_{n+1}-u_n}{2\pi}=\frac{1}{4} \tag{3.13}$$

Now, integrating along Λ, we have

$$X_0(w_{n+1})-X_0(w_n)=\int_{u_n}^{u_{n+1}}\left[\frac{\partial X_0}{\partial u}+\frac{\partial X_0}{\partial v}\psi'(u)\right]du$$

and by use of the (generalized) mean value theorem we obtain

$$Q_n=\frac{X_0(w_{n+1})-X_0(w_n)}{\pi\displaystyle\int_{u_n}^{u_{n+1}}\frac{du}{\theta(u)}}=\frac{\theta(u_n')}{\pi}\left[\frac{\partial X_0}{\partial u}+\frac{\partial X_0}{\partial v}\psi'(u)\right]_{u=u_n',v=\psi(u_n')}$$

since $\psi'(u)$ is continuous (even constant) on (u_n, u_{n+1}); $u_n<u_n'<u_{n+1}$. If we write $A(w)=\operatorname{Arg} F_0'(w)$, we obtain

$$Q_n=\frac{\theta(u_n')}{\pi}|F_0'(w_n')|(\cos A(w_n')-\sin A(w_n')\psi'(u_n')),\qquad w_n'=u_n'+i\psi(u_n'). \tag{3.14}$$

We now use estimates from [5] for $|\operatorname{Arg} F_0'(w)|$ in the Remark to Theorem 1 (at the end of its proof) and for $\sigma(u)$ in [5], (2.3). We apply these inequalities with $L=2\pi$, $p=\varepsilon$, $p'=5\varepsilon/4$ and obtain using (3.4) in Lemma 3.1 of the present paper

$$|A(w)|\leqq 2K_1\tilde{\eta}\left(\frac{u}{1+5\varepsilon/2}-1\right)\leqq 2K_1\tilde{\eta}\left(\frac{u}{1+3\varepsilon}\right)\quad\text{for}\quad u\geqq M_4(\varepsilon)\geqq M_2(\varepsilon). \tag{3.15}$$

We can also choose $M_4(\varepsilon)$ so large that, by [5], (2.3),

$$\sigma(u)\leqq 4\pi\cdot K_1\tilde{\eta}\left(\frac{u}{1+3\varepsilon}\right),\qquad u\geqq M_4(\varepsilon). \tag{3.16}$$

Furthermore $|\psi'(u)| \leqq \varepsilon(u) \leqq K_1\tilde{\eta}(u-1) \leqq K_1\tilde{\eta}(u/(1+3\varepsilon))$ for $u \geqq M_2(\varepsilon)$. Writing $\cos A = 1-2\sin^2(A/2)$ we obtain

$$1-K_5\tilde{\eta}^2\left(\frac{u}{1+3\varepsilon}\right) \leqq \cos A(w_n') - \sin A(w_n')\psi'(u_n')$$

$$\leqq 1+K_5\tilde{\eta}^2\left(\frac{u}{1+3\varepsilon}\right), \qquad u \geqq M_4. \tag{3.17}$$

We can determine $M_5(\varepsilon) > M_4(\varepsilon)$ such that

$$1-K_5\tilde{\eta}^2\left(\frac{u}{1+3\varepsilon}\right) > \frac{1}{2} \quad \text{for} \quad u \geqq M_5. \tag{3.18}$$

From (3.11), (3.12), and (3.13) we have

$$1-4[\sigma(u_n)+\sigma(u_{n+1})] \leqq Q_n \leqq 1+\tilde{\eta}^2(u_{n-1})+4[\sigma(u_n)+\sigma(u_{n+1})]$$

and using (3.16), (3.17), and (3.18) we obtain for $u_n \geqq M_5$

$$1-K_6\tilde{\eta}\left(\frac{u_n}{1+3\varepsilon}\right) \leqq \frac{\theta(u_n')}{\pi}|F_0'(w_n')| \leqq 1+K_6\tilde{\eta}\left(\frac{u_n}{1+3\varepsilon}\right),$$

and we may assume $1-K_6\tilde{\eta}(u/(1+3\varepsilon)) > 0$ for $u > M_5$. Finally, by (3.6) we find

$$1-K_7\tilde{\eta}\left(\frac{u_n}{1+3\varepsilon}\right) \leqq |F_0'(w_n')| \leqq 1+K_7\tilde{\eta}\left(\frac{u_n}{1+3\varepsilon}\right), \qquad u_n' > u_n \geqq M_5(\varepsilon).$$

or

$$\big||F_0'(w_n')|-1\big| \leqq K_7\tilde{\eta}\left(\frac{u_n}{1+3\varepsilon}\right) \quad \text{for} \quad u_n > M_5(\varepsilon). \tag{3.19}$$

