Holomorphic equivalence and proper mapping of bounded Reinhardt domains not containing the origin.

Autor(en): **Barrett, David E.**

Objekttyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **59 (1984)**

PDF erstellt am: **22.07.2024**

Persistenter Link: https://doi.org/10.5169/seals-45409

Comment. Math. Helvetici 59 (1984) 550–564 0010-2571/84/040550-15$01.50+0.20/0

Holomorphic equivalence and proper mapping of bounded Reinhardt domains not containing the origin

DAVID E. BARRETT[1]

§1. Introduction

A bounded domain in $\mathbb{C}^n$ is said to be a Reinhardt domain if it is invariant under the action of the n-torus $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$ on $\mathbb{C}^n$ given by $(\theta_1, \ldots, \theta_n) \cdot (z_1, \ldots, z_n) = (e^{\sqrt{-1}\theta_1} z_1, \ldots, e^{\sqrt{-1}\theta_n} z_n)$. In this paper we shall study bounded Reinhardt domains D satisfying the condition

$$\begin{array}{l} \text{there is an integer } k,\ 0 \le k \le n, \text{ such that } D \cap \{z_j = 0\} \ne \emptyset \text{ for } j = 1, \ldots, k \\ \text{and } \bar{D} \cap \{z_j = 0\} = \emptyset \text{ for } j = k+1, \ldots, n. \end{array} \qquad (*)$$

The integer k above will sometimes be denoted $k(D)$.

All domains in this paper are assumed to be connected unless otherwise specified.

First we shall study the holomorphic equivalence problem for such domains. The result is summarized in the following theorem:

THEOREM 1. *If D_1 and D_2 are holomorphically equivalent bounded Reinhardt domains in $\mathbb{C}^n$ satisfying $(*)$ then $k(D_1) = k(D_2)$ and there is a biholomorphic map $F: D_1 \to D_2$ of the form*

$$F(z', z'') = (c_1 z''^{\beta_1} z_{\sigma(1)}, \ldots, c_k z''^{\beta_k} z_{\sigma(k)}, c_{k+1} z''^{\alpha_{k+1}}, \ldots, c_n z''^{\alpha_n}),$$

where

$k = k(D_1) = k(D_2)$,

$z' = (z_1, \ldots, z_k), \qquad z'' = (z_{k+1}, \ldots, z_n)$,

$\beta_j \in \mathbb{Z}^{n-k}$ *for* $j = 1, \ldots, k$,

$$\begin{pmatrix} \alpha_{k+1} \\ \vdots \\ \alpha_n \end{pmatrix} \in GL(n-k, \mathbb{Z}),$$

σ is a permutation of $\{1, \ldots, k\}$, and $c_1, \ldots, c_n$ are positive constants.

[1] Supported by NSF Grant MCS-8211330.

The maps described in the theorem consist precisely of the rational monomial maps F which map D_1 biholomorphically onto the bounded Reinhardt domain $F(D_1)$. It is not hard to see that D_1 can be chosen so that the domains $F(D_1)$ are distinct for distinct choices of F. This shows that the list of maps in the theorem cannot be shortened.

In the case of bounded Reinhardt domains containing the origin the conclusion of Theorem 1 is due to Sunada [7]. (For $n=2$ the result goes back to Thullen [8].) In the other extreme where $k(D_1)=k(D_2)=0$ the result may be viewed as a special case of rigidity results due to Bedford [1]; in this case all biholomorphic maps are of the type described in the theorem up to rotations in $\mathbb{T}^n$.

In the course of the proof of Theorem 1 we learn many things about the automorphism groups of the domains in question, but the proof does not yield the complete description of the automorphism groups furnished by Sunada in the case of bounded Reinhardt domains containing the origin [7].

In the case of proper maps we prove the following qualitative result.

THEOREM 2. *If $F: D_1 \to D_2$ is a proper holomorphic map of bounded Reinhardt domains in $\mathbb{C}^n$ and if D_1 satisfies* (*) *then F extends holomorphically to a neighborhood of $\bar{D}_1$.*

In the case where D_1 and D_2 are complete this result was proved by Bell in [2]. In the case where $k(D_1)=k(D_2)=0$, F is biholomorphic, and all boundary points of D_1 and D_2 are simple this result was proved by Kaup [4]. (This last result is of course subsumed by the more quantitative conclusions of Bedford mentioned above.)

Kaup's example $F:\{(z_1, z_2): |z_1|<|z_2|<1\} \to \{(z_1, z_2): |z_1|<1, 0<|z_2|<1\}$, $F(z_1, z_2)=(z_1/z_2, z_2)$ shows that the hypothesis (*) cannot be dropped altogether from the statement of Theorem 2.

The annulus $\{z: r_1<|z|<r_2\}$ can be mapped properly onto the unit disk in $\mathbb{C}$ by the use of elliptic functions. This example shows that a precise classification of the proper maps in Theorem 2 will necessarily be more involved than the corresponding result on biholomorphic maps.

Theorems 1 and 2 are proved in sections 2 and 3 below, respectively.

Background material used in this article and not otherwise cited may be found in [6] and section 1.4 of [5].

§2. Holomorphic equivalence

This section contains the proof of Theorem 1.

For any bounded Reinhardt domain D in $\mathbb{C}^n$ satisfying (*) let A_D denote the set of points in D of the form $(0', z'')$ and let $\tilde{D}$ denote the envelope of holomorphy of D. $\tilde{D}$ is also a bounded Reinhardt domain in $\mathbb{C}^n$ and is known to be logarithmically convex (meaning that the set $\{(x_1, \ldots, x_n) \in \mathbb{R}^n : (e^{x_1}, \ldots, e^{x_n}) \in \tilde{D}\}$ is convex) and z'-complete (meaning that $(z_1, \ldots, z_n) \in \tilde{D}$ and $|w_1| \leq |z_1|, \ldots, |w_k| \leq |z_k|$ imply $(w_1, \ldots, w_k, z_{k+1}, \ldots, z_n) \in \tilde{D}$). $\tilde{D}$ also satisfies (*) with $k(\tilde{D}) = k(D)$.

