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Autor(en): **Bank, Steven B. / Laine, Ilpo**

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On the zeros of meromorphic solutions of second-order linear differential equations

STEVEN B. BANK AND ILPO LAINE[1]

1. Introduction and main results

This paper is concerned with the differential equation,

$$f''+A(z)f=0, \tag{1.1}$$

where $A(z)$ is a meromorphic function on the plane. In an earlier paper [2] the authors investigated this equation in the case where $A(z)$ is an entire function, mainly from the point of view of determining the distribution of zeros of solutions. (Of course, in this case all solutions of (1.1) are entire.) The following theorem summarizes these results, and also includes some well-known facts (see [2] for references). As in [2], we will use the notation $\sigma(f)$ to denote the order of growth of a meromorphic function f, and $\lambda(f)$ to denote the exponent of convergence of the zero-sequence of f.

THEOREM A. *Let $A(z)$ be an entire function, and let f_1 and f_2 be any two linearly independent solutions of* (1.1). *Then:*

(A) *If $A(z)$ is a polynomial of degree $n \geq 1$, then the following hold:* (i) *Any solution $f \not\equiv 0$ of* (1.1) *is of order* $(n+2)/2$, *and* (ii) *At least one of the numbers $\lambda(f_1)$, $\lambda(f_2)$ is $(n+2)/2$.*

(B) *If $A(z)$ is transcendental, any solution $f \not\equiv 0$ of* (1.1) *is of infinite order of growth.*

(C) *If $A(z)$ is transcendental, and $\sigma(A)$ is finite but not a positive integer, then* $\max\{\lambda(f_1), \lambda(f_2)\} \geq \sigma(A)$ *if $\sigma(A) \geq \frac{1}{2}$, while if $\sigma(A) < \frac{1}{2}$, then* $\max\{\lambda(f_1), \lambda(f_2)\} = \infty$.

(D) *For any σ, $0 \leq \sigma \leq \infty$, there exists an entire transcendental function $A(z)$ of order σ such that* (1.1) *possesses a solution with no zeros. If σ is a positive integer or*

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∞, *there exists an entire function* $A(z)$ *of order* σ *such that* (1.1) *possesses two linearly independent solutions each having no zeros.*

(E) *If* $0<\sigma(A)\leq\infty$, *and if* $A(z)$ *has the property that* $\lambda(A)<\sigma(A)$, *then for any solution* $f\not\equiv 0$ *of* (1.1), *the inequality* $\lambda(f)\geq\sigma(A)$ *holds.*

The main technique used in the proofs of Parts (A) and (C) consisted of looking at the product $f_1 f_2$ of the solutions. The proof of Part (E) mainly used the Tumura–Clunie theory (see [6; §3.5]).

In the present paper, we consider the case of equation (1.1) where $A(z)$ is a meromorphic function on the plane, and we seek to determine to what extent results analogous to those in Theorem A hold. Of course, when $A(z)$ is meromorphic, there are some immediate difficulties. For example, it is well-known (see [4; p. 205]) that if $A(z)$ is entire, then the growth of any solution of (1.1) can be estimated in terms of the growth of $A(z)$ alone. However, this is not true if $A(z)$ is meromorphic (see [1] and [3]). but there are more basic difficulties in the case where $A(z)$ is meromorphic. For example, it is possible that no solution of (1.1) except the zero solution is single-valued on the plane. This obstacle can easily be handled since necessary and sufficient conditions on $A(z)$ can be found which guarantee that all solutions of (1.1) are meromorphic functions on the plane. Two such types of conditions are used in this paper. The primary one for our purposes is to represent $A(z)$ in terms of another meromorphic function $E(z)$ which will be the product of two meromorphic solutions of (1.1) (see Lemma B below). The second way is to represent $A(z)$ in terms of another meromorphic function $g(z)$ which will be the quotient of two solutions of (1.1). Of course, the latter is the classical technique of using the Schwarzian derivative of a meromorphic function $g(z)$, which we will denote by $\{g, z\}$. (See Fuchs [5; §2] or Hille [8; Chapter 10].) The necessary and sufficient conditions for single-valued meromorphic solutions are found in Lemmas A and B below.

It was shown in [2; § 5(a)] that the obvious meromorphic analogue of Part A(ii) of Theorem A does not hold, since one can construct a rational function $A(z)$ having a pole of order $n\geq 1$ at ∞, for which equation (1.1) possesses two linearly independent meromorphic solutions f_1, f_2 on the plane such that $\max\{\lambda(f_1), \lambda(f_2)\}<(n+2)/2$. (We remark that Part A(i) of Theorem A is valid for any meromorphic solution $f\not\equiv 0$ of (1.1) when $A(z)$ has a pole of order n at ∞ (see [2; §5(a)] or [13]).) In the following theorem (which is proved in §4), we determine all rational functions $A(z)$ having a pole of some order n at ∞, for which (1.1) possesses two linearly independent meromorphic solutions f_1 and f_2 on the plane such that $\lambda(f_j)<(n+2)/2$ for $j=1, 2$.

THEOREM 1. (a) *Let* $A(z)$ *be a rational function having a pole at* ∞ *of any* *order n, and assume that* (1.1) *possesses two linearly independent meromorphic solutions f_1, f_2 in the plane such that $\lambda(f_j)<(n+2)/2$ for $j=1,2$. Then $A(z)$ must have the form,*

$$A=((E')^2-c^2-2EE'')/4E^2, \tag{1.2}$$

where c is a nonzero constant, and where $E(z)$ is a rational function with the following properties:

(i) *$E(z)\not\equiv 0$ and $E(z)\to 0$ as $z\to\infty$;*

(ii) *All zeros of $E(z)$ in the complex plane are simple;*

(iii) *All poles of $E(z)$ are of even order;*

(iv) *At any finite zero z_0 of $E(z)$, the number $c/E'(z_0)$ is an odd integer.*

In addition, it is also true that f_1 and f_2 each have only finitely many zeros and finitely many poles in the plane, and any solution f_3 of (1.1) *which is not a constant multiple of either f_1 or f_2 has the property that $\lambda(f_3)=(n+2)/2$. Finally, n must be even, $n\geq 2$.*

(b) *Conversely, let c be a nonzero constant, and let $E(z)$ be a rational function which possess properties* (i)–(iv). *Then, if $A(z)$ is the rational function defined by* (1.2), *then $A(z)$ has a pole at ∞, and the equation* (1.1) *possesses two linearly independent meromorphic solutions f_1, f_2 in the plane with the following properties:*

(v) *f_1 and f_2 each have only finitely many zeros;*

(vi) *$E=f_1f_2$ and c is the Wronskian of f_1 and f_2;*

(vii) *$f_1'/f_1=(\frac{1}{2})((E'/E)-(c/E))$, and $f_2'/f_2=(\frac{1}{2})((E'/E)+(c/E))$.*

The example produced in [2; §5a] illustrating the phenomenon described in Theorem 1 corresponds to applying Part (b) to $E(z)=(z-2)(z-1)^{-2}$ and $c=1$. However, the general method of Part (b) allows us to obtain a simpler example, by choosing $E(z)=z^{-2}$, and taking $c\neq 0$ to be arbitrary. In this case, we find that the functions $z^{-1}\exp(\pm(c/6)z^3)$ are both solutions of (1.1) where $A(z)$ is given by $-2z^{-2}-(c^2/4)z^4$.