Now we come to the proof of (3.8) and (3.9). Here we make use of Lemma 2 in [5]. According to this lemma we can determine an $M_6 > M_5 + \pi$ such that for any $w^* = u^* + iv^* \in \Lambda$ with $u^* \geqq M_6$ there is an $r = r(u^*)$ such that the disk $\{|w-w^*| < r\} \subset S_0$. Moreover, we can, because of (3.4) and (3.6) and the expression for $r(u^*)$, assume that

$$r(u^*) \geqq \frac{\pi}{2}\,\theta = r_0 \quad \text{for} \quad u^* > M_6(\varepsilon).$$

Thus the disk $\{|w-w^*| \leqq \theta\pi/2\} \subset S_0$ for $w^* \in \Lambda$, $u^* \geqq M_6(\varepsilon)$.

We consider now $g(w)=\log(F_0'(w)/|F_0'(w^*)|)$ with $\operatorname{Im} g(w)=\operatorname{Arg} F_0'(w)$, $w^*\in\Lambda$, $u^*>M_6(\varepsilon)$. By (3.15),

$$|\operatorname{Im} g(w)|\leqq 2K_1\tilde{\eta}\left(\frac{u^*-\pi}{1+3\varepsilon}\right) \quad \text{for} \quad |w-w^*|\leqq r_0$$

and, therefore, we have in $|w-w^*|\leqq \alpha r_0=\alpha^2(\pi/2)$.

$$\left|\log\left|\frac{F_0'(w)}{F_0'(w^*)}\right|\right|\leqq\frac{2}{\pi}2K_1\tilde{\eta}\left(\frac{u^*-\pi}{1+3\varepsilon}\right)\log\frac{1+\alpha}{1-\alpha}. \tag{3.20}$$

Given $w^*\in\Lambda$ we can find a u_n' in the sequence determined above such that $|u_n'-u^*|\leqq\frac{1}{2}$. Since $|\psi'(u)|\leqq\frac{1}{2}$ in the interval between u_n' and u^* (except when u is a corner-point), we have $|w_n'-w^*|<\sqrt{5}/4<\alpha r_0$ for $\alpha>3/4$. Hence we may apply (3.20) for $w=w_n'$ and obtain thus

$$\left|\log\left|\frac{F_0'(w_n')}{F_0'(w^*)}\right|\right|\leqq\frac{4}{\pi}K_1\tilde{\eta}\left(\frac{u^*-\pi}{1+3\varepsilon}\right)\log\frac{1+\alpha}{1-\alpha}. \tag{3.21}$$

From (3.20) and (3.21) we have for $|w-w^*|\leqq\alpha r_0$, $|w_n'-w^*|\leqq\sqrt{5}/4$:

$$\left|\log\left|\frac{F_0'(w)}{F_0'(w_n')}\right|\right|\leqq K_8\tilde{\eta}\left(\frac{u^*-\pi}{1+3\varepsilon}\right)\left(K_8=\frac{8K_1}{\pi}\log\frac{1+\alpha}{1-\alpha}\right)$$

Hence

$$\left|\frac{F_0'(w)}{F_0'(w_n')}-1\right|\leqq e^{K_8}K_8\tilde{\eta}\left(\frac{u^*-\pi}{1+3\varepsilon}\right).$$

We use (3.19) to estimate $|F_0'(w_n')|$. We have $u_n\geqq u^*-\frac{1}{2}>u^*-\pi$. We choose now $M_7(\varepsilon)>M_6(\varepsilon)$ so large that $K_7\tilde{\eta}(u^*-\pi/(1+3\varepsilon))<1$ for $u^*>M_7$. Then we have $|F_0'(w_n')|\leqq 2$ and therefore

$$\big||F_0'(w)|-|F_0'(w_n')|\big|\leqq 2K_9\tilde{\eta}\left(\frac{u^*-\pi}{1+3\varepsilon}\right), \qquad K_9=e^{K_8}K_8. \tag{3.22}$$

We take now $\operatorname{Re} w=\operatorname{Re} w^*=u$. Writing

$$w\in S(\alpha)=\left\{u\geqq M_7,\ \psi(u)-\alpha^2\frac{\pi}{2}<v<\psi(u)+\alpha^2\frac{\pi}{2}\right\} \tag{3.23}$$

we have by (3.22) and (3.19)

$$\begin{aligned}||F_0'(w)|-1| &\leqq ||F_0'(w)|-|F_0'(w_n')||+||F_0'(w_n')|-1| \\ &\leqq (2K_9+K_7)\tilde{\eta}\left(\frac{u-\pi}{1+3\varepsilon}\right).\end{aligned}$$

Using this in conjunction with (3.15) we obtain

$$|F_0'(w)-1| \leqq K\tilde{\eta}\left(\frac{u-\pi}{1+3\varepsilon}\right) \leqq K\tilde{\eta}\left(\frac{u}{1+4\varepsilon}\right)$$

for $w \in S(\alpha)$ and $u \geqq M_8(\varepsilon)$. If we now set $a(\varepsilon)=\max(M_7(\varepsilon), M_8(\varepsilon))$ we obtain (3.9).