Let Aut (D) denote the group of automorphisms of D and let $\text{Aut}_0(D)$ denote the identity component of Aut (D). By a theorem of H. Cartan Aut (D) is a finite-dimensional Lie group and $\text{Aut}_0(D)$ is generated by the Lie algebra $\mathfrak{G}(D)$ of complete (real) vector fields on D which are holomorphic as maps into $\mathbb{C}^n$. (It is helpful to note that vector fields of this type may be written in the form $2 \operatorname{Re} X$, where X is a uniquely determined complex vector field of type $(1,0)$ with holomorphic coefficients. The identity $2 \operatorname{Re}[X, Y] = [2 \operatorname{Re} X, 2 \operatorname{Re} Y]$ for holomorphic vector fields X and Y of type $(1,0)$ is useful in this context.) Furthermore, $\text{Aut}_0(D)$ acts naturally and smoothly on $\tilde{D}$.

Our first step in the proof of Theorem 1 will be to study the invariance properties of $A_{\tilde{D}}$. Let D be a bounded Reinhardt domain in $\mathbb{C}^n$ satisfying (*).

LEMMA 1. *For each* $g \in \text{Aut}_0(D)$ *there are real numbers* $\theta_{k(D)+1}, \ldots, \theta_n$ *such that* $(w', w'') = g(0', z'')$ *satisfies* $w_j = e^{\sqrt{-1}\theta_j} z_j$ *for each* $(0', z'') \in A_{\tilde{D}}$, $j = k(D)+1, \ldots, n$.

Proof. Let $p: \tilde{D} \to A_{\tilde{D}}$ be the canonical projection. Then $p \circ g: A_{\tilde{D}} \to A_{\tilde{D}}$ is holomorphic and homotopic to the identity. But Theorem 3 below shows that any such map is given by a rotation in $\mathbb{T}^{n-k}$. This proves the lemma.

THEOREM 3. *Let* Ω_1 *and* Ω_2 *be pseudoconvex Reinhardt domains compactly contained in* $(\mathbb{C}^*)^d$. *Suppose that there are holomorphic maps* $F: \Omega_1 \to \Omega_2$ *and* $G: \Omega_2 \to \Omega_1$ *such that* $G_* F_*: H_1(\Omega_1, \mathbb{R}) \to H_1(\Omega_1, \mathbb{R})$ *is the identity map. Then* F *and* G *are biholomorphic maps of the form* $(z_1, \ldots, z_d) \mapsto (c_1 z^{\alpha_1}, \ldots, c_d z^{\alpha_d})$ *where*

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_d \end{pmatrix} \in GL(d, \mathbb{Z})$$

and $c_1, \ldots, c_d \in \mathbb{C}^*$.

Proof. This is a special case of Theorem 2 in Bedford [1].

LEMMA 2. $M_D = \{g(z) : g \in \text{Aut}_0(D), z \in A_{\tilde{D}}\}$ *is a closed analytic subset of* $\tilde{D}$.

Proof. Let Φ be the map $\Phi:\mathrm{Aut}_0(D)\times A_{\tilde D}\to \tilde D$, $\Phi(g, p)=g\cdot p$, so that M_D is the image of Φ. If we can show that Φ is proper and has constant rank, and that the image of the tangent map Φ' is complex at every point, then M_D will be a properly immersed complex submanifold of $\tilde D$ and the lemma will be proved.

Let Ψ be the map $\Psi:\mathrm{Aut}(\tilde D)\times \tilde D\to \tilde D\times\tilde D$, $\Psi(g, z)=(g\cdot z, z)$. H. Cartan proved that $\mathrm{Aut}(\tilde D)$ acts properly on $\tilde D$, which means precisely that Ψ is proper. Let $p_1:\tilde D\times\tilde D\to\tilde D$ be the projection onto the first factor. Then $\Phi=p_1\circ(\Psi\mid \mathrm{Aut}_0(D)\times A_{\tilde D})$, and Lemma 1 shows that p_1 is proper on $\Psi(\mathrm{Aut}_0(D)\times A_{\tilde D})$. Hence Φ is proper.

Now the image of the tangent map $\Phi'_{(g,z)}$ is the subspace $g_*(V_z)$ of $T_{g\cdot z}\tilde D$, where V_z is the real span of $\{X_z: X\in\mathfrak{G}(D)\}\cup(\{0'\}\times\mathbb{C}^{n-k(D)})$. Since g_* is a $\mathbb{C}$-linear isomorphism of $T_z\tilde D$ onto $T_{g\cdot z}\tilde D$, we will be done if we show that V_z is closed under the complex structure tensor J and has dimension independent of $z\in A_{\tilde D}$.

Let R be the vector field

$$2\,\mathrm{Re}\sqrt{-1}\left(z_1\frac{\partial}{\partial z_1}+\cdots+z_k\frac{\partial}{\partial z_k}\right)$$

which generates the rotations $(z', z'')\mapsto(e^{\sqrt{-1}\theta}z', z'')$, $\theta\in\mathbb{R}$. Let

$$X=2\,\mathrm{Re}\sum_{j=1}^{n} f_j(z)\frac{\partial}{\partial z_j}\in\mathfrak{G}(D).$$

Then $[R, X]\in\mathfrak{G}(D)$ and

$$[R, X]_z=-2\,\mathrm{Re}\sqrt{-1}\sum_{j=1}^{k(D)} f_j(z)\frac{\partial}{\partial z_j}\equiv -JX_z \bmod \{0'\}\times\mathbb{C}^{n-k(D)}$$

since $z'=0'$. Thus V_z is closed under J.

