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Foliation dynamics and leaf invariants

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§1. Statement of results

Let \mathcal{F} be a codimension- n foliation of a smooth manifold M without boundary. M may be either compact or open, and assume \mathcal{F} is transversally C^2 . The purpose of this note is to examine the relation between the linear holonomy of the leaves of \mathcal{F} and the growth rates of the leaves.

THEOREM 1. *Let \mathcal{F} and M be as above. Given a leaf $L \subset M$ of \mathcal{F} , suppose its linear holonomy group $\Gamma_L \subset GL(n, \mathbb{R})$ is not amenable. Then \mathcal{F} has a leaf L' which contains L in its closure, and for all Riemannian metrics on M , L' has exponential growth.*

Amenability is taken in the sense of topological groups, where Γ_L is endowed with the topology from $GL(n, \mathbb{R})$.

We actually prove a slightly more general result, from which Theorem 1 follows by standard methods.

THEOREM 2. *Let \mathcal{G} be a pseudogroup of local diffeomorphisms of \mathbb{R}^n , all of whose elements are defined at and fix the origin $0 \in \mathbb{R}^n$, and are C^2 in a neighborhood of 0 . Let Γ denote the linear group of Jacobians at 0 of the elements of \mathcal{G} . If Γ is not amenable, then the action of \mathcal{G} on \mathbb{R}^n has an orbit with exponential growth and which contains 0 in its closure.*

The normal bundle to \mathcal{F} is denoted by Q . The restriction of Q to a leaf L is well-known to be a flat \mathbb{R}^n -vector bundle, to which there are associated characteristic classes [12] obtained from the relative Lie algebra cohomology of (\mathfrak{gl}_n, O_n) . They are given by a map

$$\chi_L : H^*(\mathfrak{gl}_n, O_n) \rightarrow H^*(L).$$

The leaf classes of L consist of the image of χ_L .

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THEOREM 3. *Let \mathcal{F} be a foliation of M as above. Suppose there exists $y \in H^m(\mathfrak{gl}_n, O_n)$ with $m > 1$ and $\chi_L(y) \neq 0$. Then the linear holonomy group Γ_L of L is not amenable.*

COROLLARY 4. *Let \mathcal{F} and M be as above. Suppose that all leaves of \mathcal{F} have non-exponential growth. Then for every leaf L of \mathcal{F} , the linear holonomy group Γ_L is amenable, and all leaf classes of L in degrees greater than one are zero.*

The hypothesis $m > 1$ is necessary. For example, a flow on M with a linearly attracting closed orbit L has $\chi_L(y_1) \neq 0$, where y_1 is the standard generator of $H^1(\mathfrak{gl}_n, O_n)$. All orbits of the flow have at most linear growth, hence non-exponential, and the holonomy group of L is \mathbb{Z} , which is amenable.

Corollary 4 can be viewed as a generalization to all of the characteristic classes for flat bundles of a result due to Hirsch and Thurston. The Main Theorem of [7] implies that the Euler class of the restriction $Q|L \rightarrow L$ is zero if the foliated normal sphere bundle to L has an invariant transverse measure. This will be the case, for example, when \mathcal{F} has a leaf L' of non-exponential growth with L contained in the closure of L' .

Theorem 1 is complementary to a result of Zimmer (Theorem 5.5 of [20]; see also Corollary 4.3 of [10]): If \mathcal{F} is amenable, then there exists a measurable framing s of $Q \rightarrow M$ such that for almost every leaf L , there is a closed amenable subgroup $G_L \subset GL(n, \mathbb{R})$ for which the linear holonomy along L , with respect to s , takes values in G_L . For example, \mathcal{F} will be amenable if almost every leaf has subexponential growth.

Note that the set of leaves of \mathcal{F} with non-trivial linear holonomy has measure zero (Lemma 7.2 of [10]), so Zimmer's theorem does not imply our Theorem 1. With the stronger hypothesis that every leaf of \mathcal{F} has nonexponential growth, Theorem 1 implies that for every leaf L , there exists a framing s_L of $Q|L \rightarrow L$ for which the linear holonomy along L , with respect to s_L , takes values in an amenable subgroup G_L . It is an open problem to find sufficient conditions on the dynamics of \mathcal{F} that imply $Q \rightarrow M$ has a measurable framing s , with respect to which every leaf has amenable linear holonomy.

This work arose out of the study [9], and was motivated by an attempt to generalize to all codimensions the results of Duminy [2] relating the Godbillon-Vey class in codimension-one with leaf dynamics. For a further discussion, see [10].

We now give an idea of the proofs. Theorem 3 is based on the observation that the well-known explicit Lie algebra forms, representing the generators of $H^*(\mathfrak{gl}_n, O_n)$, are exact when restricted to the Lie algebra of a maximal amenable subgroup of $GL(n, \mathbb{R})$. This is proven in §3. The heart of this paper is the proof of

Theorem 2. It is useful to compare Theorem 2 with Tits' Theorem [19]: a non-amenable linear group Γ contains a free non-abelian subgroup on two generators. From this it is easy to see that the linear action of Γ on \mathbb{R}^n has orbits of exponential growth. Two problems arise when one tries to use this to show the pseudogroup \mathcal{G} has orbits of exponential growth. First, control must be maintained over the domains of the appropriate holonomy maps from \mathcal{G} . This is achieved by finding an element $\gamma_0^{-1} \in \mathcal{G}$ with non-trivial contracting stable manifold, and then applying our elements from \mathcal{G} to some power of γ_0^{-1} . The second, more delicate problem is to control how well the orbit under \mathcal{G} of a given point is "shadowed" by the corresponding orbits under Γ . This latter problem occupies §5, and is where the C^2 -assumption on \mathcal{G} is needed. It is doubtful that Theorem 2 holds if we are just given that \mathcal{G} is C^1 . Finally, we remark that the proof of Theorem 2 is reminiscent of the proof given in [5] of a special case of Tits' Theorem.

