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On the existence of extremal Teichmüller mappings

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1. Introduction

Let μ be an orientation preserving homeomorphism of the unit circumference $|z|=1$ onto $|w|=1$ which admits a quasiconformal extension into the disk $D:|z|<1$. Let K_0 be the maximal dilatation of an extremal q.c. extension of μ into D, while H denotes the dilatation of μ, i.e. the infimum of the dilatations of all the extensions of μ into arbitrarily small annuli $1-\varepsilon<|z|\leq 1$, $\varepsilon>0$. If $H<K_0$, then there exists a unique extremal Teichmüller mapping associated with a quadratic differential of finite norm ([9], [10]). To estimate H one can introduce a local dilatation $H(\zeta)$ of μ at a point $\zeta\in\partial D$: It is the infimum of the maximal dilatations of all extensions of μ into arbitrarily small neighborhoods of ζ with respect to $\bar{D}$ (see [5]). R. Fehlmann recently showed that $H=\max_{|\zeta|=1}H(\zeta)$. In the present paper we use this result to give an upper estimate of H which, together with a lower estimate of K_0, allows the conclusion $H<K_0$. (So far, in the literature, this has been done only in the case $H=1$, see [9] and [10].)

In order to carry out the program, two quantities are introduced in sections (2) and (3) respecively. The first one is motivated by the maximal dilatation $q(\rho)$ of the extremal selfmapping of the upper halfplane with boundary values $\mu(x)=x$ for $x\geq 0$ and $\mu(x)=\rho x$, $\rho\geq 1$ for $x<0$, which is easily computed from [7]:

$$q(\rho)=1+\frac{1}{2\pi^2}\log^2\rho+\frac{1}{\pi}(\log\rho)\sqrt{1+\frac{1}{4\pi^2}\log^2\rho}.$$

The second one (section (3)) is obtained in the usual way (see [1]) as the ratio of the moduli of certain quadrilaterals in $|z|<1$ and $|w|<1$ respectively.

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2. An estimate of the local dilatation of μ

In what follows, we will write the given boundary correspondence μ in a real form: $\vartheta = h_\mu(\theta)$, where $\vartheta = \arg w$ and $\theta = \arg z$: namely $h_\mu(\theta) := \arg \mu(e^{i\theta})$. Then $\vartheta = h_\mu(\theta)$ is a ρ-quasisymmetric homeomorphism of $\mathbf{R}$ onto itself. Assume μ is absolutely continuous and satisfies the condition:

$$m \leq h_\mu'(\theta) \leq M, \quad \text{almost everywhere,} \tag{1}$$

where m and M are constant, $0 < m < M < \infty$. We introduce the functions:

$$q(\lambda, \omega) := \tfrac{1}{2}(\omega + \omega^{-1}) + \frac{\omega}{2\pi^2} \log^2 \lambda + \frac{\omega}{2} \sqrt{\left[(\omega - 1)^2 + \frac{1}{\pi^2} \log^2 \lambda\right]\left[(\omega + 1)^2 + \frac{1}{\pi^2} \log^2 \lambda\right]}, \tag{2}$$

$$\lambda(\theta) := \lim_{\varepsilon \to 0} \sup_{0<|t|<\varepsilon} \left\{\frac{h_\mu(\theta + t) - h_\mu(\theta)}{h_\mu(\theta) - h_\mu(\theta - t)}\right\}, \tag{3}$$

and

$$\omega(\theta) := \lim_{\varepsilon \to 0+} \max\left(\frac{M(\theta, \theta + \varepsilon)}{m(\theta, \theta + \varepsilon)}, \frac{M(\theta - \varepsilon, \theta)}{m(\theta - \varepsilon, \theta)}\right) \tag{4}$$

where

$$m(x, y) := \operatorname*{ess\,inf}_{(x,y)} \{h_\mu'\}, \qquad M(x, y) := \operatorname*{ess\,sup}_{(x,y)} \{h_\mu'\}.$$