V_z is now seen to be the *complex* span at z of a family of holomorphic vector fields. It follows that the set S of $z\in A_{\tilde D}$ where $\dim V_z$ fails to be maximal is a proper analytic subset of $A_{\tilde D}$. But S is clearly invariant under the action of $\mathbb{T}^{n-k(D)}$ on $A_{\tilde D}$, hence S must be empty. Thus $\dim V_z$ is constant on all of $A_{\tilde D}$. This completes the proof of the lemma.

It is not hard to show that M_D is actually a manifold.

In the case $k(D)=n$ Lemma 2 is a special case of a result of Kaup [4].

LEMMA 3. *Every* $\mathrm{Aut}_0(D)$*-invariant closed analytic subset of* $\tilde D$ *contains* M_D.

Proof. If M is such a subset and $(z', z'') \in M$ then M contains all points of the form $(e^{\sqrt{-1}\theta}z', z'')$. Since $\tilde{D}$ is z'-complete it follows that M contains $(0', z'') \in A_{\tilde{D}}$. Hence M contains the real $(n-k(D))$-torus in $A_{\tilde{D}}$ obtained by rotating $(0', z'')$. But this implies that $A_{\tilde{D}} \subset M$, which implies the lemma.

Now let $F: D_1 \to D_2$ be a bihomomorphic map of bounded Reinhardt domains in $\mathbb{C}^n$ satisfying $(*)$. F extends to a biholomorphism of $\tilde{D}_1$ onto $\tilde{D}_2$ which will also be called F. Since $\tilde{D}_\mu$ retracts onto $A_{\tilde{D}_\mu}$, and $A_{\tilde{D}_\mu}$, being logarithmically convex, is diffeomorphic to the product of an $(n-k(D_\mu))$-dimensional torus with an open $(n-k(D_\mu))$-cell $(\mu = 1, 2)$, it follows that $n - k(D_1) = \dim H_1(\tilde{D}_1, \mathbb{R}) = \dim H_1(\tilde{D}_2, \mathbb{R}) = n - k(D_2)$; let $k = k(D_1) = k(D_2)$.

By Lemmas 2 and 3 $F(M_{D_1}) = \{g(z): g \in \operatorname{Aut}_0(D_2), z \in F(A_{\tilde{D}_1})\}$ contains $A_{\tilde{D}_2}$. Hence after composition with an automorphism of D_2 we may assume that F maps a point $(0', a'') \in A_{\tilde{D}_1}$ to a point $(0', b'') \in A_{\tilde{D}_2}$.

Let K_1 be the isotropy subgroup of $(0', a'')$ in $\operatorname{Aut}_0(D_1)$ and let K_2 be the isotropy subgroup of $(0', b'')$ in $\operatorname{Aut}_0(D_2)$. Then K_1 and K_2 are compact and $F^*K_2 = K_1$.

Let $\mathcal{O}^{K_\mu}$ be the subspace of $\mathcal{O}(D_\mu)$ consisting of functions f satisfying $f \circ g = f$ for all $g \in K_\mu$ $(\mu = 1, 2)$; clearly $F^*\mathcal{O}^{K_2} = \mathcal{O}^{K_1}$. Let $d\sigma_\mu$ denote the Haar probability measure on K_μ. Then for each $f \in \mathcal{O}(D_\mu)$ we can form a function $f^{K_\mu}: z \mapsto \int_{K_\mu} f(g(z))\, d\sigma_\mu(g)$ in $\mathcal{O}^{K_\mu}$, and $f \in \mathcal{O}^{K_\mu}$ if and only if $f = f^{K_\mu}$.

Let $p: \tilde{D}_\mu \to A_{\tilde{D}_\mu}$ be the canonical projection.

LEMMA 4.

$\mathcal{O}^{K_\mu} = p^*\mathcal{O}(A_{\tilde{D}_\mu})$ *for* $\mu = 1, 2$.

Proof. K_μ contains $\mathbb{T}^k$ acting on the z'-variables, so the mean value property shows that $f(z', z'') = f(0', z'')$ for $f \in \mathcal{O}^{K_\mu}$; thus $\mathcal{O}^{K_\mu} \subset p^*\mathcal{O}(A_{\tilde{D}_\mu})$.

Now let $f \in p^*\mathcal{O}(A_{\tilde{D}_\mu})$; we must show that $f^{K_\mu} = f$ so that $f \in \mathcal{O}^{K_\mu}$. But $f^{K_\mu} \in p^*\mathcal{O}(A_{\tilde{D}_\mu})$ by the preceding paragraph so that

$$
\begin{aligned}
f^{K_\mu}(z', z'') &= f^{K_\mu}(0', z'') \\
&= \int_{K_\mu} f(g(0', z''))\, d\sigma_\mu(g) \\
&= \int_{K_\mu} f(pg(0', z''))\, d\sigma_\mu(g) \\
&= \int_{K_\mu} f(0', z'')\, d\sigma_\mu(g) \\
&= f(0', z'') \\
&= f(z', z'')
\end{aligned}
$$

since $pg(0', z'') = (0', z'')$ for $g \in K_\mu$ by Lemma 1 and the definition of K_μ. This proves the lemma.

COROLLARY. *$F_j(z) = c_j z''^{\alpha_j}$ $(j = k+1, \dots, n)$ for suitable*

$$\begin{pmatrix} \alpha_{k+1} \\ \vdots \\ \alpha_n \end{pmatrix} \in GL(n-k, \mathbb{Z})$$

and $c_{k+1}, \dots, c_n \in \mathbb{C}^*$.

Proof. $F^* p^* \mathcal{O}(A_{\tilde{D}_2}) = p^* \mathcal{O}(A_{\tilde{D}_1})$ by Lemma 4 and the definitions of K_1 and K_2, so each coordinate function F_j depends only on $z_{k+1}, \dots, z_n$ $(j = k+1, \dots, n)$. Thus $p \circ F$ maps $A_{\tilde{D}_1}$ biholomorphically onto $A_{\tilde{D}_2}$ with inverse given by $p \circ F^{-1}$, and the conclusion now follows from Theorem 3.

