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Cut locus and parallel circles of a closed curve on a Riemannian plane admitting total curvature

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0. Introduction

A Riemannian plane is a complete Riemannian manifold homeomorphic to a plane R^2. The total curvature $c(M)$ of a Riemannian plane M is defined to be an improper integral $\int_M G\,dv$ of the Gaussian curvature G with respect to the volume element dv of M. A well known theorem due to Cohn–Vossen [1] states that if such an M admits total curvature, then $-\infty \leqq c(M) \leqq 2\pi$. Since total curvature is not a topological invariant but depends on the choice of Riemannian metric, it is natural to ask what is the geometric significance of total curvature. Indeed it was proved in [4] and in [5] that the existence of total curvature on a Riemannian plane imposes strong restrictions to the mass of rays emanating from an arbitrary fixed point. Moreover, for a finitely connected Riemannian 2-manifold admitting total curvature the behavior of Busemann functions on it is in some sense controlled by total curvature (see [6], [7]).

The purpose of this paper is to investigate certain restrictions of the existence of total curvature on M to the distance function d induced from the Riemannian metric. Throughout this paper let M be a Riemannian plane admitting total curvature, let $\mathfrak{C}$ be a simply closed regular smooth curve on M and let M_1 be the complement of the open disk bounded by $\mathfrak{C}$. The cut locus $C(\mathfrak{C})$ of $\mathfrak{C}$ in M_1 is discussed throughout. Geodesics are parametrized by arclengths. For a point x on M_1 a minimizing geodesic $\sigma:[0, a] \to M$ is called a *segment from x to $\mathfrak{C}$* if $\sigma(0) = x$, $\sigma(a) \in \mathfrak{C}$ and the length $L(\sigma)$ of σ is $d(x, \mathfrak{C})$. Let $\rho: M_1 \to R$ be the distance function to $\mathfrak{C}$. If $\mathfrak{C}$ is a geodesic circle around a fixed point p with radius less than the injectivity radius of the exponential map $\exp_p: M_p \to M$ at p, then ρ is the distance function to p. A *cut point x to $\mathfrak{C}$* along a segment σ is by definition a point with the property that any extension of σ beyond x is not a segment to $\mathfrak{C}$. Let $L = L(\mathfrak{C})$, let $s \in [0, L]$ be the arclength parameter of $\mathfrak{C}$, and let N be the unit

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[1] Dedicated to Professor W. Klingenberg on his 60th birthday.

outer normal field to $\mathfrak{C}$. A point x on M_1 is a *focal* point to $\mathfrak{C}$ if x is a critical value of the normal exponential map $(s, t) \to \exp_{\mathfrak{C}(s)} tN(s)$, $s \in [0, L]$, $t \geqq 0$. For each $t \geqq 0$ set $S(t) := \{x \in M_1; \rho(x) = t\}$ and $B(t) := \{x \in M_1; \rho(x) < t\}$. $S(t)$ is a smooth curve for all sufficiently small t such that it is contained inside the cut locus of $\mathfrak{C}$.

A complete description on the cut locus and focal locus of $\mathfrak{C}$ on a Riemannian plane which is not assumed to admit total curvature was established by Hartman [3], and his results are the C^∞ extension of Fiala's results in which Fiala assumed that a closed curve and Riemannian metric are analytic. Here, the same notations as used in [2] and [3] will be employed. A cut point x to $\mathfrak{C}$ is called *normal* if there exist exactly two distinct segments from x to $\mathfrak{C}$ such that x is not a focal point to $\mathfrak{C}$ along any of the two segments. A cut point x to $\mathfrak{C}$ is called *anormal* if it is not normal. A number $t > 0$ is called *anormal* if there is an anormal point on $S(t)$. A number $t > 0$ is called *exceptional* if it is either anormal or if it is not anormal and there is a normal point on $S(t)$ at which the angle between the two vectors tangent to the segments to $\mathfrak{C}$ is π. It was proved by Hartman [3] that the set of all exceptional values is closed and of measure zero on $[0, \infty)$, and that if $t > 0$ is not exceptional, then $S(t)$ intersects $C(\mathfrak{C})$ only at finitely many points and $S(t)$ is a piecewise smooth curve, where the smoothness breaks at points of the intersection $C(\mathfrak{C}) \cap S(t)$. The continuity of the length $L(t)$ of $S(t)$ will break at each exceptional value t where there exists a normal point on $S(t)$ at which the angle between the two vectors tangent to the segments to $\mathfrak{C}$ is π. The cut locus forms a smooth curve in a small neighborhood of each normal cut point, and the curve bisects the two segments to $\mathfrak{C}$.

Under the assumption that M admits total curvature, more precise observations on the cut locus of $\mathfrak{C}$ will be provided. The following notions play an important role in our discussion on $C(\mathfrak{C})$. For each $x \in M_1$ at which there are more than one segment to $\mathfrak{C}$, let $E(x) \subset M_1$ be the maximal compact set bounded by a subarc of $\mathfrak{C}$ and two segments from x to $\mathfrak{C}$ such that every segment from x to $\mathfrak{C}$ lies on $E(x)$ and that $E(x)$ is homeomorphic to a closed 2-disk. The segments lying in the boundary of $E(x)$ will be denoted by σ_x^-, $\sigma_x^+ : [0, \rho(x)] \to M_1$. If there is a unique segment σ_x from x to $\mathfrak{C}$, then $E(x)$ consists of the points on the segment. Let $\beta : M_1 \to [0, 2\pi)$ be defined as the angle between $\dot\sigma_x^-(0)$ and $\dot\sigma_x^+(0)$ measured with respect to $E(x)$. Then $\beta(x) = 0$ if and only if there is a unique segment from x to $\mathfrak{C}$. A geodesic $\gamma : [0, \infty) \to M$ is called a *ray from* $\mathfrak{C}$ if $\gamma(0) \in \mathfrak{C}$ and $\rho(\gamma(t)) = t$ for all $t \geqq 0$. It is elementary that there exists at least a ray from $\mathfrak{C}$. Let $F \subset M_1$ be the set of all points on rays from $\mathfrak{C}$. Then $F \neq \emptyset$ and is closed since the limit of a converging sequence of rays from $\mathfrak{C}$ is a ray from $\mathfrak{C}$. The complement $M_1 - F$ consists of a countable disjoint union $\bigcup_{\lambda \in \Lambda} D_\lambda$ of relatively open sets in M_1, each D_λ is connected, noncompact and bounded by two (possibly one ray, if there is a unique ray from $\mathfrak{C}$) rays from $\mathfrak{C}$ and an open subarc $\mathfrak{C}_\lambda$ of $\mathfrak{C}$, where Λ is an index set. Each component of $C(\mathfrak{C})$ is contained in some D_λ and each D_λ contains a component of $C(\mathfrak{C})$. Our first observation on $C(\mathfrak{C})$ is the following