Turning to Part (B) of Theorem A, it is very easy to see that this can fail to hold if $A(z)$ is a transcendental meromorphic function. For example, it is easy to verify that when $A(z)=-2\sec^2 z$, equation (1.1) possesses the solutions $f_1(z)=\tan z$, and $f_2(z)=1+z(\tan z)$, which are linearly independent, and of finite order. In our next theorems (which are proved in §5), we determine all meromorphic functions $A(z)$ on the plane for which all solutions of (1.1) are meromorphic on the plane, and of finite order of growth. The constructions are stated in terms of both the quotient approach (Theorem 2A), and the product approach (Theorem 2B). As an application of these results (see the Remark in §5), it is shown that examples of the phenomenon can occur for any finite choice of $\sigma(A)$. We now state the results:

THEOREM 2A. (a) *Let* $A(z)$ *be meromorphic on the plane, and assume that* (1.1) *possesses two linearly independent meromorphic solutions* f_1 *and* f_2 *on the plane, each having finite order of growth. Then* $g = f_1/f_2$ *is a non-constant meromorphic function of finite order with the following properties:*

(i) *All poles of* g *are of odd order*;

(ii) *All zeros of* g' *are of even multiplicity*;

(iii) $A \equiv (\frac{1}{2})\{g, z\}$.

(b) *Conversely, suppose* $g(z)$ *is a nonconstant meromorphic function on the plane having finite order of growth and satisfying* (i) *and* (ii) *above. Then, with* A *defined by* (iii), *the equation* (1.1) *possesses two linearly independent meromorphic solutions* f_1, f_2 *on the plane, each having finite order of growth, and such that* $g = f_1/f_2$. *In addition, if either* g' *has infinitely many zeros, or if* g *has infinitely many multiple poles, then* $A(z)$ *has infinitely many poles* (*and so is not rational*).

THEOREM 2B. (a) *Let* c *be a nonzero constant, and let* $E(z) \not\equiv 0$ *be a meromorphic function on the plane having finite order of growth, and satisfying the following properties:*

(i) *All zeros of* E *are simple*;

(ii) *All poles of* E *are of even order*;

(iii) *If* $(z_1, z_2, \ldots)$ *is the zero-sequence of* $E(z)$, *then each number* $q_n = c/E'(z_n)$ *is an odd integer*;

(iv) *If* $s_n = (1 + |q_n|)/2$, *then the sequence obtained from* $(z_1, z_2, \ldots)$ *by letting* z_n *appear* s_n *times, has a finite exponent of convergence*;

(v) $m(r, 1/E) = 0(\log r)$ *n.e. as* $r \to \infty$.

Then, with $A(z)$ *defined by* (1.2), *the equation* (1.1) *possesses two linearly independent meromorphic solutions* f_1, f_2 *on the plane, each having finite order of growth. In addition, properties* (vi), (vii) *in Theorem* 1 *hold. Furthermore, if either* E *has infinitely many poles, or if for infinitely many* z_n *we have* $q_n \neq \pm 1$, *then* $A(z)$ *has infinitely many poles* (*and so is not rational*).

(b) *Conversely, let* $A(z)$ *be a meromorphic function on the plane, and assume that* (1.1) *possesses two linearly independent meromorphic solutions* f_1, f_2 *on the plane, each of finite order of growth. Then there exist a nonzero constant* c, *and a meromorphic function* $E(z) \not\equiv 0$ *of finite order on the plane such that* A *has the form* (1.2), *and* (i)–(v) *above hold.*

As a simple example of the construction given in Part (a) of Theorem 2B, we can take $E(z) = -\sin z$, and c to be any odd integer. Then the conditions (i)–(v) are fulfilled. From (1.2), we find $A(z)$ to be $(\frac{1}{4})+((1-c^2)/4\sin^2 z)$, and from the formulas for f_1 and f_2 in (vii) of Theorem 1, we find the meromorphic solutions $\sqrt{2}\sin(z/2)((1-\cos z)/\sin z)^{(c-1)/2}$, and $-\sqrt{2}\sin(z/2)((1-\cos z)/\sin z)^{-(c+1)/2}$ of (1.1). (Other examples are found in §5.)

The result in Part (C) of Theorem A shows that for entire transcendental functions $A(z)$, the only way for an equation (1.1) to possibly possess two linearly independent solutions each having no zeros, is in the case where $\sigma(A)$ is a positive integer or ∞. (Of course, such examples do exist from Part (D) of Theorem A.) However, as an application of our next result, we show that for transcendental meromorphic functions $A(z)$, there are examples of equations (1.1) for any choice of $\sigma(A)$ which possess two linearly independent meromorphic solutions on the plane each having no zeros. The following results give the general construction of all such equations, and the application mentioned above can be found in §6 along with the proofs. For completeness, we include both the product approach (Theorem 3A) and the quotient approach (Theorem 3B).

THEOREM 3A. (a) *Let $A(z)$ be meromorphic on the plane, and assume* (1.1) *possesses two linearly independent meromorphic solutions f_1 and f_2 on the plane, each having no zeros. Then, there exist a nonzero constant c and an entire function $\psi \not\equiv 0$ with the following properties:*

(i) $A=(4\psi\psi''-8(\psi')^2-c^2\psi^6)/4\psi^2$;

(ii) *If $H(z)$ denotes a primitive of $-(c/2)\psi^2$ on the plane, then there are nonzero constants c_1 and c_2 such that,*

$$f_1=(c_1/\psi)e^{H}, \quad \textit{and} \quad f_2=(c_2/\psi)e^{-H}. \tag{1.3}$$

In addition, if $A(z)$ is transcendental, the following two properties hold:

(iii) *Every solution $f\not\equiv 0$ of* (1.1) *is of infinite order of growth on the plane*;

(iv) *Any solution $f\not\equiv 0$ of* (1.1) *which is linearly independent with each of f_1 and f_2, satisfies $\lambda(f)=\infty$.*

(b) *Conversely, let $\psi\not\equiv 0$ be an entire function, and let c be a nonzero constant. Define $A(z)$ by* (i), *and let H denote a primitive of $-(c/2)\psi^2$. Then for any nonzero constants c_1 and c_2, the meromorphic functions f_1 and f_2 defined by* (1.3) *are linearly independent solutions of* (1.1), *each having no zeros. In addition, any zero of ψ is a pole of $A(z)$, and so if ψ has infinitely many zeros, then A cannot be rational.*

THEOREM 3B. *Let $A(z)$ be meromorphic on a simply-connected region D. Then* (1.1) *possesses two linearly independent meromorphic solutions on D, each having no zeros on D, if and only if there exists a nonconstant analytic function*

$g(z)$ on D such that,

(i) *g has no zeros on D;*

(ii) *All zeros of g' on D are of even multiplicity;*

(iii) *$A = (\frac{1}{2})\{g, z\}$.*

The reason why the result in Part (C) of Theorem A can fail to hold for meromorphic coefficients $A(z)$ is explained by the next result which shows that what is actually occurring in the meromorphic case is a balance between zeros and poles of a solution. If poles as well as zeros are taken into consideration, then we have the following direct analogue of the first part of Part (C) of Theorem A, (to be proved in §7) where we use the notation $\bar{\lambda}(f)$ to denote the exponent of convergence of the sequence of zeros of f, each counted only once. (Of course, in this notation, $\lambda(1/f)$ is the exponent of convergence of the sequence of poles of f.)

THEOREM 4. *Let $A(z)$ be a transcendental meromorphic function on the plane of finite order σ, where σ is not a positive integer, and assume that f_1 and f_2 are two linearly independent meromorphic solutions on the plane of* (1.1). *Then, if $\sigma > 0$, we have*

$$\max\{\bar{\lambda}(f_1), \bar{\lambda}(f_2), \lambda(1/f_1)\} \geq \sigma. \tag{1.4}$$

If $\sigma = 0$, then at least one of the following three sets must be infinite: the set of zeros of f_1; the set of zeros of f_2; the set of poles of f_1.