By composing F with a rotation in $\mathbb{T}^{n-k}$ we may assume that $c_{k+1}, \dots, c_n$ are positive.

Let H be the map

$$H(z', z'') = (z', c_{k+1} z''^{\alpha_{k+1}}, \dots, c_n z''^{\alpha_n}) \tag{1}$$

and let $D_3 = H^{-1}(D_2)$. Then D_3 also satisfies $(*)$ and it suffices now to study $G = H^{-1} \circ F : D_1 \to D_3$ which is of the form $G(z', z'') = (G'(z', z''), z'')$.

Note that $A_{D_3} = A_{D_1}$ and $A_{\tilde{D}_3} = A_{\tilde{D}_1}$.

Next we shift our point of view somewhat to study G using the Bergman kernel function.

LEMMA 5. *For any Reinhardt domain D in $\mathbb{C}^n$ let $S_D = \{\gamma \in \mathbb{Z}^n : z^\gamma \in \mathcal{O}(D), \|z^\gamma\|_{L^2(D)} < \infty\}$. Then the functions z^γ, $\gamma \in S_D$ form a complete orthogonal set in the Bergman space $L^2(D) \cap \mathcal{O}(D)$ so that the Bergman kernel function for D is of the form*

$$K_D(z, w) = \sum_{\gamma \in S_D} c_\gamma z^\gamma \bar{w}^\gamma$$

with $c_\gamma > 0$ for each $\gamma \in S_D$.

Proof. By [6, p. 13] any $f \in \mathcal{O}(D)$ has a representation $f(z) = \sum a_\gamma z^\gamma$ converging uniformly on compact subsets of D. (The sum ranges over $\gamma \in \mathbb{Z}^n$ for which $z^\gamma \in \mathcal{O}(D)$.) Our conclusions follow from the formula

$$\|f\|^2_{L^2(D)} = \sum |a_\gamma|^2 \|z^\gamma\|^2_{L^2(D)}$$

which is justified by integrating over Reinhardt domains compactly contained in D using the well-known orthogonality of the z^γs and passing to the limit.

As a consequence of Lemma 5 the Bergman kernel function K_μ for D_μ may be written in the form

$$K_\mu(z, w) = \sum_{(\alpha,\beta)\in \mathbb{N}^k \times \mathbb{Z}^{n-k}} c^\mu_{\alpha\beta} z'^\alpha \overline{w'^\alpha} z''^\beta \overline{w''^\beta}$$

with all $c^\mu_{\alpha\beta} > 0$, $\mu = 1, 3$. The transformation law for the Bergman kernel function applied to the map G thus yields

$$\sum_{(\alpha,\beta)\in \mathbb{N}^k \times \mathbb{Z}^{n-k}} c^1_{\alpha\beta} z'^\alpha \overline{w'^\alpha} z''^\beta \overline{w''^\beta} = u(z)\overline{u(w)} \sum_{(\alpha,\beta)\in \mathbb{N}^k \times \mathbb{Z}^{n-k}} c^3_{\alpha\beta} G'(z)^\alpha \overline{G'(w)^\alpha} z''^\beta \overline{w''^\beta} \tag{2}$$

for all $z, w \in \tilde{D}_1$, where u is the Jacobian determinant of the map G.

In (2) let us take $w = (0', a'')$. Then $G'(w) = 0'$ so that all terms on both sides with $\alpha \neq 0$ vanish. Hence $u(z)$ must be independent of z'.

Next let $z = (z', z'')$ and $w = (0', z'')$. Then the left-hand side of (2) is independent of z', so the sum on the right-hand side must also be independent of z'. Since the map $z' \mapsto G'(z', z'')$ is a diffeomorphism for fixed z'', the coefficient

$$\overline{G'(0', z'')^\alpha} \sum_{\beta \in \mathbb{Z}^{n-k}} c^3_{\alpha\beta} z''^\beta \overline{z''^\beta}$$

of $G'(z', z'')^\alpha$ must vanish for $\alpha \neq 0$. Hence $G'(0', z'') = 0'$.

Thus, for any z''_0, G maps the k-dimensional Reinhardt domain $\{z \in \tilde{D}_1 : z'' = z''_0\}$ biholomorphically onto the Reinhardt domain $\{z \in \tilde{D}_3 : z'' = z''_0\}$ preserving the point $(0', z''_0)$. Then by H. Cartan's theorem on biholomorphic maps of circular domains G must be linear in z'. Thus we have proved

LEMMA 6. *G is of the form $(z', z'') \mapsto (\Lambda_{z''}(z'), z'')$ where $\Lambda_{z''}$ is a linear map (or $k \times k$ matrix) varying holomorphically with z''.*

Let $C^1_{z''}$ and $C^3_{z''}$ be the positive diagonal $k \times k$ matrices with jth diagonal entries given by

$$\sum_{\beta \in \mathbb{Z}^{n-k}} c^1_{(j)\beta} z''^\beta \overline{z''^\beta}$$

and

$$|u(z'')|^2 \sum_{\beta \in \mathbb{Z}^{n-k}} c^3_{(j)\beta} z''^{\beta} \overline{z''^{\beta}}$$

respectively. (Here (j) is the multi-index for which $z'^{(j)} = z_j$.)

Take $w'' = z''$ in (2) and equate the terms which are linear in z' and w' to get

$$C^1_{z''} = {}^t\bar{\Lambda}_{z''} C^3_{z''} \Lambda_{z''}. \tag{3}$$

For simplicity of notation we will often drop the subscripted z'' in the sequel.