THEOREM A. *Let M be a Riemannian plane admitting total curvature and let $\mathfrak{C}$ be a simply closed smooth regular curve on M. Assume that $M_1 - F \neq \emptyset$. Then there exists for each $\lambda \in \Lambda$ a number t_λ and a curve $x:[t_\lambda, \infty) \to D_\lambda \cap C(\mathfrak{C})$ with the properties:*

(1) *$\rho(x(t)) = t$ for $t \geqq t_\lambda$.*

(2) *$x(t)$ is smooth except at a set of measure zero in $[t_\lambda, \infty)$.*

(3) *$\{E(x(t))\}$ is monotone increasing and $\lim_{t\to\infty} E(x(t)) = D_\lambda$.*

(4) *$\lim_{t\to\infty} \beta(x(t)) = 0$, and in particular, if I_λ is the domain of $\mathfrak{C}_\lambda$ and $\kappa(s)$ is the geodesic curvature of $\mathfrak{C}$ at $\mathfrak{C}(s)$ with respect to N, then*

$$c(D_\lambda) = -\int_{I_\lambda} \kappa(s)\, ds.$$

The statement (2) was already established by Hartman (see Proposition 5.6, p. 713, [3]). Note that the constant t_λ depends on a small number taken less than half of the length of $\mathfrak{C}_\lambda$. The starting point of x is not clearly stated because there are many (possibly infinitely many if t_λ is exceptional) curves with these properties. A special choice will be made from a geometric viewpoint as stated in Lemma 1.1, (2). Roughly speaking x may be considered as a "main street" of cut locus in D_λ. Note also that $\{t_\lambda\}$ contains a divergent subsequence.

THEOREM B. *Let M be a Riemannian plane admitting total curvature and let $\mathfrak{C}$ be a simply closed smooth regular curve on M. Then there exist constants $R_1 \leqq R_2$ with the following properties:*

(1) *If $t > R_1$, then $S(t)$ is arcwise connected.*

(2) *If $t > R_2$, then $S(t)$ is homeomorphic to a circle.*

The discontinuity of $L(t)$ will occur at t when there are points on $S(t)$ at each point of which there are exactly two segments to $\mathfrak{C}$ making an angle π. Such a point may still allow to exist on $S(t)$ for $t > R_1$. But there exists no such point on $S(t)$ if $t > R_2$. It is a natural consequence of Theorem B that there exists a constant $R_3 \geqq R_2$ such that $\rho(x) < \pi$ for all $x \in M_1$ with $\rho(x) > R_3$. $L(t)$ is continuous for all $t > R_3$. Note also that if the metric and $\mathfrak{C}$ are analytic, then $L(t)$ is continuous (see Théorèm 1, p. 326, [2]).

Now the function β will be discussed. It follows from Proposition 6.1 in [3] that $\beta(x(t))$ is smooth if $x(t)$ is a normal point and that it is not necessarily continuous if $x(t)$ is anormal. Whatever t is exceptional or non-exceptional, β has the following uniform property and this property plays an essential role for the estimate of the derivatives of $L(t)$ for all large normal values.

THEOREM C. *Let M be a Riemannian plane admitting total curvature and let $\mathfrak{C}$ be a simply closed smooth regular curve on M. For an arbitrary given positive ε, there exists a $t(\varepsilon)$ such that if $t > t(\varepsilon)$, then*

$$\sum_{x \in S(t)} \beta(x) < \varepsilon.$$

Note that when t is a non-exceptional value, the left hand side in the above inequality is a finite sum, and when t is exceptional, it is a countable sum. In fact, let t be exceptional and let $\{x_\mu;\ \mu \in \Lambda(t),\ x_\mu \in S(t),\ \beta(x_\mu) > 0\}$, where $\Lambda(t)$ is an index set. For each $\mu \in \Lambda(t)$ if J_μ is the closed interval corresponding to the subarc $E(x_\mu) \cap \mathfrak{C}$ of $\mathfrak{C}$, then $J_\mu \cap J_{\mu'} = \emptyset$ for $\mu \neq \mu'$ and $\sum_{\mu \in \Lambda(t)} \text{meas}\,(J_\mu) \leqq L$. This means that $\Lambda(t)$ is at most countable. The proof of Theorem C requires topological properties of $S(t)$ as stated in Theorem B.

In the final section a sharp estimate for the derivatives of $L(t)$ at non-exceptional values greater than R_3 will be obtained. And the following Theorem D will be established as a direct consequence of the estimate. The formula for the derivatives of $L(t)$ at a non-exceptional value was first established by Fiala [2] in the analytic case and later by Hartman [3] in the smooth case. They are essentially the same. Under the assumption that M has total curvature an essential improvement for the derivatives of $L(t)$ will be provided here.

THEOREM D. *Let M be a Riemannian plane admitting total curvature and let $\mathfrak{C}$ be a simply closed smooth regular curve on M. If $L(t)$ and $A(t)$ are the length of $S(t)$ and the area of $B(t)$, then*

$$\lim_{t \to \infty} \frac{L^2(t)}{B(t)} = 2(2\pi - c(M)).$$

Hartman proved this relation under a stronger assumption that $\int_M |G|\, dv < \infty$.