We remark here that in contrast to the strong result in the second part of Part (C) of Theorem A when $A(z)$ is an entire function of order less than $\frac{1}{2}$, no such result is possible in the meromorphic case as evidenced by examples constructed in §5 (following the proof of Theorem 2B).

As in the case of Part (C) of Theorem A we next show that if the poles of a solution are taken into consideration in the case when $A(z)$ is meromorphic, then a direct analogue of Part (E) of Theorem A holds for meromorphic $A(z)$. This result follows very easily from a theorem of W. Hayman [9; Theorem 4], and the theorem of Hayman permits us to obtain the conclusion under the weaker condition $\bar{\lambda}(A) < \sigma(A)$, thus answering a question raised in [2; p. 352]. The theorem (which will be proved in §7) is as follows:

THEOREM 5. *Let $A(z)$ be a transcendental meromorphic function on the plane of order σ, where $0 < \sigma \leq \pm\infty$, and assume that $\bar{\lambda}(A) < \sigma$. Then, if $f(z) \not\equiv 0$ is a meromorphic solution on the plane of* (1.1), *we have*

$$\max\{\bar{\lambda}(f), \bar{\lambda}(1/f)\} \geq \sigma. \tag{1.5}$$

In the next result, we consider the situation of an equation (1.1) where $A(z)$ is meromorphic, and where (1.1) possesses two linearly independent meromorphic solutions each of whose zero-sequences has a finite exponent of convergence. We address the question of what can be said about the distribution of zeros of other solutions. The answer is very simple, and is given by the following theorem which is proved in §7, and is followed by a simple corollary for the case when $A(z)$ is entire.

THEOREM 6. *Let $A(z)$ be a transcendental meromorphic function on the plane, and assume that* (1.1) *possesses two linearly independent meromorphic solutions f_1 and f_2 on the plane, satisfying $\bar{\lambda}(f_1)<\infty$ and $\bar{\lambda}(f_2)<\infty$. Then, any solution $f\not\equiv 0$ of* (1.1) *which is not a constant multiple of either f_1 or f_2 satisfies,*

$$\max\{\bar{\lambda}(f), \bar{\lambda}(1/f)\}=\infty, \tag{1.6}$$

unless all solutions of (1.1) *are of finite order. In the special case where $\bar{\lambda}(1/A)<\infty$ (e.g. A is of finite order), we can conclude that $\bar{\lambda}(f)=\infty$ unless all solutions of* (1.1) *are of finite order.*

COROLLARY 7. *Let $A(z)$ be a transcendental entire function, and assume that* (1.1) *possesses two linearly independent solutions f_1 and f_2 such that $\lambda(f_1)<\infty$ and $\lambda(f_2)<\infty$. Then, any solution $f\not\equiv 0$ of* (1.1) *which is not a constant multiple of either f_1 or f_2 satisfies $\lambda(f)=\infty$.*

For our final results, we return to the case where $A(z)$ in (1.1) is an entire transcendental function, and to the methods developed in [2] for dealing with this case. As seen from Theorem A, when the order of $A(z)$ is a positive integer or ∞, there seem to be no general results concerning the zeros of solutions of (1.1) except in the special case $\lambda(A)<\sigma(A)$. In our final theorem, we develop a positive result which will be proved in §8, and as a corollary, we apply this result to a special class of equations. Other applications are given in §8. We prove:

THEOREM 8. *Let $A(z)$ be an entire trancendental function of finite order σ, and let $\delta(r)=\min\{|A(z)|:|z|=r\}$ for $r>0$. Assume there is a subset U of $[1,\infty)$ having infinite logarithmic measure, and two constants c_1 and α such that*

$$c_1>0, \quad \alpha>2(\sigma-1), \quad \text{and} \quad \delta(r)\geq c_1 r^{\alpha} \text{ for } r \text{ in } U. \tag{1.7}$$

Then for any two linearly independent solutions f_1, f_2 of (1.1), *we have,*

$$\max\{\lambda(f_1), \lambda(f_2)\}\geq 1+(\alpha/2). \tag{1.8}$$

Remark. The original proof given in [2] of Theorem A(C) for the case $\sigma(A)<\frac{1}{2}$, follows from Theorem 8 and the well-known minimum modulus theorem of P. Barry.

COROLLARY 9. *Let $P(\rho)$ be a nonconstant polynomial such that $P(0)\neq 0$. Let β be a nonzero complex number, and let m be a positive integer. Then if f_1 and f_2 are any two linearly independent solutions of*

$$f''+z^m P(e^{\beta z})f=0, \tag{1.9}$$

we have

$$\max\{\lambda(f_1), \lambda(f_2)\}\geq 1+(m/2). \tag{1.10}$$

We remark that in light of an example constructed in [2; p. 356], the conclusion of Corollary 9 can fail to hold if $m=0$.

Finally, the authors would like to acknowledge valuable conversations with their colleagues, Robert P. Kaufman and Günter Frank. The authors would also like to thank the referee for very helpful comments.

2. Preliminaries

(a) For a meromorphic function $f(z)$ on the plane, we will use the standard notation of the Nevanlinna theory (see [6] or [9]) including the notation $\bar{N}(r, f)$ for the counting function for the distinct poles of f, as well as the notations $\sigma(f)$, $\lambda(f)$, and $\bar{\lambda}(f)$, which were introduced in §1. Following Hayman [7], we use the abbreviation "n.e." (nearly everywhere) to mean "everywhere in $(0, \infty)$ except in a set of finite measure."

(b) For a nonconstant meromorphic function $g(z)$ in a region D, we will use the standard notation $\{g, z\}$ for the Schwarzian derivative of $g(z)$,

$$\{g, z\}=(g'''/g')-(\tfrac{3}{2})(g''/g')^2. \tag{2.1}$$

(c) If $E(z)\not\equiv 0$ is meromorphic on a region D, and c is a nonzero constant, we will use the notation,

$$\langle E, c\rangle=((E')^2-c^2-2EE'')/4E^2. \tag{2.2}$$

It is very easy to verify that for any nonconstant meromorphic function $g(z)$, we have,

$$\langle\pm(cg/g'), c\rangle=(\tfrac{1}{2})\{g, z\}. \tag{2.3}$$

(d) For any two meromorphic functions, f and g, we will denote their Wronskian by $W(f, g)$.

(e) We will require the following elementary fact: If $A(z)$ is meromorphic on the plane, and if $f_1 \not\equiv 0$ and $f_2 \not\equiv 0$ are meromorphic functions on the plane which satisfy (1.1), then $\sigma(f_1) = \sigma(f_2)$. (The proof is very simple: It is obvious if f_1 and f_2 are linearly dependent.) In the case of linear independence, we have,

$$d((f_2/f_1))/dz = c/f_1^2, \tag{2.4}$$

where $c = W(f_1, f_2)$. This relation immediately shows that $\sigma(f_1) \geq \sigma(f_2)$ in the light of Whittaker's result (see [6; p. 104]) that $\sigma(g) = \sigma(g')$ for meromorphic functions g. Reversing the roles of f_1 and f_2 now proves the statement.

3. Single-valued solutions

In this section we give necessary and sufficient conditions for all solutions of equation (1.1) to be meromorphic (and hence single-valued) in a simply-connected region D. (We remark that all regions considered are subsets of the finite plane, and hence do not contain the point at infinity.)