To prove (3.10) note that for $|w-w^*| \leqq \alpha r_0$, $u^* > a(\varepsilon)$

$$\left|\frac{F_0''(w)}{F_0'(w)}\right| \leqq \frac{2}{r_0(1-\alpha)^2} \operatorname*{Max}_{|W-w^*|\leqq r_0} |\operatorname{Arg} F_0'(W)| \leqq \frac{8K_1}{\pi\alpha(1-\alpha)^2}\,\tilde{\eta}\left(\frac{u^*}{1+4\varepsilon}\right)$$

by (3.15). Taking here $\operatorname{Re} w = u = u^*$ we obtain the inequality (3.10).

Let $\Lambda_\pm = \{u > a(\varepsilon),\ v = \psi(u) \pm \alpha^2\pi/2\}$. For $w = u + iv_+ \in \Lambda_+$ and $w = u + iv_- \in \Lambda_-$ we have

$$\begin{aligned}Y_0(u, v_+) - Y_0(u, v_-) &= \int_{v_-}^{v_+} \frac{\partial Y_0(u, v)}{\partial v}\,dv = \int_{v_-}^{v_+} \frac{\partial X_0(u, v)}{\partial u}\,dv \\ &= \int_{v_-}^{v_+} |F_0'(u+iv)| \cos A(u+iv)\,dv.\end{aligned}$$

By the first inequality in (3.9) and (3.15) the integrand

$$|F_0'(u+iv)|\left(1-2\sin^2\frac{A(u+iv)}{2}\right) \geqq \left(1-K\tilde{\eta}\left(\frac{u}{1+4\varepsilon}\right)\right)\left(1-2K_1^2\tilde{\eta}^2\left(\frac{u}{1+3\varepsilon}\right)\right) > \alpha$$

for $u \geqq M_9$ for a sufficiently large $M_9 > a(\varepsilon)$. Hence for $u \geqq M_9$

$$Y_0(u, v_+) - Y_0(u, v_-) \geqq (v_+ - v_-)\alpha = \pi\alpha^2 \cdot \alpha = \pi\alpha^3.$$

Since $\alpha^3 = \frac{4}{5}$ this implies that, for $u \geqq M_9$, $F_0(u+iv_+)$ lies *above* the line $y = -\pi/2 + 4\pi/5 = 3\pi/10$ and $F_0(u+iv_-)$ lies *below* the line $y = \pi/2 - 4\pi/5 = -3\pi/10$.

By Lemma 3.2 we can determine an $M_{10}(\varepsilon) \geqq M_9(\varepsilon)$ so large that

$|Y(w)-Y_0(w)|<\pi/20$ for $\operatorname{Re} w>M_{10}(\varepsilon)$ $(w\in S)$. If $w=u+iv_x$, $u>M_{10}(\varepsilon)$, then $F(u+iv_+)$ lies *above* the line $y=(3\pi/10)-\pi/20=\pi/4$ and $F(u+iv_-)$ lies below the line $y=-\pi/4$. Hence the substrip $\{u>M_{10}(\varepsilon),\ \psi(u)-\alpha^2\pi/2<v<\psi(u)+\alpha^2\pi/2\}$ of $S(\alpha)$ contains the image C_0 of a part of the real axis $\{x\geqq x_0(\varepsilon),\ y=0\}$ under the mapping $z\mapsto F^{-1}(z)$. That this $x_0(\varepsilon)$ can be determined uniformly for all $\omega_0\in\Gamma$ and depends only $M_{10}(\varepsilon)$ and f follows from the uniform continuity of f on $\partial\mathbf{D}$ and the application of the mappings (3.1).