Let ∂ and $\bar{\partial}$ denote the usual holomorphic and anti-holomorphic differentials with respect to z'' acting on functions (and forms) with values in the space of $k \times k$ complex matrices. Then differentiation of (3) yields

$$\partial C^1 = {}^t\bar{\Lambda} C^3 \partial \Lambda + {}^t\bar{\Lambda}(\partial C^3)\Lambda,$$

hence substituting from (3) we have

$$\begin{aligned} \partial \Lambda &= ({}^t\bar{\Lambda} C^3)^{-1} \partial C^1 - (C^3)^{-1}(\partial C^3)\Lambda \\ &= \Lambda(\partial \log C^1) - (\partial \log C^3)\Lambda. \end{aligned} \tag{4}$$

Differentiating again we have

$$0 = \bar{\partial}\partial\Lambda = \Lambda(\bar{\partial}\partial \log C^1) - (\bar{\partial}\partial \log C^3)\Lambda.$$

Thus if at some point $z''_0 \in A_{\tilde{D}_1}$ the $(1, 1)$-forms in the ith diagonal entry of $\bar{\partial}\partial \log C^1$ and the jth diagonal entry of $\bar{\partial}\partial \log C^3$ are unequal we must have $\Lambda_{ij} = 0$ at z''_0. Since $\bar{\partial}\partial \log C^1$ and $\bar{\partial}\partial \log C^3$ are invariant under the $\mathbb{T}^{n-k}$ action on the z''-variables, the relation $\Lambda_{ij} = 0$ must persist on the real $(n-k)$-torus in $A_{\tilde{D}_1}$ obtained by rotating z''_0. Since Λ_{ij} is a holomorphic function of z'' it follows that Λ_{ij} vanishes for all $z'' \in A_{\tilde{D}_1}$.

Let the set $\{1, \ldots, k\}$ be partitioned into coordinate blocks $\{E^{\mu}_{\nu}\}_{\nu}$ by the equivalence relation $i \equiv_{\mu} j$ if and only if

$$(\bar{\partial}\partial \log C^{\mu})_{ii} = (\bar{\partial}\partial \log C^{\nu})_{jj}$$

for all $z'' \in A_{\tilde{D}_\mu}$ $(\mu = 1, 3)$. Then the calculations above show that after permuting the z'-coordinates of D_3 we may assume that the two partitions above are the same and that Λ maps V_{E_ν} isomorphically onto itself for each ν and for all

$z'' \in A_{\tilde{D}_1}$. (Here the V_{E_ν} are the summands in the direct sum decomposition $\mathbb{C}^k = \bigoplus_\nu V_{E_\nu}$ associated with the partition $\{E_\nu\}_\nu = \{E^1_\nu\}_\nu = \{E^3_\nu\}_\nu$.) Thus it will suffice now to study the behavior of Λ on each V_{E_ν} separately.

LEMMA 7. *$\Lambda_{ij} = \rho_{ij} z''^{\gamma_{ij}}$ for suitable constants $\rho_{ij} \in \mathbb{C}$ and multi-indices $\gamma_{ij} \in \mathbb{Z}^{n-k}$. Furthermore, $\rho_{ij} = 0$ if i and j fail to lie in the same coordinate block of the partition $\{E_\nu\}_\nu$.*

Proof. From the work above it suffices to prove the first statement.

To simplify notation we make the temporary assumption that $E_\nu = \{1, \dots, k\}$ for some ν so that

$$\bar{\partial}\partial \log C^1 = \bar{\partial}\partial \log C^3 = \Phi$$

is a scalar matrix of (1, 1) forms for all $z'' \in A_{\tilde{D}_1}$.

Let W be a positive scalar solution of $\bar{\partial}\partial W = \Phi$ depending only on $|z_{k+1}|, \dots, |z_n|$. (For example, we may take W to be the first diagonal entry of $\log C^1$.) Then $\log C^\mu$ differs from W by a diagonal solution T^μ of $\bar{\partial}\partial T^\mu = 0$ with T^μ depending only on $|z_{k+1}|, \dots, |z_n|$. Let (r_s, θ_s) give polar coordinates for z_s, $s = k+1, \dots, n$. Then in particular we have

$$0 = \frac{\partial^2}{\partial \bar{z}_s\, \partial z_s} T^\mu = \frac{\partial^2}{\partial r_s^2} T^\mu + (r_s)^{-1} \frac{\partial}{\partial r_s} T^\mu$$

so that T^μ is necessarily of the form

$$\Gamma^\mu + \sum_{s=k+1}^{n} 2A^\mu_s \log r_s;$$

the Γ^μs and A^μ_ss are constant diagonal matrices. Let $B^\mu = \exp \Gamma^\mu$ and $U = \exp W$. Then

$$\begin{aligned} C^\mu &= UB^\mu \prod_s r_s^{2A^\mu_s} \\ &= UB^\mu \prod_s z_s^{A^\mu_s} \bar{z}_s^{A^\mu_s}. \end{aligned} \tag{5}$$

(The use of diagonal matrices as exponents above may be interpreted componentwise.)

Putting (5) into (4) we have

$$\begin{aligned}\partial\Lambda &= \Lambda(\partial \log U) + \sum_s z_s^{-1}\Lambda A_s^1\, \partial z_s \\ &\quad - (\partial \log U)\Lambda - \sum_s z_s^{-1} A_s^3 \Lambda\, \partial z_s;\end{aligned}$$

since U is scalar this yields

$$\frac{\partial}{\partial z_s}\Lambda_{ij} = z_s^{-1}\Lambda_{ij}\{(A_s^1)_{jj} - (A_s^3)_{ii}\}$$

so that

$$\begin{aligned}\Lambda_{ij} &= \rho_{ij} \prod_s z_s^{\{(A_s^1)_{jj} - (A_s^3)_{ii}\}} \\ &= \rho_{ij} z^{\prime\prime\gamma_{ij}} \qquad (6)\end{aligned}$$

for suitable constants ρ_{ij}. This proves the lemma.