I. The proof of Theorem A

The following Lemma 1.1 will be useful for the proof of Theorem A. This lemma was recently proved by Shiga under the restriction that $\mathfrak{C}$ is a geodesic circle around an arbitrary fixed point with radius less than the injectivity radius of the exponential map at that point (see Lemma C in [5]). For each $\lambda \in \Lambda$ let

$\gamma_\lambda^-, \gamma_\lambda^+ : [0, \infty) \to M_1$ be the rays from $\mathfrak{C}$ which are contained in the boundary of D_λ.

LEMMA 1.1. *Assume that $M_1 - F \neq \emptyset$. For any given positive ε there exists a constant $R(\varepsilon) > 0$ with the following properties:*

(1) If $x \in M_1 - F$ satisfies $\rho(x) > R(\varepsilon)$ and if $\sigma_x : [0, \rho(x)] \to M_1$ is a segment from x to $\mathfrak{C}$, then there exists a $\lambda \in \Lambda$ such that one of the subarcs of $\mathfrak{C}_\lambda$ divided by $\sigma_x(\rho(x))$ has length less than ε.

(2) If the length L_λ of $\mathfrak{C}_\lambda$ is greater than 2ε, then there exists for every $t > R(\varepsilon)$ a unique point $x(t)$ on $D_\lambda \cap S(t) \cap C(\mathfrak{C})$ such that there are at least two distinct segments from $x(t)$ to $\mathfrak{C}$ and such that each length of the two subarcs of $\mathfrak{C}_\lambda$ taken outside of $E(x(t))$ is less than ε.

Proof of Lemma 1.1. If $x \in D_\lambda$ and if L_λ is less than 2ε, then the conclusion (1) is trivial. If $x \in D_\lambda$ and if $L_\lambda \geqq 2\varepsilon$, then (1) is a direct consequence of the fact that D_λ contains no ray from $\mathfrak{C}$.

For the proof of (2), fix an $\varepsilon' \in (0, \varepsilon)$. $R(\varepsilon)$ may be replaced by $R(\varepsilon')$. For each $t > R(\varepsilon')$ let $c_t : [0, 1] \to S(t) \cap D_\lambda$ be a curve such that $c_t(0) = \gamma_\lambda^-(t)$, $c_t(1) = \gamma_\lambda^+(t)$ and such that the image of c_t bounds $D_\lambda - B(t)$. Let $J^- := \{u \in [0, 1]$; every segment $\sigma_{c_t(u)} : [0, t] \to M_1$ from $c_t(u)$ to $\mathfrak{C}$ has the property that the subarc of $\mathfrak{C}_\lambda$ between $\gamma_\lambda^-(0)$ and $\sigma_{c_t(u)}(t)$ has length less than $\varepsilon\}$, and similarly let $J^+ := \{u \in [0, 1]$; every segment $\sigma_{c_t(u)} : [0, t] \to M_1$ from $c_t(u)$ to $\mathfrak{C}$ has the property that the subarc of $\mathfrak{C}_\lambda$ between $\sigma_{c_t(u)}(t)$ and $\gamma_\lambda^+(0)$ has length less than $\varepsilon\}$. Clearly J^- (respectively, J^+) contains a small interval around 0 (respectively, 1), and has the property that if $u \in J^-$ (respectively, $u \in J^+$), then $[0, u] \subset J^-$ (respectively, $[u, 1] \subset J^+$). This is an immediate consequence of the facts that any segment from $x \in c_t([0, 1])$ to $\mathfrak{C}$ does not intersect $c_t([0, 1])$ at its interior and that any two segments from distinct points on $c_t([0, 1])$ to $\mathfrak{C}$ do not intersect each other. If $J^- \cup J^+$ is a proper subset of $[0, 1]$, then a point $c_t(u')$ with $u' \in [0, 1] - J^- \cup J^+$ has the desired property. if $J^- \cup J^+ = [0, 1]$, then $J^- \cap J^+ = \emptyset$ implies that one of them is open and the other closed, and the point $c_t(u')$ with $u' = \sup\{u; u \in J^-\}$ has the desired property.

The uniqueness of such a point $x(t) = c_t(u')$ follows from the fact that for any $u \in [0, u')$ and for any segment $\sigma_{c_1(u)} : [0, t] \to M_1$ from $c_t(u)$ to $\mathfrak{C}$ its endpoint belongs to the subarc of $\mathfrak{C}_\lambda$ between $\gamma_\lambda^-(0)$ and $\sigma_{x(t)}^-(t)$, and the similar property holds for any $u \in (u', 1]$ and for any segment from $c_t(u)$ to $\mathfrak{C}$.

This completes the proof of Lemma 1.1.

Proof of Theorem A. Let $\varepsilon > 0$ be chosen so as to satisfy that $2\varepsilon < L_\lambda$ and set $t_\lambda := R(\varepsilon')$ for some ε' in $(0, \varepsilon)$. (1) is obvious from the previous lemma. (2) has already been established by Hartman, as stated in the introduction. To prove (3) let $t_1 > t_2 > t_\lambda$. Then $\sigma^-_{x(t_1)}([0, t_1])$ intersects $c_{t_2}([0, 1])$ at $\sigma^-_{x(t_1)}(t_1 - t_2)$ which belongs to the subarc of c_{t_2} between $c_{t_2}(0) = \gamma^-_\lambda(t_2)$ and $x(t_2)$. Therefore $\sigma^-_{x(t_1)}(t_1)$ belongs to the subarc of $\mathfrak{C}_\lambda$ between $\gamma^-_\lambda(0)$ and $\sigma^-_{x(t_2)}(t_2)$. Similarly, $\sigma^+_{x(t_1)}(t_1)$ lies on the subarc of $\mathfrak{C}_\lambda$ between $\sigma^+_{x(t_2)}(t_2)$ and $\gamma^+_\lambda(0)$. This proves $E(x(t_1)) \supset E(x(t_2))$ and the monotone property of $\{E(x(t))\}$ is proved. Since $\varepsilon > 0$ is any, $\lim_{t\to\infty} E(x(t)) = D_\lambda$ is obvious. This proves (3).