LEMMA A. (a) *Let $A(z)$ be meromorphic in a region D, and assume that* (1.1) *possesses two linearly independent meromorphic solutions f_1, f_2 in D. Then $g = f_1/f_2$ possesses the following properties:*

(i) *All poles of $g(z)$ in D are of odd order;*

(ii) *All zeros of $g'(z)$ in D are of even multiplicity.*

(iii) $A \equiv (\frac{1}{2})\{g, z\}$.

(b) *Conversely, let $g(z)$ be a nonconstant meromorphic function in a simply-connected region D, which possesses properties* (i) *and* (ii), *and define $A(z)$ by* (iii). *Then the equation* (1.1) *possesses two linearly independent meromorphic solutions f_1, f_2 in D such that $g = f_1/f_2$.*

Proof. Part (a). It is well-known [5; p. 6] that (iii) holds. Denoting $c = W(f_1, f_2)$, we have $g' = -c/f_2^2$ from which (i) and (ii) follow.

Part (b). Since the zeros (resp. poles) of g' in D are of even multiplicity (resp. even order), and since D is simply-connected, clearly there exists in D a meromorphic branch $\phi(z)$ of $(g'(z))^{-1/2}$. *With* $A(z)$ defined by (iii), it is well-known [5; p. 6] that ϕ and $g\phi$ are linearly independent solutions of (1.1) which proves Part (b).

LEMMA B. (a) *Let $A(z)$ be meromorphic in a region D, and assume that* (1.1) *possesses two linearly independent meromorphic solutions* f_1, f_2 *in* D. *Set* $E = f_1 f_2$ *and* $c = W(f_1, f_2)$. *Then,*

(i) *All zeros of* $E(z)$ *in* D *are simple;*

(ii) *All poles of* $E(z)$ *in* D *are of even order;*

(iii) *At any zero* z_0 *of* E *in* D, *the number* $c/E'(z_0)$ *is an odd integer;*

(iv) $A \equiv \langle E, c \rangle$.

(b) *Conversely, let* $E(z) \not\equiv 0$ *be a meromorphic function in a simply-connected region* D, *and let* c *be a nonzero constant such that* (i), (ii), *and* (iii) *above hold. Then, if* $A(z)$ *is defined by* (iv), *the equation* (1.1) *possesses two linearly independent meromorphic solutions* f_1, f_2 *in* D *such that*

(v) $E = f_1 f_2$ *and* $c = W(f_1, f_2)$,

and

(vi) $f_1'/f_1 = (\frac{1}{2})((E'/E) - (c/E))$; $f_2'/f_2 = (\frac{1}{2})((E'/E) + (c/E))$.

Proof. Part (a). Set $g = f_1/f_2$, and so g satisfies (i)–(iii) of Lemma A, as well as,

$$g'/g = -c/E. \tag{3.1}$$

Conclusion (i) now follows immediately. Furthermore, any pole of E of order m must be a zero of g' of order m, and hence m is even by Lemma A(ii) proving conclusion (ii). From (3.1), it follows that at any zero z_0 of $E(z)$ in D, the function $g(z)$ has either a zero, say of multiplicity m, or a pole, say of order n, and in addition,

$$-c/E'(z_0) = \text{Residue of } g'/g \text{ at } z = z_0. \tag{3.2}$$

If g has a zero at z_0, then either $m = 1$ or $m - 1$ is even by Lemma A(ii). In any case, m is odd, and since the right-hand side of (3.2) is m in this case, we obtain conclusion (iii). If g has a pole at z_0, then n is odd by Lemma A(i), and again $c/E'(z_0)$ is odd by (3.2). This proves conclusion (iii). Finally (iv) follows immediately from (2.3), (3.1), and Lemma A(iii), since

$$A = (\tfrac{1}{2})\{g, z\} = \langle -E, c \rangle = \langle E, c \rangle. \tag{3.3}$$

Conversely, if $E(z) \not\equiv 0$ is meromorphic in a simply-connected region D and satisfies (i)–(iii), define $H_1 = (\frac{1}{2})((E'/E) - (c/E))$, and $H_2 = (\frac{1}{2})((E'/E) + (c/E))$. Any pole z_0 of H_1 or H_2 must clearly be a zero or pole of E. If E has a zero at z_0, then the zero is simple, and we have,

$$H_1(z) = (\tfrac{1}{2})(1 - (c/E'(z_0)))(z - z_0)^{-1} + \phi_1(z), \tag{3.4}$$

where $\phi_1(z)$ is analytic in a neighborhood of z_0. In view of condition (iii), we see that $H_1(z)$ is either analytic at z_0 or it has a simple pole with integer residue. The same statment holds for $H_2(z)$. Now assume that $E(z)$ has a pole at z_0. Then c/E is analytic at z_0 and so by condition (ii) we see that H_1 and H_2 have simple poles at z_0 with integer residue. Hence all poles of H_1 and H_2 in D are simple with integer residues. Since D is simply-connected it follows from standard techniques that H_1 and H_2 are the logarithmic derivatives of certain meromorphic functions f_1 and f_2 in D so that (vi) holds. By simple calculation from (vi), we see that f_1 and f_2 are solutions of (1.1) when A is defined by (iv). Adding the two relations (vi), it easily follows that for some constant $K \neq 0$, we have $E = Kf_1f_2$. Subtracting the two relations in (vi), we see that $W(f_1, f_2) = c/K$. It thus follows that the two solutions Kf_1 and f_2 satisfy all the conditions in (v) and (vi) proving Part (b).

Remark. Lemma B can be interpreted as giving a complete answer to the question of determining when a meromorphic function $E(z) \not\equiv 0$ in a simply connected region D is the product of two linearly independent meromorphic solutions of an equation (1.1) where A is meromorphic in D. The corresponding question when we replace "meromorphic" by "analytic" throughout, is answered by the following result:

LEMMA C. *Let $E(z) \not\equiv 0$ be analytic in a simply-connected region D. Then $E(z)$ is the product of two linearly independent analytic solutions in D of an equation* (1.1) *where $A(z)$ is analytic in D, if and only if there is a nonzero constant c such that at every zero of $E(z)$ in D, the value of $E'(z)$ is either c or $-c$.*

Proof. If $E(z)$ is the product f_1f_2 of two analytic solutions in D of (1.1) where A is analytic on D, then from Lemma B, we know that $A \equiv \langle E, c\rangle$ for some nonzero constant c. Since A is analytic on D, it follows from (2.2) that if $E(z_0) = 0$ then $E'(z_0) = \pm c$.

Conversely, if $E(z)$ has the property that at every zero, the value of E' is c or $-c$ for some fixed $c \neq 0$, then E satisfies (i)–(iii) of Lemma B so $E(z)$ is the product of two linearly independent meromorphic solutions f_1, f_2 of (1.1) where A is given by $\langle E, c\rangle$. To show A is analytic on D, we can write,

$$A(z) = h(z)/4(E(z))^2, \quad \text{where} \quad h = (E')^2 - c^2 - 2EE''. \tag{3.5}$$

Since $h' = -2EE'''$, it follows that at any zero of E, the analytic function h has at least a double zero, and so $A(z)$ is analytic on D by (3.5). Of course, then f_1 and f_2 are also analytic on D by standard results. This proves Lemma C.