Because the function h_μ is ρ-quasisymmetric, there always exists the limit (3). We denote by $H(\theta)$ the local dilatation of μ at the point $e^{i\theta}$. We have

THEOREM 1. *Under the above assumption we have*

$$H(\theta) \leq q(\lambda(\theta), \omega(\theta)), \quad \textit{for all } \theta. \tag{5}$$

Proof. Without loss of generality, we only look at $\theta = 0$ and assume that $h_\mu(0) = 0$. We are going to show the estimate (5) for the point $\theta = 0$. Applying the mapping $\zeta = \log z$, we pass to the strip Σ:

$$\Sigma := \{\zeta = \xi + i\eta : 0 < \eta < \pi\} \tag{6}$$

which is the image of the upper half-plane U. The boundary correspondence h_μ

of U becomes a boundary correspondence of Σ as follows:

$$\begin{cases} \xi \to \log(h_\mu(e^\xi)), & \text{for the lower boundary,} \\ \xi+\pi i \to \log(-h_\mu(-e^\xi))+\pi i, & \text{for the upper boundary.} \end{cases} \tag{7}$$

We construct a function $f(\zeta)$ in Σ with the boundary correspondence (7):

$$f(\zeta):=\left(1-\frac{\eta}{\pi}\right)\log(h_\mu(e^\xi))+\frac{\eta}{\pi}\log(-h_\mu(-e^\xi))+i\eta. \tag{8}$$

It is easily seen that f ia a $1-1$ mapping of Σ onto itself. Since h_μ is absolutely continuous, f is absolutely continuous along every line $\xi=\text{const.}$ or $\eta=\text{const.}$ in Σ. A simple computation shows that

$$2f_{\bar\zeta}=E(\zeta)-1+iL(\zeta), \tag{9}$$

and

$$2f_\zeta=E(\zeta)+1-iL(\zeta) \tag{10}$$

where E and L are real functions in Σ:

$$E(\zeta):=\left(1-\frac{\eta}{\pi}\right)\frac{e^\xi h_\mu'(e^\xi)}{h_\mu(e^\xi)}+\frac{\eta}{\pi}\cdot\frac{e^\xi h_\mu'(-e^\xi)}{-h_\mu(-e^\xi)} \tag{11}$$

and

$$L(\zeta):=\frac{1}{\pi}\log\frac{-h_\mu(-e^\xi)}{h_\mu(e^\xi)}. \tag{12}$$

Therefore we have

$$k(\zeta)^2=\left|\frac{f_{\bar\zeta}}{f_\zeta}(\zeta)\right|^2=\frac{(E(\zeta)-1)^2+L(\zeta)^2}{(E(\zeta)+1)^2+L(\zeta)^2}=1-\frac{4E(\zeta)}{(E(\zeta)+1)^2+L(\zeta)^2}.$$

By the condition (1) it is easy to check that $\|k(\zeta)\|_\infty<1$ and hence f is a quasiconformal mapping of Σ onto itself. It is easily seen that

$$\begin{aligned} H(0) &\leq \sup_{\operatorname{Re}\zeta<l}\frac{1+k(\zeta)}{1-k(\zeta)} \\ &= \sup_{\operatorname{Re}\zeta<l}\frac{1}{2E}(1+E^2+L^2+\sqrt{[(E+1)^2+L^2][(E-1)^2+L^2]}) \end{aligned} \tag{13}$$

for any real number l. On the other hand, a simple computation shows that

$$\frac{1}{\omega(0)}+o(1)\leq E(\zeta)\leq \omega(0)+o(1), \quad \text{as} \quad \operatorname{Re}\zeta\to-\infty \tag{14}$$

and

$$|L(\zeta)|\leq\frac{1}{\pi}\log\lambda(0)+o(1), \quad \text{as} \quad \operatorname{Re}\zeta\to-\infty. \tag{15}$$

Setting $l\to-\infty$, from (13), (14) and (15) we prove the inequality (5) for $\theta=0$.