The existence of total curvature of M (and hence of D_λ) is essential for the proof of (4). The proof technique is based on the situation that γ^-_λ and γ^+_λ are distinct. Taking account of the case where there is a unique ray from $\mathfrak{C}$, one uses the preimage $\tilde{D}_\lambda \subset \tilde{M}_1$ of D_λ in the fundamental domain of M_1 under the covering map $\pi: \tilde{M}_1 \to M_1$, where $\tilde{M}_1$ is the universal Riemannian covering. Set $\tilde{\mathfrak{C}} := \pi^{-1}(\mathfrak{C})$. The boundary of $\tilde{D}_\lambda$ consists of two distinct rays $\tilde{\gamma}^-_\lambda, \tilde{\gamma}^+_\lambda : [0, \infty) \to \tilde{M}_1$ from $\tilde{\mathfrak{C}}$ such that $\pi(\tilde{\gamma}^-_\lambda) = \pi(\tilde{\gamma}^+_\lambda) = \gamma_\lambda$ and of the subarc $\tilde{\mathfrak{C}}_\lambda$ of $\tilde{\mathfrak{C}}$ whose endpoints are $\tilde{\gamma}^-_\lambda(0)$ and $\tilde{\lambda}^+_\lambda(0)$. If there are more than one ray from $\mathfrak{C}$, then $\tilde{D}_\lambda$ is identical with D_λ. Thus the arguments developed below covers the case where the boundary of D_λ contains two distinct rays.

For a point x on D_λ let $\tilde{x} \in \tilde{D}_\lambda$ be such that $\pi(\tilde{x}) = x$. Also let $\tilde{E}(x) \subset \tilde{D}_\lambda$ be the compact domain such that $\pi(\tilde{E}(x)) = E(x)$ and let $\tilde{\sigma}^-_x$ and $\tilde{\sigma}^+_x$ be the preimages of σ^-_x and σ^+_x through $\tilde{x}$. Then $\lim_{t\to\infty} \tilde{E}(x(t)) = \tilde{D}_\lambda$ implies that the limit of $\beta(x(t))$ as $t \to \infty$ exists and satisfies:

$$c(D_\lambda) = c(\tilde{D}_\lambda) = \lim_{t\to\infty} c(\tilde{E}(x(t))) = \lim_{t\to\infty} \beta(x(t)) - \int_{I_\lambda} \kappa(s)\, ds.$$

On the other hand for an arbitrary given $\eta > 0$ and for a given divergent sequence $\{t_j\}$, a monotone increasing sequence $\{\tilde{Q}_j\}$ of compact domains in $\tilde{\tilde{D}}_\lambda$ will be constructed below in such a way that $\lim_{j\to\infty} \tilde{Q}_j = \tilde{\tilde{D}}_\lambda$ and that

$$\lim_{j\to\infty} c(\tilde{Q}_j) \leqq \eta - \int_{I_\lambda} \kappa(s)\, ds.$$

If the above construction has been achieved, it will then follow that $\lim_{j\to\infty} c(\tilde{Q}_j) = \lim_{t\to\infty} c(\tilde{E}(x(t)))$ and hence (4) will be proved.

The construction of $\{\tilde{Q}_j\}$ is done as follows. Let $\tilde{d}_\lambda : \tilde{D}_\lambda \times \tilde{D}_\lambda \to R$ be the distance function induced from the Riemannian metric on $\tilde{D}_\lambda$. It is elementary that $\tilde{d}_\lambda \geqq d \circ \pi$, and also that every two points $\tilde{x}$ and $\tilde{y}$ in $\tilde{D}_\lambda$ can be joined by a curve in $\tilde{D}_\lambda$ whose length realizes $\tilde{d}_\lambda(\tilde{x}, \tilde{y})$. Such a curve will be called a $\tilde{d}_\lambda$-segment. A $\tilde{d}_\lambda$-segment is not necessarily the preimage of a geodesic in M_1. If a $\tilde{d}_\lambda$-segment does not pass through a point on $\tilde{\mathfrak{C}}_\lambda$, then its image under π is a geodesic in M_1, but not necessary minimizing. Note that every $\tilde{d}_\lambda$-segment does not intersect $\tilde{\gamma}^-_\lambda$ and $\tilde{\gamma}^+_\lambda$ at any point on their interior.