LEMMA D. *Let $A(z)$ be meromorphic on a simply-connected region D, and assume that equation* (1.1) *possesses two linearly independent meromorphic solutions f_1 and f_2 in D. Set $E = f_1 f_2$ and $c = W(f_1, f_2)$. Then:*

(a) *If f_1 has a zero z_0 in D of multiplicity n, then f_2 is analytic and nonzero at z_0 if $n = 1$, while if $n > 1$, then f_2 has a pole at z_0 of order $n-1$.*

(b) *If f_1 has a pole at a point z_0 in D of order n, then either f_2 has a zero at z_0 of multiplicity $n+1$, or f_2 has a pole at z_0 of order n.*

(c) *$E(z)$ has a zero at a point z_0 in D if and only if exactly one of the functions f_1, f_2 has a zero at z_0.*

(d) *For any constant c_1, the equation $A = \langle F, c_1 \rangle$ possesses a meromorphic solution $F \not\equiv 0$ in D. Any function $F(z) \not\equiv 0$ which is meromorphic in a subregion of D and satisfies $A = \langle F, c_1 \rangle$ for some constant c_1 is a product of two solutions of* (1.1) *whose Wronskian is c_1.*

(e) *If D is the whole complex plane, then the Nevanlinna characteristic of E satisfies the following estimate n.e. as $r \to \infty$:*

$$T(r, E) = 0(\bar{N}(r, 1/E) + T(r, A) + \log r). \tag{3.6}$$

Proof. Part (a). This follows immediately from the relation,

$$d((f_2/f_1))/dz = c/f_1^2, \tag{3.7}$$

since f_2/f_1 must have a pole at z_0 of order $2n-1$.

Part (b). We have $f_1'/f_1 = -n(z-z_0)^{-1} + \phi_1(z)$, where ϕ_1 is analytic at z_0. From this we obtain

$$-A = f_1''/f_1 = (n^2+n)(z-z_0)^{-2} + (z-z_0)^{-1}\psi_1(z), \tag{3.8}$$

where ψ_1 is analytic at z_0. From (3.8) we see that the indicial equation for (1.1) at z_0 has roots $n+1$ and $-n$, and so from standard results [8; pp. 155–161] the equation (1.1) possesses an analytic solution $f_3(z)$ in a neighborhood of z_0 which has a zero at z_0 of multiplicity $n+1$. Then $f_3 = c_1 f_1 + c_2 f_2$ for some constants c_1 and c_2, and we must have $c_2 \neq 0$ since f_1 has a pole at z_0. Writing $f_2 = c_2^{-1}(f_3 - c_1 f_1)$, we see that f_2 has a zero of multiplicity $n+1$ at z_0 if $c_1 = 0$, while if $c_1 \neq 0$, f_2 has a pole of order n at z_0. This proves Part (b). (An alternate proof of Part (b) which is analogous to the proof of Part (a), can be given using (3.7) with f_1 and f_2 reversed.)

Part (c). This follows easily from Part (a) noting that any zero of E is a zero of one of the functions f_1 or f_2.

Part (d). If c_1 is given and is nonzero, then $F = (c_1/c)E$ will satisfy $A = \langle F, c_1 \rangle$ by Lemma B(iv). If $c_1 = 0$, it is easily verified that $F = f_1^2$ satisfies $A = \langle F, c_1 \rangle$. Now assume that $F \not\equiv 0$ satisfies $A = \langle F, c_1 \rangle$ for some constant c_1. It is easily verified that each of F, f_1^2, $f_1 f_2$, f_2^2 satisfies the linear differential equation,

$$w''' + 4Aw' + 2A w = 0. \tag{3.9}$$

Since f_1^2, $f_1 f_2$, and f_2^2, are linearly independent, the function F is a linear combination of them, and so is a product of two linear combinations of f_1 and f_2. If c_2 is the Wronskian of these two linear combinations, it is easy to see (see Lemma B(iv)) that $A = \langle F, c_2 \rangle$ and so $c_1 = \pm c_2$. This proves Part (d).

Part (e). By Lemma B(iv), we have $A = \langle E, c \rangle$. We rewrite this equation in the form,

$$E^2 = c^2 / ((E'/E)^2 - 2(E''/E) - 4A). \tag{3.10}$$

We now apply the Nevanlinna theory (including the lemma on the logarithmic derivative) to (3.10), and we obtain,

$$T(r, E) = 0(\bar{N}(r, E) + \bar{N}(r, 1/E) + T(r, A) + \log r) \tag{3.11}$$

holding n.e. as $r \to \infty$. However, any pole of E must be a pole of f_1 or f_2 and hence (see (3.8)) at most a double pole of A. It now follows from (3.11) that (3.6) holds n.e. as $r \to \infty$.

4. Proof of Theorem 1

Part (a). Let $A(z)$ be a rational function having a pole of order n at ∞, and assume that (1.1) possesses two linearly independent meromorphic solutions f_1, f_2 in the plane such that $\lambda(f_j) < (n+2)/2$ for $j = 1, 2$. Set $E = f_1 f_2$ and $c = W(f_1, f_2)$, so that $E(z) \not\equiv 0$ and c is a nonzero constant. Then, from Lemma B, $A(z)$ has the form (1.2), and $E(z)$ possesses properties (ii), (iii), and (iv) listed in Theorem 1. From (3.8), clearly the poles of f_1 and f_2 in the plane can only occur at the poles of A, and so $E(z)$ is analytic in a neighborhood of ∞, say $|z| > K$. If $E(z)$ has an essential singularity at ∞, then the Wiman–Valiron Theory ([11: Chapter 4], [12: Chapters 9 and 10], or [14: Chapter 1]) is applicable to (1.2), and since $A(z)$ has a pole of order n at ∞, it would follows that $\sigma(E) = (n+2)/2$. But in view of (3.6), the rationality of $A(z)$, and the fact that $E(z)$ is not rational, we then obtain $\bar{\lambda}(E) = (n+2)/2$. But then at least one of the two solutions f_j would satisfy $\bar{\lambda}(f_j) = (n+2)/2$ contradicting the hypothesis. Hence the meromorphic function $E(z)$ has at most a pole at ∞, and so is rational. Since E'/E and E''/E both tend to zero as $z \to \infty$, it follows from (1.2) that $E \to 0$ as $z \to \infty$ (and, in fact, has a zero of multiplicity $n/2$ at ∞.) From Lemma D, Part (a), we see that both f_1 and f_2 have only finitely many zeros in the plane since E has only finitely many zeros. Finally, let $f_3 = c_1 f_1 + c_2 f_2$ where c_1 and c_2 are nonzero constants, and set $E_1 = f_1 f_3$. If we assume $\lambda(f_3) < (n+2)/2$, the same argument as above would show that E_1 is rational. But $E_1 = c_1 f_1^2 + c_2 E$, and so f_1^2 would be rational. This implies f_1 is rational, and so $A = -f_1''/f_1$ tends to zero as $z \to \infty$ contradicting the hypothesis. Hence $\lambda(f_3) = (n+2)/2$ proving Part (a) completely.

Part (b). Let $A(z)$ be the rational function defined by (1.2), where $c \neq 0$ and the rational function $E(z)$ satisfy (i)–(iv). Clearly $A(z)$ has a pole at ∞, and by Lemma B, the equation (1.1) possesses two linearly independent meromorphic solutions f_1, f_2 in the plane, such that conclusions (vi) and (vii) in Theorem 1 hold. Since $E = f_1 f_2$ and E is rational, it follows from Part (a) of Lemma D that each of f_1, f_2 has only finitely many zeros in the plane. This proves Part (b) completely.

5. Solutions of finite order

In this section, we prove Theorems 2A and 2B, and give examples.

Proof of Theorem 2A. Part (a) follows immediately from Lemma A(a).