3.4. *Proof of the upper estimate* (1.3). We consider $F(w)-F_0(w)$ in the disk $\{|w-w^*|\leqq\alpha\pi/2=r_0\}$, where $w^*\in\Lambda$ and $\operatorname{Re} w^*>a(\varepsilon)$ so that this disk is in S_0. By Lemma 3.2, if $w\in S_0$ and $u>M_2(\varepsilon)$, then $|\operatorname{Im}(F(w)-F_0(w))|\leqq M_1\tilde\eta(u/(1+3\varepsilon))$. Hence by the Schwarz–Poisson representation we have in the disk $\{|w-w^*|\leqq \alpha r_1=\alpha^2\pi/2\}$

$$|F'(w)-F_0'(w)|\leqq\frac{4M_1}{\pi\alpha(1-\alpha)^2}\,\tilde\eta\left(\frac{u}{1+3\varepsilon}\right)\tag{3.24}$$

and

$$|F''(w)-F_0''(w)|\leqq\frac{16M_1}{(\pi\alpha)^2(1-\alpha)^3}\,\tilde\eta\left(\frac{u}{1+3\varepsilon}\right).$$

We set $\delta_1(\varepsilon)=2e^{-x_0(\varepsilon)}/(1+e^{-x_0(\varepsilon)})$. Substracting from and adding to the left-hand side of (3.2) the term

$$\frac{F_0''(w)}{[F_0'(w)]^2}+\left[\frac{1}{F_0'(w)}-1\right]$$

and using (3.24), (3.25) and (3.9) we obtain for $\zeta=\rho\zeta_0$, $0<\rho<1$, $1-\rho\leqq\delta_1(\varepsilon)$

$$\frac{1-\rho}{2}\left|\frac{f''(\zeta)}{f'(\zeta)}\right|\leqq(1-\rho)+M\tilde\eta\left(\frac{u}{1+4\varepsilon}\right)=1-\rho+M\,[\eta(2e^{-u/(1+4\varepsilon)})+2e^{-\varepsilon u}].$$

Here M depends only on the function η. Since $e^{-u}=|f(\zeta)-f(\zeta_0)|\leqq\frac{1}{2}(1-\rho)^{1-\varepsilon}$ if $1-\rho\leqq\delta_0(\varepsilon)\leqq\delta_1(\varepsilon)$ by (2.2) we have

$$2e^{-u/(1+4\varepsilon)}\leqq(1-\rho)^{(1-\varepsilon)/(1+4\varepsilon)}<(1-\rho)^{1-5\varepsilon}\quad\text{for}\quad 1-\rho\leqq\delta_0(\varepsilon).$$

(Note that $\delta_0(\varepsilon)$ is independent of ζ_0, ω_0.) Hence, for $1-\rho=\delta$,

$$\tfrac{1}{2}\beta(\delta) \leqq \delta + M[\eta(\delta^{1-5\varepsilon}) + 2\delta^{5\varepsilon/6}] \quad \text{for} \quad \delta \leqq \delta_0(\varepsilon),$$

because $\varepsilon(1-\varepsilon) > 5\varepsilon/6$. If we replace now ε by $\varepsilon/5$ we obtain the upper estimate.

4. Consequences and examples

4.1. *Smooth curves.* We derive now some results of Lesley and the second author [4] from Theorem 1.

THEOREM 4.1. *Let f map $\mathbf{D}$ conformally onto the inner domain of Γ and let*

$$\int_0^1 \frac{\eta(t)}{t}\, dt < \infty. \tag{4.1}$$

Then Γ is smooth, $\log f'$ has a continuous extension to $\bar{\mathbf{D}}$ and

$$\max_{|\zeta_1-\zeta_2|\leqq\delta} |\log f'(\zeta_1) - \log f'(\zeta_2)| \leqq M \int_0^{\delta^{1-\varepsilon}} \frac{\eta(t)}{t}\, dt + M(\varepsilon)\delta^{\varepsilon/6} \tag{4.2}$$

for $0 < \varepsilon < \frac{1}{2}$ and $0 < \delta < 1$.

As Rubel, Shields and Taylor have shown [9], it does not matter whether the maximum is taken for $\zeta_1, \zeta_2 \in \partial\mathbf{D}$ or for $\zeta_1, \zeta_2 \in \bar{\mathbf{D}}$. The upper bound in [4, Application 1] is, instead of (4.2),

$$M(\varepsilon) \int_0^{\delta^{1-\varepsilon}} \frac{\eta(t)}{t} \log \frac{1}{\eta(t)}\, dt + M(\varepsilon)\delta^{1-\varepsilon} \int_{\delta^{1-\varepsilon}}^1 \frac{\eta(t)}{t^2} \log \frac{1}{\eta(t)}\, dt.$$

Integration of (1.6) gives the same bound with $\delta^{1-\varepsilon}$ replaced by δ. The estimate (4.2) is still better for "not too smooth" curves.