Choose an $\eta_0 > 0$ sufficiently small such that if $t_0 = R(\eta_0)$, then every geodesic in M passing through a point on $D_\lambda \cap B(1) - E(x(t_0))$ intersects the boundary of $D_\lambda - E(x(t_0))$ and goes out D_λ beyond its intersection. This is possible because the set is sufficiently narrow. For every j with $t_j > t_0$ there exist large numbers t_j^- and t_j^+ such that every $\tilde{d}_\lambda$-segment joining $\tilde{x}(t_j)$ to $\tilde{\gamma}_\lambda - (t_j^-)$ does not intersect $\mathfrak{C}_\lambda$, and such that every $\tilde{d}_\lambda$-segment joining $\tilde{x}(t_j)$ to $\tilde{\gamma}_\lambda^+(t_j^+)$ does not intersect $\mathfrak{C}_\lambda$. The minimizing property of $\tilde{\sigma}^-_{x(t_j)}$, $\tilde{\sigma}^+_{x(t_j)}$ and $\tilde{\gamma}_\lambda^-$, $\tilde{\gamma}_\lambda^+$ implies that such a $\tilde{d}_\gamma$-segment does not intersect them at any point on its interior, and hence all interior points on such a $\tilde{d}_\lambda$-segment lies in the interior of $\tilde{D}_\lambda$. This observation shows that the domain bounded by $\mathfrak{C}_\lambda$, $\tilde{\gamma}_\gamma^+([0, t_j^+])$, two $\tilde{d}_\lambda$-segments joining $\tilde{x}(t_j)$ to $\tilde{\gamma}_\lambda^-(t_j^-)$ and to $\tilde{\gamma}_\lambda^+(t_j^+)$ and $\tilde{\gamma}_\lambda^-([0, t_j^-])$ contains $\tilde{E}(x(t_j))$ as a proper subset. $\tilde{Q}_j$ is obtained by choosing t_j^- and t_j^+ sufficiently large so as to satisfy that if $\tilde{\tau}_j^+ : [0, l_j^+] \to \tilde{D}_\lambda$ and $\tilde{\tau}_j^- : [0, l_j^-] \to \tilde{D}_\lambda$ are $\tilde{d}_\lambda$-segments with $\tilde{\tau}_j^+(0) = \tilde{\tau}_j^-(0) = \tilde{x}(t_j)$, $\tilde{\tau}_j^+(l_j^+) = \tilde{\gamma}_\lambda^+(t_j^+)$, $\tilde{\tau}_j^-(l_j^-) = \tilde{\gamma}_\lambda^-(t_j^-)$, then $\tilde{Q}_j$ is the domain bounded by $\mathfrak{C}_\lambda$, $\tilde{\gamma}_\lambda^-([0, t_j^-])$, $\tilde{\gamma}_\lambda^+([0, t_j^+])$, $\tilde{\tau}_j^+([0, l_j^+])$ and $\tilde{\tau}_j^-([0, l_j^-])$, and the angles of the corners at $\tilde{\gamma}_\lambda^+(t_j^+)$ and at $\tilde{\gamma}_\lambda^-(t_j^-)$ are less than $\eta/2$. Such a choice of t_j^+ is seen as follows: If $\theta(t)$ is the angle between $\tilde{\gamma}_\lambda^+(t)$ and the tangent vector to a $\tilde{d}_\lambda$-segment joining $\tilde{x}(t_j)$ to $\tilde{\gamma}_\lambda^+(t)$, then the function $t - \tilde{d}_\lambda(x(t_j), \gamma_\lambda^+(t))$ for $t \geqq t_0$ is Lipschitz continuous with the Lipschitz constant 2 and bounded above by $\tilde{d}_\lambda(\tilde{x}(t_j), \tilde{\gamma}_\lambda^+(t_0))$. This function is expressed as

$$\int_{t_0}^{t} [1 - \cos \theta(u)]\, du + \{t_0 - \tilde{d}_\lambda(x(t_j), \gamma_\lambda^+(t_0))\}.$$

Therefore $\liminf_{u \to \infty} [1 - \cos \theta(u)] = 0$, and the existence of a desired constant is verified. The original idea of this technique was developed by Cohn–Vossen [1]. Thus a monotone increasing sequence $\{\tilde{Q}_j\}$ of compact domains is obtained by choosing a suitable subsequence, and $\lim_{j \to \infty} \tilde{Q}_j = \bar{\tilde{D}}_\lambda$ is obvious from the construction.

Finally

$$c(D_\lambda) = -\int_{I_\lambda} \kappa(s)\, ds$$

is obvious from (1) and $\lim_{t \to \infty} \beta(x(t)) = 0$.

This completes the proof of Theorem A.

II. The proof of Theorem B

The idea of the proof of Theorem B (1) is summarized as follows: Suppose that there is a divergent sequence $\{t_j\}$ such that for each j $S(t_j)$ is not arcwise connected. For each j the value t_j is replaced by a $t_j' \leqq t_j$ in such a way that $S(t_j')$ is not arcwise connected and that there exists a compact component of $M_1 - B(t_j')$

which has a nonempty interior. Thus one may suppose that for each j $M_1 - B(t_j)$ contains a compact component $W(t_j)$ having nonempty interior. Then it will be possible to find for each j a geodesic $\gamma_j : [0, 2l_j] \to M_1$ such that $\gamma_j(0)$, $\gamma_j(2l_j)$ are on $\mathfrak{C}$ and hits orthogonally to $\mathfrak{C}$. Then the compact domains $\{D_j\}$, each D_j is bounded by $\gamma_j([0, 2l_j])$ and a subarc of $\mathfrak{C}$, have an infinite subsequence $\{D_k\}$ which is monotone increasing and $\lim_{k\to\infty} D_k = D_\lambda$ for some $\lambda \in \Lambda$. Then it will turn out that D_λ does not admit total curvature, a contradiction. The proof technique of deriving the nonexistence of total curvature on D_λ has already been employed in the proof of Theorem A.

For each $t > 0$ let $c_t : [0, 1] \to S(t)$ be a simply closed curve which bounds the unique unbounded component of $M_1 - B(t)$. The proof of Theorem B, (2) will be achieved if the image of c_t coincides with $S(t)$ for all $t > R_2$. The same idea as used in the proof of (1) will be employed for the proof of (2).

The following Lemmas 2.1 and 2.2 will be useful for the proof of Theorem B.

LEMMA 2.1. *Assume that there exists a monotone divergent sequence* $\{t_j\}$ *such that for each* j $M - B(t_j)$ *has a compact component* W_j *with nonempty interior. Then there exists for each* j *a geodesic* $\gamma_j : [0, 2l_j] \to M_1$ *with the following properties:*

(1) *The endpoints of* γ_j *are on* $\mathfrak{C}$ *at which* γ_j *hits orthogonally to* $\mathfrak{C}$.

(2) $\gamma_j([0, 2l_j])$ *and a subarc* $\mathfrak{C}_j$ *(having the same endpoints as* γ_j*) of some* $\mathfrak{C}_{\lambda(j)}$, $\lambda(j) \in \Lambda$, *bounds a compact domain in* M_1 *which is homeomorphic to a closed* 2*-disk and which contains* W_j *in its interior.*

(3) $\gamma_j(l_j)$ *is a cut point to* $\mathfrak{C}$ *along* γ_j, *and in particular* l_j *is exceptional.*

Proof. Fix a number j and let $\psi_j : [0, 1] \to W_j - \mathrm{int}(W_j)$ be a simply closed curve. $M - \mathrm{int}(W_j)$ is homeomorphic to a closed half cylinder $S^1 \times [0, \infty)$ whose fundamental group is generated by $[\psi_j]$, where $[\psi_j]$ represents the free homotopy class of all closed curves in $M - \mathrm{int}(W_j)$ containing ψ_j. There is a $\lambda(j) \in \Lambda$ such that $W_j \subset D_{\lambda(j)}$.