Part (b). Assume now that g is a nonconstant meromorphic function on the plane having finite order of growth, and possessing properties (i) and (ii). Then with A defined by (iii), it follows from Lemma A(b), that (1.1) possesses two independent meromorphic solutions f_1, f_2 on the plane with $g = f_1/f_2$. Then $g' = -c/f_2^2$ where $c = W(f_1, f_2)$, and so f_2 is of finite order. Since $f_1 = gf_2$, we also have that f_1 is of finite order. To prove the last statement, we observe first that it is easy to verify (e.g. see [5; p. 5]) that if $g(z)$ has a Laurent expansion around a point z_0 of the form,

$$g(z) = c_0 + \sum_{\substack{k=p \\ k\neq 0}}^{\infty} c_k (z - z_0)^k, \tag{5.1}$$

where $p \neq 0$ is an integer, and $c_p \neq 0$, then

$$\{g(z), z\} = ((1-p^2)/2)(z-z_0)^{-2} + (z-z_0)^{-1}\phi(z), \tag{5.2}$$

where $\phi(z)$ is analytic at z_0. Since the left-hand side of (5.2) is $2A(z)$, it easily follows that any zero of g' or any multiple pole of g is a pole of A, and this proves Theorem 2A completely.

Proof of Theorem 2*B*. Part (a). Under the stated conditions (i)–(iii), on E and c, and with $A(z)$ defined by (1.2), it follows from Lemma B that (1.1) possesses two linearly independent meromorphic solutions f_1, f_2 in the plane such that $E=f_1f_2$, $c=W(f_1,f_2)$, and

$$f_1'/f_1=(\tfrac{1}{2})((E'/E)-(c/E)); \qquad f_2'/f_2=(\tfrac{1}{2})((E'/E)+(c/E)). \tag{5.3}$$

We now analyze the sequence of zeros and sequence of poles of f_1. Suppose ρ is a zero of f_1 of multiplicity q. By Lemma D(a), the point ρ is a zero of E, so $\rho=z_n$ for some n. From (5.3), the residue of f_1'/f_1 is $(1-q_n)/2$ at z_n, so $q=(1-q_n)/2$ (where $q_n=c/E'(z_n)$). Since $q\geq 1$, we see that $q_n\leq -1$, and thus $q=s_n=(1+|q_n|)/2$. Thus the zero-sequence of f_1 is contained in the sequence described in (iv) of Theorem 2B, and thus has a finite exponent of convergence. Now suppose w is a pole of f_1 of order t. Then by Lemma D(b), either E has a pole at w of order $2t$, or E has a zero at w, in which case $w=z_n$. In the latter case, it follows from (5.3) that $-t=(\tfrac{1}{2})(1-q_n)$. Hence $q_n\geq 3$ and $t=s_n-1$. It follows that the sequence of poles of f_1 is contained in the union of two sequences R_1 and R_2, where R_1 is the sequence obtained from the pole sequence of E by eliminating one-half of the occurrences of each pole, and where R_2 is the sequence obtained from $(z_1, z_2, \ldots)$ by repeating z_n only s_n-1 times. Since E is of finite order, clearly R_1 has a finite exponent of convergence. In view of condition (iv) of the theorem, R_2 has a finite exponent of convergence. Thus the sequence of poles of f_1, (like the sequence of zeros of f_1), has a finite exponent of convergence. Hence we may write, $f_1=(Q_1/Q_2)e^Q$, where Q_1 and Q_2 are canonical products of finite order, and where Q is entire. Now by the Nevanlinna theory, each of $m(r, E'/E)$, $m(r, Q_1'/Q_1)$ and $m(r, Q_2'/Q_2)$ is $0(\log r)$ as $r\to\infty$. In view of condition (v) and (5.3), it now follows that $m(r, Q')=0(\log r)$ as $r\to\infty$, and thus Q is a polynomial. Hence f_1 is of finite order. It now follows from §2(e) that f_2 is also of finite order.

The relations (vi) and (vii) in Theorem 1 also hold (see (5.3)).

To prove the last statements, we observe first that any pole of E is a pole of at least one of f_1, f_2, and hence clearly is a pole of A (see (3.8)). Now set $g=f_1/f_2$ so that (3.1) holds. Hence at any z_n, we have relation (3.2), and thus at z_n, g has either a zero of multiplicity $-q_n$ if $q_n<0$, or a pole of order q_n if $q_n>0$. Since $A=(\tfrac{1}{2})\{g, z\}$ by Lemma A, we see from (5.1) and (5.2) that,

$$A(z)=((1-q_n^2)/4)(z-z_n)^{-2}+(z-z_n)^{-1}\psi(z), \tag{5.4}$$

where ψ is analytic at z_n. Hence, if $q_n\neq\pm 1$, then A has a pole at z_n. This proves Part (a).

Part (b). Let $E=f_1f_2$, and $c=W(f_1,f_2)$. Then E is of finite order, and by Lemma B, properties (i)–(iii) hold, and A is given by (1.2). Set $g=f_1/f_2$ so that $g'/g=-c/E$. Since g is of finite order, we obtain (v) from the Nevanlinna theory. As in the proof of Part (a), we see that at each z_n, g has either a zero of multiplicity $-q_n$ if $q_n<0$, or a pole of order q_n if $q_n>0$. Since both the sequence of zeros of g, and the sequence of poles of g each have a finite exponent of convergence, it now follows easily that the sequence described in (iv) also has a finite exponent of convergence, and thus Part (b) is proved.

Remark. In this remark, we show that for any nonnegative real number α, there exists a transcendental meromorphic function $A(z)$ on the plane of order α, such that every solution $f(z)\not\equiv 0$ of (1.1) is a transcendental meromorphic function on the plane of order α. The construction is quite easy. Let ψ be a transcendental entire function of order α having only simple zeros such that $\lambda(\psi)=\alpha$. Let g denote a primitive of ψ^2. Then if $A=(\frac{1}{2})\{g,z\}$, we see by Lemma A that g is the quotient f_1/f_2 of two linearly independent meromorphic solutions on the plane of (1.1). Since $g'=-c/f_2^2$, where $c=W(f_1,f_2)$, we see that ψ^{-1} and $g\psi^{-1}$ are meromorphic solutions on the plane of (1.1). From (5.1) and (5.2), every zero of g' is a double pole of A. Hence every zero of ψ is a double pole of A, and so A is a transcendental meromorphic function of order at least α. However, since g is of order α, we also have $\sigma(A)\leq\alpha$ and so $\sigma(A)=\alpha$. The solution ψ^{-1} is of order α, ans so by §2(e), every solution (except the zero solution) is of order α. Of course, all solutions (except zero) are transcendental since A is transcendental. The examples constructed here have the property $\sigma(f)=\sigma(A)$ for all solutions $f\not\equiv 0$ of (1.1). In the following example, we construct an equation (1.1) where A is a transcendental meromorphic function of finite order on the plane, all of whose solutions $f\not\equiv 0$ are meromorphic functions of finite order on the plane satisfying $\sigma(f)>\sigma(A)$.