We note for future reference that (6) shows that $(A_s^1)_{jj} \equiv (A_s^3)_{ii} \bmod \mathbb{Z}$ when $\rho_{ij} \neq 0$. If ρ_{ii} is also non-zero it follows that $(A_s^3)_{ii} \equiv (A_s^3)_{jj} \bmod \mathbb{Z}$.

To complete the proof of Theorem 1, we now show that G can be adjusted by an automorphism of D_3 so that the ρ_{ij}s in Lemma 7 satisfy $\rho_{ij} = 0$ for $i \neq \sigma(j)$, where σ is a permutation of $\{1, \ldots, k\}$. We need the following theorem.

THEOREM 4. *Let Ω be a (possibly disconnected) bounded Reinhardt domain in $\mathbb{C}^k$. Let K be the identity component of the group of linear automorphisms of Ω. Then there is a partition $\{Q_\sigma^\Omega\}_\sigma$ of $\{1, \ldots, k\}$ into coordinate blocks along with a positive diagonal bilinear form C_σ on each $V_{Q_\sigma^\Omega}$ so that K consists precisely of linear maps of the form $\oplus_\sigma L_\sigma$, where $L_\sigma : V_{Q_\sigma^\Omega} \to V_{Q_\sigma^\Omega}$ satisfies*

$${}^t\bar{L}_\sigma C_\sigma L_\sigma = C_\sigma.$$

(Here $\mathbb{C}^k = \oplus_\sigma V_{Q_\sigma^\Omega}$ is the direct sum decomposition associated with the partition $\{Q_\sigma^\Omega\}_\sigma$.)

Proof. This is proved in slightly different form in Sunada [7]. An adapted proof is provided here for the convenience of the reader.

Since Ω has finite volume, $\det L = 1$ for all $L \in K$. Applying Lemma 5 to (a component of) Ω the Bergman Kernel transformation law for the map L reads

$$\sum_{\alpha \in S_\Omega} c_\alpha z^\alpha \bar{w}^\alpha = \sum_{\alpha \in S_\Omega} c_\alpha L(z)^\alpha \overline{L(w)}^\alpha.$$

Equating linear terms we obtain ${}^t\bar{L}CL = C$, where C is the positive diagonal matrix with jth diagonal entry equal to $c_{(j)}$. By performing dilations in each variable separately we may arrange that C is (a scalar multiple of) the identity matrix and thus K is made to be a subgroup of the unitary group $U(k)$.

In the notation of this paper the Lie algebra of $U(k)$ consists of all vector fields of the form $2\,\mathrm{Re} \sum_{i,j} a_{ij} z_i\, \partial/\partial z_j$ with $a_{ij} = -\overline{a_{ji}}$.

Let $\mathfrak{K}$ denote the Lie algebra of K. Since Ω is Reinhardt $\mathfrak{K}$ must contain the vector fields $R_j = 2\,\mathrm{Re}\sqrt{-1}\, z_j(\partial/\partial z_j)$ which generate the rotations in each of the variables. Then for

$$A = 2\,\mathrm{Re} \sum_{i,j} a_{ij} z_i \frac{\partial}{\partial z_j} \in \mathfrak{K}$$

and $i \neq j$ we have

$$2\,\mathrm{Re}\,[R_i, [R_j, A]] = 2\,\mathrm{Re}\left(a_{ij} z_i \frac{\partial}{\partial z_j} + a_{ji} z_j \frac{\partial}{\partial z_i}\right) \in \mathfrak{K}$$

and

$$2\,\mathrm{Re}\,[R_i, [R_i, [R_j, A]]] = 2\,\mathrm{Re}\sqrt{-1}\left(a_{ij} z_i \frac{\partial}{\partial z_j} - a_{ji} z_j \frac{\partial}{\partial z_i}\right) \in \mathfrak{K}.$$

Let $i \sim j$ if $a_{ij} \neq 0$ for some $A \in \mathfrak{K}$. Then the presence of the R_js shows that $\sim$ is reflexive, the relation $a_{ij} = -\overline{a_{ji}}$ shows that $\sim$ is symmetric, and the computation

$$\left[a_{ij} z_i \frac{\partial}{\partial z_j} + a_{ji} z_j \frac{\partial}{\partial z_i},\, b_{js} z_j \frac{\partial}{\partial z_s} + b_{sj} z_s \frac{\partial}{\partial z_j}\right] = a_{ij} b_{js} z_i \frac{\partial}{\partial z_s} - a_{ji} b_{sj} z_s \frac{\partial}{\partial z_i}$$

shows that $\sim$ is transitive.

Let $\{Q_\sigma^\Omega\}_\sigma$ be the partition of $\{1, \ldots, k\}$ induced by $\sim$. Then $K \subset \Pi_\sigma U(|Q_\sigma^\Omega|)$, where each unitary factor $U(|Q_\sigma^\Omega|)$ acts in the usual manner on $V_{Q_\sigma^\Omega}$. But remembering to count the R_js we have found

$$k + \sum_\sigma |Q_\sigma^\Omega|\,(|Q_\sigma^\Omega| - 1) = \dim \Pi_\sigma U(|Q_\sigma^\Omega|)$$

independent vector fields in $\mathfrak{K}$. Hence $K = \Pi_\sigma U(|Q_\sigma^\Omega|)$ (up to the dilations performed earlier) and the theorem is proved.