3. An estimate of the smallest maximal dilatation K_0 from below

Apply the mapping $g: z\mapsto i(1-z)/(1+z)$ to the disk $|z|<1$ and a fractional linear transformation G to the disk $|w|<1$, which maps $|w|<1$ onto a upper half-plane with $G(\mu(-1))=\infty$. Then the boundary homeomorphism $G\circ\mu\circ g^{-1}$ of $\mathbf{R}$ onto $\mathbf{R}$ is a ρ-quasisymmetric function, namely, there is a number ρ such that $G\circ\mu\circ g^{-1}$ satisfies the ρ-condition. The infimum of all such numbers ρ is denoted by ρ_0. Denote by $U(z_1, z_2, z_3, z_4)$ the quadrilateral formed by the upper half-plane and the vertexes z_1, z_2, z_3, and z_4. We introduce a function

$$p(\rho):=M\{U(\infty,-1,0,\rho)\}, \quad \text{for} \quad \rho>0, \tag{16}$$

where $M\{U(\infty,-1,0,\rho)\}$ is the modulus of $U(\infty,-1,0,\rho)$, the a-side and b-side of which are chosen such that $p(\rho)$ is an increasing function of ρ. It is known that $p(\rho)=1+r(\rho)\log\rho$, where $r(\rho)$ is a monotone function of ρ and

$$0.2284\cdots<r(\rho)<\frac{1}{\pi}. \tag{17}$$

(See [1] by A. Beurling and L. Ahlfors.)

We are now going to show the inequality:

$$p(\rho_0)\leq K_0. \tag{18}$$

Obviously, for any $x\in\mathbf{R}$ and $t>0$, $M\{U(\infty, x-t, x, x+t)\}=1$ and

$$M\{U(\infty, \tilde{\mu}(x-t), \tilde{\mu}(x), \tilde{\mu}(x+t))\}=M\{U(\infty,-1,0,B(x,t))\}=p(B(x,t)), \tag{19}$$

where $\tilde{\mu} := G \circ \mu \circ g^{-1}$ and

$$B(x, t) := \frac{\tilde{\mu}(x+t) - \tilde{\mu}(x)}{\tilde{\mu}(x) - \tilde{\mu}(x-t)}. \tag{20}$$

On the other hand, if f_0 is an extremal mapping of $|z|<1$ onto $|w|<1$ with the given boundary correspondence μ, then the mapping $G \circ f_0 \circ g^{-1}$ is an extremal mapping of U onto itself with the boundary correspondence $\tilde{\mu} = G \circ \mu \circ g^{-1}$ and hence we have

$$p(B(x, t)) = \frac{M\{U(\infty, \tilde{\mu}(x-t), \tilde{\mu}(x), \tilde{\mu}(x+t))\}}{M\{U(\infty, x-t, x, x+t)\}} \leq K_0. \tag{21}$$

Similarly, one can show that the inequality (21) is true for $t<0$. Noting that ρ_0 is the supremum of $B(x, t)$ for all $x \in \mathbf{R}$ and $t \neq 0$, we get the estimate (18).

One can easily prove that the estimate is sharp in the sense that for any K_0, there is a boundary correspondence μ such that the equality in (18) holds.

4. The main theorem

For a given homeomorphism μ of $|z|=1$ onto $|w|=1$, we call a boundary point $z = e^{i\theta}$ an essential boundary point if $H(\theta) = K_0$. R. Fehlmann proved that if there is a degenerating Hamilton sequence, then there exists an essential boundary point on the circle $|z|=1$ (See [2], p. 567). On the other hand, K. Strebel proved that if there is no degenerating Hamilton sequence for a complex dilatation of an extremal mapping f_0 then f_0 is a Teichmüller mapping associated with a quadratic differential of finite norm. Therefore one can conclude that if there is no essential boundary point, then the extremal Teichmüller mapping exists. By the inequality (18) and Theorem 1 we see that if $q(\lambda(\theta), \omega(\theta)) < p(\rho_0)$ for all θ, then there is no essential boundary point. We have proved