It is asserted that there exists a point y on $\psi_j([0, 1])$ such that the boundary $\partial E(y)$ is not homotopic to 0 in $M - \mathrm{int}(W_j)$. In fact, if otherwise supposed, then a contradiction is derived as follows: Note first that every segment from a point on $\psi_j([0, 1])$ to $\mathfrak{C}$ does not pass through any point on $\mathrm{int}(W_j)$. If $V = \bigcup_{u \in [0, 1]} E(\psi_j(u))$, then $V \subset M_1 - \mathrm{int}(W_j)$. It follows from the supposition that $E(\psi_j(u))$ is homeomorphic to a closed 2-disk for each $u \in [0, 1]$ with $\beta(\psi_j(u)) > 0$. Thus V is homeomorphic to $S^1 \times [0, 1]$, and in particular $M_1 = V \cup \mathrm{int}(W_j)$ leads to a contradiction that M_1 is bounded.

Since $\partial E(y)$ is not homotopic to 0 in $M - \mathrm{int}(W_j)$, it belongs to $[\psi_j]^k$ for some integer k. Since $\partial E(y)$ has no self-intersection, $k = 1$. Clearly $\psi_j([0, 1]) \subset E(y) \cap D_{\lambda(j)}$.

If $\beta(y)=\pi$, then the desired γ_j is nothing but the geodesic $\sigma_y^+ * \sigma_y^- : [0, 2t_j] \to M_1$ which is defined as

$$\sigma_y^+ * \sigma_y^-(v) := \begin{cases} \sigma_y^-(v) & 0 \leqq v \leqq t_j \\ \sigma_y^+(v-t_j) & t_j \leqq v \leqq 2j_j, \end{cases}$$

where $l_j = t_j$.

Consider the case where $\beta(y) \neq \pi$. Since $\psi_j([0,1])$ intersects $E(y)$ only at the point y and since $\partial E(y)$ is homotopic to ψ_j, W_j is contained in $E(y)$. It is asserted that $\beta(y) > \pi$. In fact, if otherwise supposed, then for a sufficiently small $h > 0$, $d(\sigma_y^-(h), \sigma_y^+(h)) < 2h$. It follows from $\beta(y) < \pi$ that for some $h > 0$ the segment τ joining $\sigma_y^-(h)$ to $\sigma_y^+(h)$ passes through a point in int (W_j) and the length of a curve obtained by joining $\sigma_y^-([h, t_j]) \cup \tau \cup \sigma_y^+([h, t_j])$ is less than $2t_j$, a contradiction.

As is seen in the above paragraph, $\beta(y) > \pi$ and $\sigma_y^+ * \sigma_y^-$ can be replaced by a shorter curve in $D_{\lambda(j)} - \text{int}(W_j)$ which is freely homotopic to it. Then a standard length-decreasing deformation procedure in $D_{\lambda(j)} - \text{int}(W_j)$ is carried out to this broken geodesic with endpoints on $\mathfrak{C}$, and the linit of the deformation exists and is a geodesic $\gamma_j : [0, 2l_j] \to D_{\lambda(j)}$, where $l_j < t_j$. This γ_j has the properties (1) and (2). If $\gamma_j(l_j)$ is not the cut point to $\mathfrak{C}$ along γ_j, then there is a segment $\sigma : [0, l_j'] \to M_1$ from $\gamma_j(l_j)$ to $\mathfrak{C}$ and $l_j' < l_j$ and its image is in $D_{\lambda(j)}$. Clearly $\sigma([0, l_j']) \cap W_j \neq \emptyset$ and one of the two broken geodesics $\sigma * \gamma_j \mid [0, l_j]$ and $\sigma * \gamma_j \mid [l_j, 2l_j]$ together with the corresponding subarcs of $\mathfrak{C}$ is freely homotopic to ψ_j, and they bound a closed 2-disk containing W_j in its interior. By iterating this procedure, the desired geodesic satisfying (3) is obtained as the limit of length-decreasing deformations.

This completes the proof of Lemma 2.1.

For each j let $D_j \subset D_{\lambda(j)}$ be the compact domain bounded by $\gamma_j([0, 2l_j])$ and the corresponding subarc $\mathfrak{C}_j$ of $\mathfrak{C}_{\lambda(j)}$. Recall that D_j contains W_j in its interior.

LEMMA 2.2. *Under the same assumption as in Lemma* 2.1, *there exists a monotone increasing subsequence* $\{D_k\}$ *of* $\{D_j\}$ *such that* $\lim_{k\to\infty} D_k = D_\lambda$ *holds for some* $\lambda \in \Lambda$.

Proof. It follows from the property (3) of γ_j that if $j \neq k$, then either $D_j \cap D_k = \emptyset$ or else one is contained in the other as a proper subset. If there are infinitely many disjoint D_j's and if I_j is the domain of the subarc $\mathfrak{C}_j$, then

$$c(D_j) = \pi - \int_{I_j} \kappa(s)\, ds$$

holds for each j, and hence the sum of $c(D_j)$ over all disjoint members is ∞. This contradicts to the assumption that M admits total curvature. Therefore, except a finite members of $\{D_j\}$ it is monotone increasing. Let $\{D_k\}$ be an infinite monotone subsequence of $\{D_j\}$. Then $\{I_k\}$ is also a monotone increasing sequence of intervals and there is a $\lambda \in \Lambda$ such that $I_k \subset I_\lambda$ for all k.

It is asserted that $\{l_k\}$ is divergent. Let $L_* := \sup_{k\to\infty} l_k$ and let $\lim_{k\to\infty} I_k = I_*$. Suppose that $l_* < \infty$. It is elementary that there exists a geodesic $\gamma_* : [0, 2l_*] \to D_\lambda$ such that γ_* is the limit of $\{\gamma_k\}$. Since D_k contains W_k and since $\{D_k\}$ is monotone increasing, the domain D_* bounded by $\mathfrak{C}(I_*)$ and $\gamma_*([0, 2l_*])$ contains a divergent sequence, a contradiction to the assumption that M is homeomorphic to a plane.