EXAMPLE. Set $E(z)=\cos(z^{1/2})$. Then $E(z)$ is an entire function having simple zeros at the points $z_n=((2n+1)\pi/2)^2$ for $n=0,1,\ldots$, and no other zeros. It is easy to verify that if we choose $c=1/\pi$, then for each n we have

$$q_n=c/E'(z_n)=(-1)^{n+1}(2n+1), \tag{5.5}$$

and so q_n is odd integer. Defining s_n as in Theorem 2B, Part (iv), we have $s_n=n+1$, and it is easy to see that the sequence obtained from $(z_0,z_1,\ldots)$ by repeating z_n s_n-times has exponent of convergence equal to 1. The function $E(z)$ satisfies the differential equation,

$$1/E(z)^2=1+4z(E'(z)/E(z))^2, \tag{5.6}$$

and since E is of order $\frac{1}{2}$, we have from (5.6) that $m(r, 1/E) = 0(\log r)$ as $r \to \infty$. Hence from Theorem 2B, Part (a), if we set $A = \langle E, c \rangle$, then (1.1) possesses two linearly independent meromorphic solutions f_1, f_2 on the plane, each having finite order of growth, satisfying $E = f_1 f_2$, $c = W(f_1, f_2)$, and such that (5.3) holds. From (5.3) and (5.5), we see that the residue at z_n of f_2'/f_2 is $(1+q_n)/2$. Hence if n is odd, then f_2 has a zero at z_n of multiplicity $n+1$. Since the exponent of convergence of the sequence obtained from $(z_1, z_3, \ldots)$ by repeating z_{2k+1} $(2k+2)$-times is obviously equal to 1, we can conclude that $\sigma(f_2) \geq 1$. Using §2(e), we can now conclude that $\sigma(f) \geq 1$ for every solution $f \not\equiv 0$ of (1.1). Of course, since E is of order $\frac{1}{2}$, we see that $A = \langle E, c \rangle$ is of order at most $\frac{1}{2}$. In fact, A is of order precisely $\frac{1}{2}$, since for $n \geq 1$ we have $q_n \neq \pm 1$, and so from the proof (see (5.4)) of Theorem 2B, the function $A(z)$ has a double pole at $z_1, z_2, \ldots$. This shows that $\sigma(A) \geq \frac{1}{2}$ and thus $\sigma(A) = \frac{1}{2}$. Hence $\sigma(A) < \sigma(f) < \infty$ for all solutions $f \not\equiv 0$.

6. Zero-free solutions

Proof of Theorem 3A. Part (a). Set $E = f_1 f_2$, and $c = W(f_1, f_2)$. By assumption, E has no zeros, and by Lemma B, all poles of E are of even order. Hence E has the form $1/\psi^2$ where ψ is an entire function, and by Lemma B the representation (i) holds since the right side of (i) is $\langle 1/\psi^2, c \rangle$. Now let H denote a primitive of $-(c/2)\psi^2$. From the relations $E = f_1 f_2$ and $c = W(f_1, f_2)$, we see that (5.3) holds, and hence

$$f_1'/f_1 = -((\psi'/\psi) + (c/2)\psi^2); \qquad f_2'/f_2 = -((\psi'/\psi) - (c/2)\psi^2). \tag{6.1}$$

Since $H' \equiv -(c/2)\psi^2$, the representations (1.3) follow immediately.

Now assume that $A(z)$ is transcendental. Then H must be transcendental, for in the contrary case, ψ would be a polynomial, and A would be rational by (i). Now, in view of (1.3), and the definition of H, we have as $r \to \infty$,

$$T(r, e^H) \leq T(r, f_1) + (\tfrac{1}{2})T(r, H') + 0(1). \tag{6.2}$$

Since $T(r, H') = 0(T(r, H))$ n.e. as $r \to \infty$, and $T(r, e^H)/T(r, H) \to +\infty$ as $r \to \infty$ (see [6; pp. 54, 55]) we see from (6.2) that,

$$T(r, e^H) \leq 2T(r, f_1) + 0(1) \text{ n.e. as } r \to \infty. \tag{6.3}$$

Since e^H is of infinite order, the same is true for f_1, and also for all solutions $f \not\equiv 0$ by §2(e).

Now let $f = \alpha f_1 + \beta f_2$ where α and β are nonzero constants, and set $E_1 = ff_1$. Since f_1 has no zeros, clearly any zero of E_1 must be a zero of f. We now apply Lemma D, Part (e), to both E and E_1. From the relation $f_1^2 = (1/\alpha)(E_1 - \beta E)$, we thus obtain,

$$T(r, f_1) = 0(T(r, A) + \bar{N}(r, 1/f) + \log r), \tag{6.4}$$

n.e. as $r \to \infty$. Since $A = -f_1''/f_1$, we have

$$m(r, A) = 0(\log T(r, f_1) + \log r), \quad \text{n.e. as } r \to \infty. \tag{6.5}$$

Since f_1 has no zeros, any pole of A must be a pole of f_1 and hence a zero of ψ by (1.3). Since A can have at most double poles (see (5.2) and Lemma A), we see $N(r, A) = 0(N(r, 1/\psi))$ as $r \to \infty$, and so from (6.4) and (6.5) we have,

$$T(r, f_1) = 0(N(r, 1/\psi) + \bar{N}(r, 1/f) + \log r), \tag{6.6}$$

n.e. as $r \to \infty$. Since $\psi^2 = -(2/c)H'$, it now follows from (6.3) and (6.6) that,

$$T(r, e^H) = 0(\bar{N}(r, 1/f) + \log r) \quad \text{n.e. as } r \to \infty. \tag{6.7}$$

Since $\sigma(e^H) = \infty$, we thus obtain $\bar{\lambda}(f) = \infty$. This proves Part (a).

Part (b). Set $E = 1/\psi^2$, and $A = \langle E, c\rangle$. Then, by Lemma B, Part (b), equation (1.1) possesses two linearly independent meromorphic solutions f_1 and f_2 on the plane such that $E = f_1 f_2$, $c = W(f_1, f_2)$, and (5.3) holds. Since $E = 1/\psi^2$, we see that (6.1) holds, and since $H' = -(c/2)\psi^2$, we now see that the functions defined by (1.3) are solutions of (1.1). Any zero of ψ is a pole of E, and by Lemma D(b), a pole of f_1. Thus (see (3.8)), any zero of ψ is a pole of A. This proves Theorem 3A.

Remark. In this remark, we show that for any α, $0 \leq \alpha \leq +\infty$, there exists a transcendental mermorphic function $A(z)$ on the plane of order α such that (1.1) possesses two linearly independent meromorphic solutions on the plane, each having no zeros. The construction is very simple. Let ψ be an entire function of order α with only simple zeros, and satisfying $\lambda(\psi) = \alpha$. Let c be a nonzero constant, and set $A = \langle(1/\psi^2), c\rangle$. Then by Theorem 3A, the equation (1.1) possesses two linearly independent meromorphic solutions, each having no zeros, and $\lambda(1/A) = \alpha$. Thus, $\sigma(A) \geq \alpha$. But obviously, $\sigma(A) \leq \sigma(\psi) = \alpha$, so $\sigma(A) = \alpha$.

Proof of Theorem 3B. If (1.1) possesses two linearly independent meromorphic solutions f_1 and f_2 on D, then setting $g = f_1/f_2$, we see that the conclusions (ii) and (iii) hold by Lemma A. Since $g' = -c/f_2^2$, we see that if f_2 has no zeros on D, then g must be analytic on D. Since $(1/g)'=c/f_1^2$, we see that if f_1 has no zeros on D, then $1/g$ is analytic on D, so (i) holds.

Conversely, under conditions (i)–(iii) and the analyticity of g, it follows from Lemma A that (1.1) possesses two linearly independent meromorphic solutions f_1, f_2 on D such that $g=f_1/f_2$. Since $g'=-c/f_2^2$, and $(1/g)'=c/f_1^2$, it follows that if g has no zeros or poles on D, then f_1 and f_2 have no zeros on D.