THEOREM 2. *Let μ be an orientation preserving homeomorphism of $|z|=1$ onto $|w|=1$ which admists a quasiconformal extension into the disk $|z|<1$. Suppose that μ is absolutely continuous and satisfies the condition* (1). *If*

$$q(\lambda(\theta), \omega(\theta)) < p(\rho_0), \quad \textit{for all } \theta, \tag{22}$$

then there exists an extremal Teichmüller mapping associated with a quadratic differential of finite norm.

5. Applications

Applying Theorem 1 and Theorem 2 to the special case that the given boundary homeomorphism μ is piecewise smooth, we get some interesting results. In this case the condition (1) requires

$$h'_+(\theta)\neq 0 \quad \text{and} \quad h'_-(\theta)\neq 0 \quad \text{for all } \theta, \tag{23}$$

where $h'_+(\theta)$ and $h'_-(\theta)$ are the right-derivative and the left-derivative of h_μ, respectively. It is easily seen that

$$\lambda(\theta)=\max\left\{\frac{h'_+(\theta)}{h'_-(\theta)}, \frac{h'_-(\theta)}{h'_+(\theta)}\right\} \quad \text{and} \quad \omega(\theta)=1. \tag{24}$$

By Theorem 1, we have

$$H(\theta)\leq q(\lambda(\theta),1):=q(\lambda(\theta)). \tag{25}$$

Moreover, by a normal family argument, one can prove that the equality $H(\theta)=q(\lambda(\theta))$ for every point. Here the function $q(\lambda)$ is given by the expression

$$q(\lambda)=1+\frac{1}{2\pi^2}\log^2\lambda+\frac{1}{\pi}(\log\lambda)\sqrt{1+\frac{1}{4\pi^2}\log^2\lambda}. \tag{26}$$

From (24), (25) and Theorem 2, we have

THEOREM 3. *If the given boundary homeomorphism μ is assumed as above and*

$$\frac{1}{q^{-1}\circ p(\rho_0)}<\frac{h'_+(\theta)}{h'_-(\theta)}<q^{-1}\circ p(\rho_0), \quad \textit{for all } \theta, \tag{27}$$

then there exists an extremal Teichmüller mapping.

COROLLARY. *If there are three points z_1, z_2, and z_3 on the circle $|z|=1$ such that the cross ratio $D(-1, z_1, z_2, z_3)=-1$, $D(\mu(-1), \mu(z_1), \mu(z_2), \mu(z_3))=-A$, $A\geq 1$, and*

$$\max(5^{-1}, A^{-1/2})\leq\frac{h'_+(\theta)}{h'_-(\theta)}\leq\min(5, A^{1/2}), \quad \text{for all } \theta, \tag{28}$$

then there exists an extremal Teichmüller mapping.

Proof. If $A = 1$, then the condition (28) implies that μ is smooth everywhere and hence the dilatation of μ is equal to one. By the result of K. Strebel [9], there is an extremal Teichmüller mapping. We assume $A > 1$. A simple computation shows that $p(\rho_0) \geq p(A)$. By the inequality (17), we have

$$p(A) > 1 + 0.2284(\log A). \tag{29}$$

On the other hand, by (28) and (26), $\lambda(\theta) \leq \min(5, A^{1/2})$, and

$$\begin{aligned} q(\lambda(\theta)) &\leq 1 + \left[\frac{1}{4\pi^2}\log 5 + \frac{1}{2\pi}\left(1 + \frac{1}{8\pi^2}\log^2 5\right)\right]\log A \\ &\leq 1 + 0.2052(\log A). \end{aligned} \tag{30}$$

Therefore $q(\lambda(\theta)) < p(\rho_0)$ for all θ and hence the corollary is proved.

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