Proof. It follows from (1.2) that, for $\zeta_1 \in \partial\mathbf{D}$ and $0 < \delta < 1$,

$$\int_{(1-\delta)\zeta_1}^{\zeta_1} \left| \frac{d}{dz} \log f'(z) \right| |dz| \leqq \int_{1-\delta}^1 \frac{\beta(1-r)}{1-r}\, dr.$$

By Theorem 1, this is

$$\leqq M_1 \int_0^{\delta} \frac{\eta(t^{1-\varepsilon})+t^{\varepsilon/6}}{t}\, dt = \frac{M_1}{1-\varepsilon} \int_0^{\delta^{1-\varepsilon}} \frac{\eta(t)}{t}\, dt + \frac{6M_1}{\varepsilon}\, \delta^{\varepsilon/6}.$$

Hence (4.1) implies that $\log f'$ is continuous in $\bar{\mathbf{D}}$ so that Γ is smooth. If $\zeta_2 \in \partial\mathbf{D}$ and $|\zeta_1-\zeta_2| \leqq \delta$, a similar argument shows that

$$\int_{(1-\delta)\zeta_1}^{\zeta_2} \left|\frac{d}{dz} \log f'(z)\right| |dz| \leqq \frac{2M_1}{1-\varepsilon} \int_0^{\delta^{1-\varepsilon}} \frac{\eta(t)}{t}\, dt + \frac{12M_1}{\varepsilon}\, \delta^{\varepsilon/6}.$$

Adding these two estimates we obtain (4.2); the range $\delta_0(\varepsilon) \leqq \delta < 1$ is trivial.

We prove now that (1.5) implies (1.6). Since $c_1 \leqq |f'(\zeta)| \leqq M_1$ for $\zeta \in \bar{\mathbf{D}}$ by Theorem 4.1, it follows (see (2.10)) that

$$c_2\delta \leqq |f(z_1)-f(z_2)| \leqq M_2\delta \quad \text{for} \quad |z_1-z_2|=\delta,\, z_1,\, z_2 \in \partial\mathbf{D}. \tag{4.3}$$

The lower estimate (1.6) is proved as in 2.3 with (2.11) replaced by (4.3). Furthermore, it was shown in [4, Cor. 1] that

$$|\arg f'(\zeta_1) - \arg f'(\zeta_2)| \leqq M_3 \int_0^{\delta} \frac{\eta(t)}{t}\, dt + \delta; \tag{4.4}$$

we have used Remark 2.1 to bring that result to this form. Now the upper estimate (1.6) follows by applying (4.4) to the derivative of the Poisson–Schwarz formula; see [14, Lemma 3].

Remark 4.1. The condition that $\log f'$ is continuous in $\bar{\mathbf{D}}$ does not conversely imply (4.1). To see this, let h be analytic in $\mathbf{D}$ and continuous in $\bar{\mathbf{D}}$ with $h(\mathbf{D}) \subset \mathbf{D}$ such that

$$\int_0^1 |h'(x)|\, dx = \infty. \tag{4.5}$$

The function f defined by $\log f' = 2+h$ satisfies $|\arg f'(z)| < \pi/4$ for $z \in \mathbf{D}$. Hence f is one-to-one in $\bar{\mathbf{D}}$ and Γ is a Jordan curve. The proof of Theorem 4.1 shows that (4.1) does not hold because of (4.5).

4.2. *Asymptotically smooth curves.* The Jordan curve Γ is called *asymptotically* *smooth* if it is rectifiable and if

$$\sup_{|\omega_1-\omega_2|\leqq\delta} \frac{l(\Gamma(\omega_1, \omega_2))}{|\omega_1-\omega_2|} \to 1 \quad \text{as} \quad \delta \to 0 \tag{4.6}$$

where l denotes the length. This is equivalent [7, Th. 2] to $\log f' \in \mathrm{VMOA}$ (vanishing mean oscillation [10]). If ρ is a positive increasing function with $\rho(\delta) \to 0$ as $\delta \to +0$, let $\mathrm{BMO}_{\partial\mathbf{D}}(\rho)$ denote the space of all $g \in L^1(\partial\mathbf{D})$ such that

$$\frac{1}{l(I)}\int_I |g(z)-g_I|\,|dz| \leqq M\rho(\delta), \qquad g_I \equiv \frac{1}{l(I)}\int g(\zeta)\,|d\zeta| \tag{4.7}$$

for all arcs $I \subset \partial\mathbf{D}$ with $l(I) \leqq \delta$. The space $H^1 \cap L^1(\partial\mathbf{D})$ is a subspace of VMOA. See [11, Chapter 5] for a discussion of these concepts.