Now let $\Omega^1(z_0'')=\{z\in D_1: z''=z_0''\}$ and $\Omega^3(z_0'')=\{z\in D_3: z''=z_0''\}$, and let us use Theorem 4 to study the linear map $\Lambda_{z_0}:\Omega^1(z_0'')\to\Omega^3(z_0'')$. Since the isotropy groups of the origin in $\mathrm{Aut}_0(\Omega^1(z_0''))$ and $\mathrm{Aut}_0(\Omega^3(z_0''))$ are related by $\Lambda_{z_0''}$ we may conclude that $\Lambda_{z_0''}$ maps each $V_{Q_\sigma^{\Omega^1(z_0'')}}$ isomorphically onto some $V_{Q_\sigma^{\Omega^3(z_0'')}}$; we shall say that $\Lambda_{z_0''}$ respects the partitions $\{Q_\sigma^{\Omega^1(z_0'')}\}_\sigma$ and $\{Q_\sigma^{\Omega^3(z_0'')}\}_\sigma$. But these partitions are invariant under the $\mathbb{T}^{n-k}$ action on z_0'', so that $\Lambda_{z_0''}$ maps $V_{Q_\sigma^{\Omega^1(z_0'')}}$ onto $V_{Q_\sigma^{\Omega^3(z_0'')}}$ for all z'' on a totally real $(n-k)$-torus, hence indeed for all $z''\in A_{\tilde{D}_1}$.

Let $\{P_\lambda^\mu\}_\lambda$ be the coarsest partition which is finer than each of the partitions $\{Q_\sigma^{\Omega^\mu(z'')}\}_\sigma$ $(z''\in A_{\tilde{D}_\mu})$ and $\{E_\nu^\mu\}_\nu$, $\mu=1,3$. Then each $\Lambda_{z_0''}$ respects the partitions $\{P_\lambda^1\}_\lambda$ and $\{P_\lambda^3\}_\lambda$; after permuting the coordinates we may assume the two partitions coincide. Let $\{P_\lambda\}_\lambda=\{P_\lambda^1\}_\lambda=\{P_\lambda^3\}_\lambda$. Then by Theorem 4 $\mathrm{Aut}_0(D_\mu)$ contains all maps of the form $(z',z'')\mapsto(L_{z''}(z'),z'')$, where (i) $L_{z''}$ is a linear map of z' which depends holomorphically upon z'', and (ii) $L_{z''}$ is of the form $\oplus_\lambda(L_\lambda)_{z''}$, where $(L_\lambda)_{z''}$ maps V_{P_λ} isomorphically onto itself and satisfies

$$ {}^t\overline{(L_\lambda)}_{z''}(C_\lambda^\mu)_{z''}(L_\lambda)_{z''}=(C_\lambda^\mu)_{z''}. $$

Here $(C_\lambda^\mu)_{z''}$ is a positive diagonal bilinear form on $V_{P_\lambda}\times V_{P_\lambda}$ which is independent of $L_{z''}$ and unique up to a positive scalar factor.

Let $\bigoplus_\lambda \Lambda_\lambda$ be the decomposition of Λ associated with the partition $\{P_\lambda\}_\lambda$. Then each Λ_λ satisfies

$$ {}^t\bar{\Lambda}_\lambda C_\lambda^3\Lambda_\lambda=C_\lambda^1 \tag{8} $$

after adjusting each $(C_\lambda^3)_{z''}$ by a scalar. The entries of Λ_λ are given by (6), so we may rewrite (8) in the form

$$ \left(\prod_s \bar{z}_s^{A_{s,\lambda}^1}\right){}^t\bar{\rho}_\lambda C_\lambda^3\left(\prod_s |z_s|^{-2A_{s,\lambda}^3}\right)\rho_\lambda\left(\prod_s z_s^{A_{s,\lambda}^1}\right)=C_\lambda^1, \tag{9} $$

where the $A_{s,\lambda}^\mu$s are constant diagonal matrices and ρ_λ is a constant invertible matrix. By permuting coordinates within the block P_λ we may arrange that the diagonal elements of ρ_λ are non-zero.

Since ${}^t\bar{\rho}_\lambda$ is invertible, (9) shows that

$$ C_\lambda^3\left(\prod_s |z_s|^{-2A_{s,\lambda}^3}\right)\rho_\lambda\left(\prod_s |z_s|^{2A_{s,\lambda}^1}\right)(C_\lambda^1)^{-1} $$

is independent of z''. Since the diagonal elements of ρ_λ are non-zero and all other matrices above are diagonal it follows that

$$ \Delta_\lambda=C_\lambda^3\left(\prod_s |z_s|^{2(A_{s,\lambda}^1-A_{s,\lambda}^3)}\right)(C_\lambda^1)^{-1} $$

is a constant diagonal matrix. Thus by (9) the matrices

$$(L_\lambda)_{z''} = \left(\prod_s z_s^{A^3_{s,\lambda}}\right) \Delta_\lambda^{1/2} \rho_\lambda \left(\prod_s z_s^{-A^3_{s,\lambda}}\right)$$

satisfy (i) and (ii) above and thereby yield an automorphism Φ of D_3. (The exponents occurring in the expansion of the right hand side above are integers by the remark following the proof of Lemma 7.)

Now replace $G: D_1 \to D_2$ by $\tilde{G} = \Phi^{-1} \cdot G$. By inspection $\tilde{G}$ is of the form

$$\tilde{G}(z', z'') = (c_1 z''^{\gamma_1} z_1, \ldots, c_k z''^{\gamma} z_k, z'')$$

for suitable positive constants $c_1, \ldots, c_k$ and multi-indices $\gamma_1, \ldots, \gamma_k \in \mathbb{Z}^{n-k}$.

Finally, $H \circ \tilde{G}$ maps D_1 biholomorphically onto D_2 and by equations (1) and (7) $H \circ \tilde{G}$ is of the form given in the statement of the theorem after adjusting constants and rescrambling coordinates.

This completes the proof of Theorem 1.

§3. Proper mapping

This section contains the proof of Theorem 2.