7. Distribution of zeros or poles of solutions

Proof of Theorem 4. We are given that $\sigma=\sigma(A)$ is finite, but not a positive integer. Set $E=f_1f_2$, and consider first the case $\sigma>0$. Assume that (1.4) fails to hold. Since the zeros of E are all simple, we then obtain $\lambda(E)<\sigma$. In view of Lemma D(b), any pole of E, say of order k, must be a pole of f_1 of order $k/2$. Hence by our assumption, we also obtain $\lambda(1/E)<\sigma$. Since $\lambda(E)<\sigma$, it follows from Lemma D(e), that $\sigma(E)\leq\sigma(A)=\sigma$. However, from Lemma B we also have $A=\langle E, c\rangle$, where $c=W(f_1, f_2)$, and so $\sigma(A)\leq\sigma(E)$. Thus $\sigma(E)=\sigma$. Now we may write $E=(G_1/G_2)e^G$, where G_1 and G_2 are entire canonical products of order less than σ, and G is a polynomial. Hence we obtain $\sigma(e^G)=\sigma$ which is absurd since σ is not an integer. This contradiction proves (1.4) if $\sigma>0$.

Now suppose $\sigma=0$ but the conclusion fails. Then as above, E has only finitely many zeros, and finitely may poles. By Lemma D(e), E is of order zero, and so E is rational. However, this implies $A=\langle E, c\rangle$ is rational contrary to hypothesis. This proves Theorem 4 completely.

Proof of Theorem 5. Since $f(z)$ is a solution of (1.1) where $\sigma(A)>0$, it is obvious that $f(z)$ cannot be rational, nor be of the form e^{az+b} for constants a and b. Hence we can invoke [7; Theorem 4], and we obtain n.e. as $r\to\infty$,

$$T(r, f/f')=0(\bar{N}(r, f)+\bar{N}(r, 1/f)+\bar{N}(r, 1/f'')). \tag{7.1}$$

In addition, since f satisfies (1.1), we have,

$$\bar{N}(r, 1/f'')\leq\bar{N}(r, 1/f)+\bar{N}(r, 1/A). \tag{7.2}$$

By assumption, $\bar{\lambda}(A)<\sigma$. Hence, if we assume that (1.5) fails to hold, then it follows from (7.1) and (7.2), that f/f' is of order less than σ. By Jensen's formula, we then see that if $\psi=f'/f$, then $\sigma(\psi)<\sigma$. However, from (1.1) it easily follows that $-A=\psi'+\psi^2$, and so we would obtain $\sigma(A)\leq\sigma((\psi)<\sigma=\sigma(A)$ which is absurd. This contradiction proves Theorem 5.

Proof of Theorem 6. Assume (1.1) possesses linearly independent meromorphic solutions f_1 and f_2 such that $\bar{\lambda}(f_1)<\infty$ and $\bar{\lambda}(f_2)<\infty$. Set $E_1=f_1f_2$, and let $f=\alpha f_1+\beta f_2$ where α and β are nonzero constants, and set $E_2=ff_1$. Assume that (1.6) fails to hold, so that $\bar{\lambda}(f)<\infty$ and $\bar{\lambda}(1/f)<\infty$. From these relations we easily see that $\bar{\lambda}(E_1)<\infty$, and $\bar{\lambda}(E_2)<\infty$. By Lemma D(e), there is a constant $b>0$ such that n.e. as $r\to\infty$,

$$T(r, E_j)=0(r^b+T(r, A)) \quad \text{for } j=1,2. \tag{7.3}$$

Since $E_2=\alpha f_1^2+\beta E_1$, we thus obtain,

$$T(r, f_1)=0(r^b+T(r, A)) \quad \text{n.e. as } r\to\infty. \tag{7.4}$$

Since $A=-f''/f$, we see that (6.5) holds, and since any pole of A is at most double (see (3.8)) and is either a zero or pole of f, we also have,

$$N(r, A)\leq 2(\bar{N}(r, 1/f)+\bar{N}(r, f)). \tag{7.5}$$

Hence by assumption, $N(r, A)=0(r^a)$ as $r\to\infty$ for some $a>0$. Together with (7.4) and (6.5), we obtain $T(r, f_1)=0(r^{a+b})$ n.e. as $r\to\infty$, from which it follows (see [2; §2(A), p. 353)] that f_1 is of finite order. Hence by §2(e), all solutions are of finite order if (1.6) fails to hold.

Now assume that $\bar{\lambda}(1/A)<\infty$. Then since any pole of f is a pole of A (see (3.8)), we have $\bar{\lambda}(1/f)\leq\bar{\lambda}(1/A)<\infty$, and so (1.6) takes the form $\bar{\lambda}(f)=\infty$. This proves Theorem 6.

Proof of Corollary 7. This result follows immediately from the last statement in Theorem 6 together with the fact (see Part (B) of Theorem A) that when A is a transcendental entire function, all solutions $f\not\equiv 0$ of (1.1) have infinite order.

8. New results when A is entire

Proof of Theorem 8. Let f_1 and f_2 be linearly independent solutions of (1.1), and set $E=f_1f_2$. Then by Lemma B, we have

$$4A=(E'/E)^2-2(E''/E)-(c/E)^2, \tag{8.1}$$

where $c=W(f_1, f_2)$. Of course E cannot be a polynomial since A is transcendental. Hence we can apply the Wiman–Valiron theory to (8.1), and we obtain the existence of a set D in $[1,\infty)$ of finite logarithmic measure such that if r does not belong to D, and z is a point on $|z|=r$ at which $|E(z)|=M(r,E)$, then

$$2\,|A(z)|\le (v(r)/r)^2 \tag{8.2}$$

where $v(r)$ denotes the central index of E. Since U is of infinite logarithmic measure, we can find a sequence $\{r_n\}\to+\infty$ such that r_n belongs to U but not to D. From (8.2) and (1.7), we then obtain,

$$2c_1r_n^{\alpha}\le (v(r_n)/r_n)^2 \quad \text{for all } n, \tag{8.3}$$

and it now follows (see [11; p. 34]) that $\sigma(E)\ge 1+(\alpha/2)$. Of course by (1.7), we also have $\sigma(A)<1+(\alpha/2)$. In view of (3.6), we then obtain $\lambda(E)\ge 1+(\alpha/2)$ from which (1.8) immediately follows. This proves Theorem 8.

Application of Theorem 8. We consider the differential equations

$$f''+z^m \sin^p (z^q)f=0, \qquad f''+z^m \cos^p (z^q)f=0, \tag{8.4}$$

where m, p, and q are positive integers. Then from Theorem 8 we can conclude that if $m>2(q-1)$, and f_1 and f_2 are two linearly independent solutions of either the first equation in (8.4) or the second equation, then

$$\max\{\lambda(f_1), \lambda(f_2)\}\ge 1+(m/2). \tag{8.5}$$

We will indicate the proof for the first equation, the second being similar. For any $\varepsilon>0$, there is a constant $K_\varepsilon>0$ such that

$$|\sin (z^q)|\ge K_\varepsilon, \tag{8.6}$$

if $|z^q-n\pi|\ge\varepsilon$ for all integers n (see [10; p. 71]). If V denotes the union of all intervals $((n\pi-\varepsilon)^{1/q}, (n\pi+\varepsilon)^{1/q})$ for $n=0, 1, \ldots$, and if U denotes the complement of V with respect to $[1,\infty)$, then it is easy to see that U has infinite logarithmic measure, and for $A(z)=z^m \sin^p (z^q)$, the minimum modulus $\delta(r)$ of A satisfies $\delta(r)\ge r^mK_\varepsilon^p$ for r in U. Since $\sigma(A)=q$, the conclusion (8.5) now follows from Theorem 8 if $m>2(q-1)$.

Proof of Corollary 9. This is similar to the proof above, and we omit it.

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Department of Mathematics
University of Illinois
Urbana, Illinois 61801
U.S.A.

and

Department of Mathematics
University of Joensuu
SF-80101 Joensuu 10
Finland

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