THEOREM 4.2. *Let f map $\mathbf{D}$ conformally onto the inner domain of Γ. If*

$$\int_0^1 t^{-1}\eta(t)^2\,dt < \infty \tag{4.8}$$

then Γ is asymptotically smooth and

$$\log f' \in \mathrm{BMO}_{\partial\mathbf{D}}(\rho_\varepsilon) \tag{4.9}$$

for $0<\varepsilon<\frac{1}{3}$ where

$$\rho_\varepsilon(\delta) = \int_0^{\delta^{1-\varepsilon}} t^{-1}\eta(t)^2\,dt + \delta^{\varepsilon/5} \qquad (0<\delta<1). \tag{4.10}$$

We need a lemma on functions of bounded mean oscillation.

LEMMA 4.1. *Let g be analytic in $\mathbf{D}$ and let*

$$|g'(z)| \leqq \varphi(\delta) \qquad (|z| \leqq 1-\delta) \tag{4.11}$$

for $0<\delta<1$. If

$$\rho(\delta) \equiv \left(\int_0^1 \frac{t\delta}{t+\delta}\varphi(t)^2\,dt\right)^{1/2} < \infty \tag{4.12}$$

then $g \in \mathrm{BMO}_{\partial\mathbf{D}}(\rho)$ *and moreover, for* $1-\delta \leqq |\zeta| < 1$,

$$\frac{1}{2\pi}\int_{\partial\mathbf{D}} |g(z)-g(\zeta)|^2 \frac{1-|\zeta|^2}{|z-\zeta|^2}|dz| \leqq K_1\rho(\delta)^2. \tag{4.13}$$

Proof. For $\zeta \in \mathbf{D}$, let

$$g_\zeta(z) = g\left(\frac{z+\zeta}{1+\bar{\zeta}z}\right) - g(\zeta) \qquad (z \in \mathbf{D}). \tag{4.14}$$

It easily follows from Parseval's formula that

$$\|g_\zeta\|_2^2 \leqq \frac{2}{\pi}\iint\limits_{\mathbf{D}} (1-|z|^2)|g_\zeta'(z)|^2\, dx\, dy. \tag{4.15}$$

Substituting $z \mapsto (z-\zeta)/(1-\bar{\zeta}z)$ we therefore obtain from (4.14) that

$$\|g_\zeta\|_2^2 \leqq \frac{2}{\pi}\iint\limits_{\mathbf{D}} \frac{(1-|z|^2)(1-|\zeta|^2)}{|1-\bar{\zeta}z|^2}|g'(z)|^2\, dx\, dy$$

$$\leqq \int_0^1 (1-r^2)(1-|\zeta|^2)\varphi(1-r)^2\left(\frac{4}{2\pi}\int_0^{2\pi}\frac{dt}{|1-r\bar{\zeta}e^{it}|^2}\right) r\, dr$$

by (4.11). Hence it follows from the Poisson integral formula that, for $|\zeta| \geqq 1-\delta$,

$$\|g_\zeta\|_2^2 \leqq K_1 \int_0^1 \frac{(1-r)\delta}{1-r+\delta}\varphi(1-r)^2\, dr.$$

Another substitution shows that this estimate is equivalent to (4.13).

Given an arc $I \subset \partial\mathbf{D}$ we choose $\zeta \in \partial\mathbf{D}$ such that $1-|\zeta| = 2l(I)$ and $\zeta/|\zeta|$ is the midpoint of I. Then we obtain from (4.13) that

$$\frac{1}{l(I)}\int_I |g(z)-g(\zeta)|^2\, |dz| \leqq K_2 \int_I |g(z)-g(\zeta)|^2 \frac{1-|\zeta|^2}{|z-\zeta|^2}|dz| \leqq K_3\rho(\delta)^2$$

for $|\zeta| \geqq 1-\delta$. Since the left-hand side is not increased if we replace $g(\zeta)$ by the mean value g_I we see that (4.7) holds.

Proof of Theorem 4.2. Let $g = \log f'$. It follows from Theorem 1 that, for

$|z| \leqq 1 - \delta$,

$$|g'(z)| = \left|\frac{f''(z)}{f'(z)}\right| \leqq \frac{M(\varepsilon)}{\delta}(\eta(\delta^{1-\varepsilon}) + \delta^{\varepsilon/6}).$$

Hence the function defined in (4.12) satisfies

$$\rho(\delta)^2 \leqq 2M(\varepsilon)^2 \int_0^1 \frac{\delta}{t(t+\delta)}[\eta(t^{1-\varepsilon})^2 + t^{\varepsilon/3}]\,dt.$$

Writing $\tau = \delta^{1-\varepsilon/3}$ we deduce that

$$\begin{aligned}\rho(\delta)^2 &\leqq M_1(\varepsilon) \int_0^\tau \left[\frac{\eta(t^{1-\varepsilon})^2}{t} + t^{\varepsilon/3-1}\right] dt + M_2(\varepsilon) \int_\tau^1 \frac{\delta}{t^2}\,dt \\ &\leqq M_3(\varepsilon) \int_0^{\tau^{1-\varepsilon}} \frac{\eta(t)}{t}\,dt + M_4(\varepsilon)\left[\tau^{\varepsilon/3} + \frac{\delta}{\tau}\right].\end{aligned}$$

Since $(1-\varepsilon/3)(1-\varepsilon) > 1 - 4\varepsilon/3$ we obtain (4.10) replacing ε by $3\varepsilon/4$.