Finally $D_* = D_\lambda$ is an immediate consequence of the facts that the boundary of D_* contains two (or possibly one) rays from $\mathfrak{C}$ and that D_λ contains no ray from $\mathfrak{C}$.

This completes the proof of Lemma 2.2.

The proof of Theorem B. Suppose (1) is false. Then there exists a divergent sequence $\{t_j\}$ such that for each j, $M_1 - B(t_j)$ has a compact component W_j with nonempty interior. There is a $\lambda \in \Lambda$ and an infinite sequence $\{D_k\}$ of compact domains, the boundary of each D_k containing the geodesic γ_k with the properties (1), (2) and (3) in Lemma 2.1, and $\lim D_k = D_\lambda$. Therefore

$$c(D_\lambda) = \lim_{k\to\infty} c(D_k) = \pi - \int_{I_\lambda} \kappa(s)\, ds.$$

This contradicts to Theorem A, (4).

Suppose (2) is false. Then there exists a divergent sequence $\{t_j\}$ such that $t_j > R_1$ for all j and such that $c_{t_j}([0, 1])$ is contained in $S(t_j)$ as a proper subset. As is proved in (1), $S(t_j)$ is arcwise connected, and hence there exists a nontrivial curve $b_j : [0, 1] \to S(t_j) - c_{t_j}([0, 1])$. Since no point on the image of b_j is on the boundary of $M_1 - \bar{B}(t_j)$, ρ takes a local maximum on each point of the image of b_j. Let N be a small ball around a point $q = b_j(1/2)$ which is contained entirely in the interior of $\bar{B}(t_j)$ such that N is divided by $b_j([0, 1])$ into two components N_1 and N_2. Every segment from a point on $N_1 \cup N_2$ to $\mathfrak{C}$ does not intersect $b_j([0, 1])$, and hence there exist two distinct segments from q to $\mathfrak{C}$ which makes an angel π at q. In particular t_j is an exceptional value. This means that for each j there exists a geodesic $\gamma_j : [0, 2t_j] \to M_1$ having the properties (1), (2) and (3) in Lemma 2.1. Thus a contradiction is derived by developing the same arguments as in the proof of (1).

This completes the proof of Theorem B.

The following Corollary is a direct consequence of the above arguments, and the proof of it will be omitted here.

COROLLARY TO THEOREM B. *Assume that a Riemannian plane M admits total curvature and let $\mathfrak{C}$ be a simply closed regular smooth curve on M.*

Then there exists a constant $R_3 \geqq R_2$ such that $\beta(x) < \pi$ holds for all x with $\rho(x) > R_3$. Or equivalently, the function ρ has no critical point on the set $M_1 - B(R_3)$.

III. The proofs of Theorems C and D

The proof of Theorem C is based on the fact that if $t > R_3$ and if $\{x_\mu ; \mu \in \Lambda(t)\}$ is the set of all cut points to $\mathfrak{C}$ on $S(t)$, where $\Lambda(t)$ is an index set, then the set $E(t) := \bigcup_{\mu \in \Lambda(t)} E(x_\mu)$ is strictly monotone increasing with t. This fact will be guaranteed by the property that ρ has no critical point on $M_1 - B(R_3)$. The monotone increasing property of $\{E(t)\}$ will be established in Lemma 3.1 below. Assuming Lemma 3.1 for the moment, Theorem C is proved as follows.

The proof of Theorem C by assuming Lemma 3.1. For each $t > R_3$ let $\mathfrak{C}_t := E(t) \cap \mathfrak{C}$ and let $I(t) \subset [0, L]$ be such that $\mathfrak{C}(I(t)) = \mathfrak{C}_t$.

Let ε be an arbitrary given positive number. Then there are at most finite elements $\lambda_1, \lambda_2, \ldots, \lambda_m$ in Λ such that

$$\left|\sum_{\lambda \in \Lambda} L_\lambda - \sum_{i=1}^{m} L_{\lambda_i}\right| < \varepsilon/2.$$

Theorem A implies that there is a number $t'_\varepsilon > \max\{t_{\lambda_1}, \ldots, t_{\lambda_m}, R_3\}$ such that for each $t > t'_\varepsilon$

$$\left|\sum_{i=1}^{m} L_{\lambda_i} - L(\mathfrak{C}_t)\right| < \varepsilon/2.$$

It then follows that $\lim_{t\to\infty} I(t) = \bigcup_{\lambda \in \Lambda} I_\lambda$ and that $\lim_{t\to\infty} E(t) = \bigcup_{\lambda \in \Lambda} D_\lambda$. In particular, there exists a number t_ε such that for every $t > t_\varepsilon$,

$$\int_{\bigcup_{\lambda \in \Lambda} I_\lambda - I(t)} |\kappa(s)|\, ds < \varepsilon/2$$

and

$$\left|\sum_{\mu \in \Lambda(t)} c(E(x_\mu)) - \sum_{\lambda \in \Lambda} c(D_\lambda)\right| < \varepsilon/2.$$

Applying the Gauss–Bonnet theorem for each $E(x_\mu)$, $\mu \in \Lambda(t)$ and summing up over $\Lambda(t)$, one obtains

$$\sum_{\mu \in \Lambda(t)} \beta(x_\mu) = \sum_{\mu \in \Lambda(t)} c(E(x_\mu)) + \int_{I(t)} \kappa(s)\, ds < \varepsilon$$

This proves Theorem C.

LEMMA 3.1. *If a cut point p satisfies* $t := \rho(p) > R_3$, *then every point y in* $E(p)$ *has the property that* $\rho(y) \leqq t$, *and* $\rho(y) = t$ *holds if and only if* $y = p$. *In particular* $C(\mathfrak{C}) \cap E(p) \cap S(t) = \{p\}$. *Moreover there exists a unique curve* $x_p : [0, \infty) \to C(\mathfrak{C}) \cap D_\lambda$ *such that* $x_p(0) = p$ *and such that* $\{E(x_p(u))\}$ *is monotone increasing with* u *and* $\rho(x_p(u)) = u + \rho(p)$ *for all* $u \geqq 0$.