LEMMA 8. *Let D be a bounded Reinhardt in $\mathbb{C}^n$. Then for each $\alpha \in \mathbb{N}^n$ there is a function $\phi_\alpha \in C_0^\infty(D)$ such that $P\phi_\alpha = z^\alpha$, where P is the Bergman projection from $L^2(D)$ onto $L^2(D) \cap \mathcal{O}(D)$.*

Proof. By Lemma 5 the Bergman kernel function for D is of the form

$$K_D(z, w) = \sum_{\alpha \in S_D} c_\alpha z^\alpha \bar{w}^\alpha$$

with each c_α positive. Let $\phi \in C_0^\infty(D)$ be nonnegative and $\mathbb{T}^n$-invariant with $\int_D \phi \, dv = 1$, and let $\phi_\alpha(z) = (c_\alpha \alpha!)^{-1} \cdot (-\partial/\partial\bar{z})^\alpha \phi$. Then

$$\begin{aligned} P\phi^\alpha(z) &= \int_D \sum_{\beta \in S_D} c_\beta z^\beta \bar{w}^\beta \phi_\alpha(w) \, dv(w) \\ &= \int_D \sum_{\beta \in S_D} c_\beta (c_\alpha \alpha!)^{-1} z^\beta \frac{\partial^\alpha \overline{w^\beta}}{\partial w^\alpha} \phi(w) \, dv(w) \\ &= \int_D z^\alpha \phi(w) \, dv(w) \\ &= z^\alpha \end{aligned}$$

by integration by parts and the mean value property. This proves the lemma.

LEMMA 9. *If D is a bounded Reinhardt domain in $\mathbb{C}^n$ satisfying $(*)$ then the Bergman kernel function $K_D(z, w)$ for D extends holomorphically in z and anti-holomorphically in w to a neighborhood of $\bar{D} \times D$ in $\mathbb{C}^{2n}$.*

Proof. Since K_D is holomorphic in z and $\bar{w}$ K_D extends automatically to $\tilde{D} \times \tilde{D}$, where $\tilde{D}$ is the envelope of holomorphy of D. Furthermore, it follows from Lemma 5 that if $\lambda_1, \ldots, \lambda_n$ are any positive numbers and $\hat{z} = (\lambda_1 z_1, \ldots, \lambda_n z_n)$, $\hat{w} = (\lambda_1^{-1} w_1, \ldots, \lambda_n^{-1} w_n)$ then $K_D(z, w) = K_D(\hat{z}, \hat{w})$ wherever both sides are defined.

Let $z_0 \in b\tilde{D}$. Without loss of generality we may assume that $z_{0j} = 0$ precisely for $1 \leq j \leq r$ $(0 \leq r \leq k(D))$. Let

$$A = \{(z_{r+1}, \ldots, z_n) \in \mathbb{C}^{n-r} : (0, \ldots, 0, z_{r+1}, \ldots, z_n) \in \tilde{D}\}.$$

Then $(z_{r+1}, \ldots, z_n) \in A$ for all $z \in \tilde{D}$ so that $(z_{0r+1}, \ldots, z_{0n}) \in bA$. Hence $\hat{z} \in \tilde{D}$ for suitable choices of $\lambda_1, \ldots, \lambda_n$ close to 1. Fix any point $w_0 \in \tilde{D}$. Then we may choose connected neighborhoods U_{z_0} of z_0 and U_{w_0} of w_0 and positive numbers $\lambda_1, \ldots, \lambda_n$ so that $\hat{z}$ and $\hat{w}$ are in $\tilde{D}$ for all $z \in U_{z_0}$, $w \in U_{w_0}$; we may define $K_D(z, w) = K_D(\hat{z}, \hat{w})$ for such z and w. Since these extensions are all based on the same series expansion they patch together in a consistent way to provide the desired global extension. This proves the lemma.

LEMMA 10. *P_D maps $C_0^0(D)$ into $\mathcal{O}(\bar{D})$.*

Proof. This is an immediate consequence of Lemma 9 and the formula

$$P_D f(z) = \int_D K_D(z, w) f(w) \, dv(w).$$

We are now ready to prove Theorem 2. Let $F^\alpha = F^* z^\alpha$ and $u = \det F'$. Then by the transformation law for the Bergman projection under proper mapping [2] we have

$$uF^\alpha = u \cdot P_{D_2} \phi_\alpha \circ F = P_{D_1}(u \cdot \phi_\alpha \circ F) \in \mathcal{O}(\bar{D})$$

using Lemma 8 and 10. This and the fact that $\mathcal{O}(\bar{D})$ is a unique factorization domain together imply that each coordinate function of F is in $\mathcal{O}(\bar{D})$. (For details of this argument see Bell [3].)

This completes the proof of Theorem 2.

REFERENCES

[1] E. Bedford, *Holomorphic mapping of products of annuli in* $\mathbb{C}^n$, Pacific J. Math. *87* (1980), 271–281.

[2] S. Bell, *The Bergman kernel function and proper holomorphic mappings*, Trans. Amer. Math. Soc. *270* (1982), 685–691.

[3] S. Bell, *Analytic hypoellipticity of the* $\bar{\partial}$*-Neumann problem and extendability of holomorphic mappings*, Acta. Math. *147* (1981), 109–116.

[4] W Kaup, *Über das Randverhalten von holomorphen Automorphismen beschränkter Gebiete*, Manuscripta Math. *3* (1970), 257–270.

[5] S. Krantz, *Function Theory of Several Complex Variables*, John Wiley, New York, 1982.

[6] R. Narasimhan, *Several Complex Variables*, University of Chicago Press, Chicago, 1971.

[7] T. Sunada, *Holomorphic equivalence problem for bounded Reinhardt domains*, Math. Ann. *235* (1978), 111–128.

[8] P. Thullen, *Zu den Abbildungen durch analytische Funktionen mehrerer Veränderlichen*, Math. Ann. *104* (1931), 244–259.

Department of Mathematics
Princeton University
Princeton, NJ 08544

Received July 11, 1983