Remark 4.2. We mention that (4.8) implies

$$\sup_{|\omega_1-\omega_2|\leqq\delta} \frac{l(\Gamma(\omega_1, \omega_2))}{|\omega_1 - \omega_2|} \leqq 1 + M \int_0^\delta \frac{\eta(t)^2}{t}\,dt. \tag{4.16}$$

We shall not give the proof; it is purely geometric and proceeds by successive subdivisions of $\Gamma(\omega_1, \omega_2)$.

4.3. *A class of examples.* We show now that β can be prescribed up to multiplicative bounds and that the assumptions (4.1) of Theorem 4.1 and (4.8) of Theorem 4.2 cannot be replaced by weaker conditions of the same general type. Note that β is subadditive, by (2.1).

THEOREM 4.3. *For every increasing subadditive function* $\varphi(\delta)$ $(0 < \delta \leqq 1)$, *a univalent function* $f(z)$ $(z \in \mathbf{D})$ *can be constructed such that*

(i) $c\varphi(\delta) \leqq \beta(\delta) \leqq M\varphi(\delta)$ *for* $0 < \delta \leqq 1$;

(ii) $\displaystyle\int_0^1 \frac{\eta(t)}{t}\,dt < \infty \Leftrightarrow \Gamma$ *smooth* $\Leftrightarrow \log f'$ *continuous in* $\bar{\mathbf{D}}$;

(iii) $\displaystyle\int_0^1 \frac{\eta(t)^2}{t}\,dt < \infty \Leftrightarrow \Gamma$ *rectifiable* $\Leftrightarrow \log f' \in \mathrm{VMOA}$.

Proof. Let b be a positive constant to be chosen later. We define

$$b_0 = b\varphi(1), \qquad b_k = b\varphi\left(\frac{1}{2^k}\right) - \frac{b}{2}\varphi\left(\frac{1}{2^{k-1}}\right) \qquad (k=1,2,\ldots); \tag{4.17}$$

it follows from the subadditivity that $b_k \geqq 0$. Induction shows that

$$\sum_{k=0}^{n} 2^{k-n} b_k = b\varphi(2^{-n}) \qquad (n=0,1,\ldots). \tag{4.18}$$

We define now f by $f(0)=0$ and

$$\log f'(z) = g(z) = \sum_{k=0}^{\infty} b_k z^{2k} \qquad (z \in \mathbf{D}); \tag{4.19}$$

this is a lacunary series with Hadamard gaps. If $0<r<1$ then

$$\max_{|z|=r} \left|\frac{f''(z)}{f'(z)}\right| = g'(r) = \sum_{k=0}^{\infty} 2^k b_k r^{2^k-1}. \tag{4.20}$$

Let $2^{-n-1} \leqq 1-r < 2^{-n}$ $(n \geqq 1)$. We see from (4.20) and (4.18) that

$$\begin{aligned}(1-r)rg'(r) &\leqq \sum_{k=0}^{\infty} 2^{k-n} b_k (1-2^{-n-1})^{2^k} \\ &\leqq b\varphi(2^{-n}) + \sum_{k=n+1}^{\infty} 2^{k-n} b_k \exp(-2^{k-n-1}).\end{aligned}$$

Since $b_k \leqq b\varphi(2^{-k}) \leqq b\varphi(1-r)$ for $k \geqq n+1$, we conclude that

$$(1-r)rg'(r) \leqq 2b\left[1+\sum_{j=0}^{\infty} 2^{j+1}\exp(-2^j)\right]\varphi(1-r). \tag{4.21}$$

Hence we see from (4.20) that $(1-|z|^2)\,|f''(z)/f'(z)| < \frac{1}{2}$ for $z \in \mathbf{D}$ if b is chosen sufficiently small, and Becker's criterion [2] shows that f maps $\mathbf{D}$ conformally onto the inner domain of a quasiconformal Jordan curve Γ. We verify now that this function f satisfies (i)–(iii).