Proof. The first conclusion is obvious if there exists a unique segment from p to $\mathfrak{C}$. (In this case p is an isolated focal point to $\mathfrak{C}$). For the proof of the first conclusion assume that the interior of $E(p)$ is nonempty. Suppose that there is a point in $\operatorname{int}(E(p))$ at which ρ is greater than t. Then there exists a point in $\operatorname{int}(E(p))$ at which ρ takes a local maximum, contradicting to the Corollary to Theorem B. Assume that $\rho(y) = t$ holds for some $y \in E(p)$. Suppose that $y \neq p$. Then y must belong to $\operatorname{int}(E(p))$. It follows from Theorem B, (2) that there exists a proper subarc of $S(t)$ containing in $E(p)$ whose endpoints are p. This means that $S(t)$ has a self-intersection at p, a contradiction to Theorem B, (2).

For a point $p \in C(\mathfrak{C})$ with $\rho(p) > R_3$, let $x_p : [0, \infty) \to C(\mathfrak{C}) \cap D_\lambda$ be the curve obtained in Theorem A such that $x_p(0) = p$. The monotone property of $\{E(x_p(u))\}$ has already been established in the proof of Theorem A. The uniqueness of x_p is seen as follows.

Suppose that there are two curves x_p, $x'_p : [0, \infty) \to C(\mathfrak{C}) \cap D_\lambda$ having the properties required. Note that there is no closed curve in $C(\mathfrak{C})$ which bounds an open bounded set. Therefore if $\{u_i\}$ is a decreasing sequence such that $\lim u_i = u_0 \geqq 0$, and such that $x_p(u_i) \neq x'_p(u_i)$ for all $i = 1, 2, \ldots$, and that $x_p(u_0) = x'_p(u_0)$, then $x_p(u) \neq x'_p(u)$ holds for all $u > u_0$. It follows from what is supposed that there is a $u_0 \geqq 0$ such that $x_p(u) \neq x'_p(u)$ for all $u > u_0$ and such that $x_p(u) = x_p(u)$ for all $u \in [0, u_0]$. Without loss of generality one may assume that $u_0 > 0$. Then there exists a small ball N around $x_p(u_0)$ which is divided by the curves $x_p([0, u_0])$, $x_p([u_0, u_1])$ and $x'_p([u_0, u_1])$ for some $u_1 > u_0$ into three components. There are at least three distinct segments from $x_p(u_0)$ to $\mathfrak{C}$ each of which passes through points on each of the three distinct components in N. Two of the three curves in $C(\mathfrak{C}) \cap N$ pass through points in the interior of $E(x_p(u_0))$. But this contradicts to the first conclusion since $\rho(x_p(u)) = \rho(x'_p(u)) > \rho(p)$ holds for $u > u_0$.

This completes the proof of Lemma 3.1.

Note that if $p, q \in S(t) \cap C(\mathfrak{C})$, $p \neq q$ and if $t > R_3$, then there exists a number $u_0 > 0$ such that $x_p(u) \neq x_q(u)$ for $0 \leqq u < u_0$ and $x_p(u) = x_q(u)$ for all $u \geqq u_0$.

The proof of Theorem D. The continuity of $L(t)$ for $t > R_3$ is a direct consequence of the fact that

$$\lim_{h \downarrow 0} S(t+h) = \lim_{h \downarrow 0} S(t-h) = S(t).$$

As was already shown by Hartman $L(t)$ is differentiable at each non-exceptional t. The derivative of $L(t)$ at such a t was first given by Fiala (see p. 330, [2]) as follows: Let t be a non-exceptional value and let $x_1(t), \ldots, x_k(t)$ be all the normal cut points on $S(t)$ and let $m(t)$ be the number of components of $S(t)$. Then

$$\frac{dL(t)}{dt} = 2\pi - c(\bar{B}(t)) - 2\pi(m(t)-1)$$
$$- \sum_{i=1}^{k} [2 \tan (\beta(x_i(t))/2) - \beta(x_i(t))].$$

For an arbitrary given ε in $(0, \pi)$ let $T'(\varepsilon) := \max (R_3, t(\varepsilon))$, where $t(\varepsilon)$ is a constant obtained in Theorem C. If $t > T'(\varepsilon)$ is non-exceptional, then $m(t) = 1$ follows from Theorem B, and from Theorem C one obtains

$$0 \leqq \sum_{i=1}^{k} [2 \tan (\beta(x_i(t))/2) - \beta(x_i(t))] < \varepsilon.$$

Therefore if $t > T'(\varepsilon)$ is non-exceptional, then

$$2\pi - c(\bar{B}(t)) - \varepsilon < \frac{dL(t)}{dt} \leqq 2\pi - c(\bar{B}(t)).$$

On the other hand the area $A(t)$ of $\bar{B}(t)$ is given as

$$A(t) - A(T) = \int_T^t L(u)\, du.$$

Now, if $c(M) = -\infty$, then $\lim_{t\to\infty} dL(t)/dt = \infty$, and the proof of Theorem D in this case is an immediate consequence of the L'Hospital theorem.

Consider the case where $c(M) > -\infty$. Let $T''(\varepsilon)$ be a number such that $|c(\bar{B}(t)) - c(M)| < \varepsilon$ holds for all $t > T''(\varepsilon)$ and set $T(\varepsilon) := \max (T'(\varepsilon), T''(\varepsilon))$. Then it is clear that for every $t > T(\varepsilon)$

$$2\pi - c(M) - 2\varepsilon \leqq \lim_{t\to\infty} \frac{L(t)}{t} \leqq 2\pi - c(M) + \varepsilon,$$

and

$$2\pi - c(M) - 2\varepsilon \leqq \lim_{t\to\infty} \frac{A(t)}{2t^2} \leqq 2\pi - c(M) + \varepsilon.$$

Since E is any positive, this completes the proof of Theorem D.

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