(i) It follows from (4.20) and (4.21) that

$$\beta(\delta) = \sup_{1-\delta \leqq r<1} (1-r)g'(r) \leqq M \sup_{1-\delta \leqq r<1} \varphi(1-r) = M\varphi(\delta).$$

Furthermore we see from (4.20) that, if $2^{-n-1}<1-r\leqq 2^{-n}$ $(n\geqq 1)$,

$$(1-r)rg'(r)\geqq \sum_{k=0}^{n} 2^{k-n-1}b_k\left(1-\frac{1}{2^n}\right)^2\geqq\frac{b}{8}\varphi\left(\frac{1}{2^n}\right)\geqq\frac{b}{8}\varphi(1-r).$$

Hence the lower estimate (i) also holds.

(ii) In view of Theorem 4.1, it is sufficient to show that the smoothness of Γ implies (4.1). If Γ is smooth then, by Lindelöf's theorem [6, p. 295],

$$\arg f'(z)=\sum_{k=0}^{\infty} b_k r^{2^k}\sin(2^k\theta)\qquad (z=re^{i\theta})$$

is continuous in $\bar{\mathbf{D}}$. Hence Szidon's theorem [1, p. 246] shows that $\sum b_k<\infty$. Since $\eta(t)\leqq M_1\beta(t^{1/2})\leqq M_2\varphi(t^{1/2})$ by Theorem 1 and by (i), we see from (4.17) that

$$\int_0^1 \frac{\eta(t)}{t}\,dt\leqq 2M_2\int_0^1\frac{\varphi(t)}{t}\,dt\leqq 2M_2\sum_{k=0}^{\infty}\varphi\left(\frac{1}{2^k}\right)$$
$$=\frac{4M_2}{b}\sum_{k=0}^{\infty} b_k<\infty.$$

(iii) Because of Theorem 4.2 and the remarks preceding it, we have only to show that the rectifiability of Γ implies (4.8). If Γ is rectifiable then $\log f'(re^{i\theta})$ has a limit as $r\to 1-0$ for almost all θ [6, p. 320]. Hence it follows from Zygmund's theorem [1, p. 237] applied to the lacunary series (4.19) that $\sum b_k^2<\infty$. As above we deduce that

$$\int_0^1 \frac{\eta(t)^2}{t}\,dt\leqq M_3\int_0^1\frac{\varphi(t)^2}{t}\,dt\leqq M_3\sum_{n=0}^{\infty}\varphi(2^{-n})^2,$$

and, by (4.18) and Schwarz's inequality, this is

$$\leqq 2M_3\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}2^{k-n}b_k^2\right)=4M_3\sum_{k=0}^{\infty}b_k^2<\infty.$$

Remark 4.3. A smooth curve Γ is called *Dini-smooth* if the modulus of continuity $\omega(\delta)$ of the tangent angle (as a function of arc length) satisfies

$$\int_0^1\frac{\omega(t)}{t}\,dt<\infty. \tag{4.22}$$

It is easy to see that $\eta(\delta) \leqq K_1 \omega(\delta)$. Hence (4.22) implies (4.1). This gives a new proof of the well-known fact [12] that $\log f'$ is continuous in $\bar{\mathbf{D}}$ if Γ is Dini-smooth.

We show now that (4.1) does not conversely imply (4.22). Let f again be defined by (4.19) where $b_k > 0$ and

$$\sum_{k=1}^{\infty} b_k < \infty, \qquad \sum_{k=1}^{\infty} k b_k = \infty. \tag{4.23}$$

The proof of Theorem 4.3(ii) shows that (4.1) holds. If $\omega^*(t)$ denotes the modulus of continuity of $\arg f'(e^{i\theta})$, it follows from Theorem 4.1 that $\omega(t) \geqq \omega^*(c_1 t)$. By Szidon's theorem [1, p. 246],

$$\begin{aligned} \omega^*(t) &= \sup_\theta |\operatorname{Im}[\log f'(e^{i\theta+it/2}) - \log f'(e^{i\theta-it/2})]| \\ &= 2 \sup_\theta \left| \sum_{k=1}^{\infty} b_k \cos(2^k\theta) \sin(2^{k-1}t) \right| \geqq c_2 \sum_{k=1}^{\infty} b_k |\sin(2^{k-1}t)|. \end{aligned}$$

Hence

$$\int_0^1 \frac{\omega^*(t)}{t} \, dt \geqq c_2 \sum_{k=1}^{\infty} b_k \int_0^1 \frac{|\sin(2^{k-1}t)|}{t} \, dt \geqq c_3 \sum_{k=1}^{\infty} k b_k = \infty$$

because of (4.23), so that (4.22) does not hold.

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Fachbereich Mathematik
Technische Universität
D 1000 *Berlin* 12
West Germany

Department of Mathematics
University of California, San Diego
La Jolla, CA 92093
U.S.A.

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