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§2. Growth types and leaf classes

Let \mathcal{F} denote a fixed codimension n , transversally C^2 foliation on a manifold M , L a fixed leaf of \mathcal{F} , and h a Riemannian metric on M . Given a basepoint $x \in L$, let $B(x, r) \subset L$ denote the ball of radius r in the submanifold metric on L . The metric h induces a volume element on L , and $\text{vol}\{B(x, r)\}$ will denote the total volume of $B(x, r)$. The growth function of L is $G(x, h, r) = \text{vol}\{B(x, r)\}$.

With respect to the choice of x and h , the growth type of L is said to be:

subexponential if $\limsup_{r \rightarrow \infty} \frac{1}{r} \log G(x, h, r) = 0$

nonexponential if $c_L \equiv \liminf_{r \rightarrow \infty} \frac{1}{r} \log G(x, h, r) = 0$

exponential if $c_L > 0$.

If M is compact, then the growth type of L is independent of the choices of x and h , [6], [16], and thus is an invariant of the way L is embedded in M .

The growth rate of a finitely generated group H is defined in a similar way (cf. [14]). Let $\{g_1, \dots, g_s\}$ be a reflexive generating set for H ; reflexive means that some g_i is the identity element, and for each i , $g_i^{-1} = g_j$ for some j . The word metric on H is then defined by

$$|g| \leq p \text{ if } g = g_{i_1} \cdots g_{i_p} \text{ for some integers } 1 \leq i_1, \dots, i_p \leq s.$$

Set $H_p = \{g \in H \text{ with } |g| \leq p\}$. Let $\#S$ denote the cardinality of a set S . We say H has subexponential growth if

$$c_H = \limsup_{p \rightarrow \infty} \frac{1}{p} \log \# H_p = \liminf_{p \rightarrow \infty} \frac{1}{p} \log \# H_p$$

is zero, and exponential growth if $c_H > 0$.

For a countable pseudogroup \mathcal{G} of local diffeomorphisms of \mathbb{R}^n , all of which are defined at and fix $0 \in \mathbb{R}^n$, we define the orbit growth type of \mathcal{G} as in Plante [16]. First, assume \mathcal{G} is finitely generated with reflexive generating set $\{\gamma_1, \dots, \gamma_s\}$. For $\gamma \in \mathcal{G}$ with y in the domain of γ , we say $|\gamma y|_y \leq p$ if there are integers $1 \leq i_1, \dots, i_p \leq s$ with γ_{i_k} defined at $\gamma_{i_{k-1}} \circ \dots \circ \gamma_{i_1}(y)$ and $\gamma(y) = \gamma_{i_p} \circ \dots \circ \gamma_{i_1}(y)$. Then set

$$\text{Orbit}(y, \mathcal{G}, p) = \{\gamma y \text{ such that } \gamma \in \mathcal{G} \text{ with } |\gamma y|_y \leq p\}$$

$$c(y, \mathcal{G}) = \liminf_{p \rightarrow \infty} \frac{1}{p} \log \# \text{Orbit}(y, \mathcal{G}, p).$$

We say \mathcal{G} has exponential orbit growth at y if $c(y, \mathcal{G}) > 0$ and nonexponential otherwise. For a non-finitely generated groupoid \mathcal{G} , we say it has exponential orbit growth at y if this is true for some finitely generated subpseudogroup $\mathcal{G}_0 \subset \mathcal{G}$.

Given a regular foliation chart $\phi: U \rightarrow \mathbb{R}^m$ with $\phi(x) = 0$ (cf. §4 of [16]), a closed path ξ in L based at x determines a holonomy map $\gamma_\xi: (V, 0) \rightarrow (W, 0)$ for some open neighborhoods V and W of $0 \in \mathbb{R}^n$, [3], [6], [16]. Given a finitely generated subgroup $H \subset \pi_1(L, x)$, choose closed paths $\{\xi_1, \dots, \xi_d\}$ representing a generating set of H , let \mathcal{G} denote the pseudogroup generated by the elements $\{\gamma_{\xi_1}, \dots, \gamma_{\xi_d}\}$. We extend the generating set to a reflexive set $\{\gamma_{\xi_1}, \dots, \gamma_{\xi_d}\}$. We extend the generating set to a reflexive set $\{\gamma_{\xi_1}, \dots, \gamma_{\xi_d}\}$, and let V be an open neighborhood of $0 \in \mathbb{R}^n$ on which all of the γ_{ξ_i} are defined. The following result is then implicit in §4 of [16]; see also Chapter IX of [6]:

PROPOSITION 2.1. *Let $y \in V$ and suppose \mathcal{G} has exponential orbit growth at y . Then for all Riemannian metrics on M , the leaf L' of \mathcal{F} through y has exponential growth.*

It is clear that Theorem 1 follows from Proposition 2.1 and Theorem 2.

Given a foliation chart $\phi: U \rightarrow \mathbb{R}^m$ centered at x , the linear holonomy map of L is given by $dh: \pi_1(L, x) \rightarrow GL(n, \mathbb{R})$, where for $\alpha \in \pi_1(L, x)$ choose a closed path ξ in L representing α , let γ_ξ denote the holonomy map associated to ξ , then set $dh(\alpha) = J_0\gamma_\xi$, the Jacobian matrix at 0. The image $\Gamma = \Gamma_L$ of dh is the linear holonomy group of L with respect to the chart (U, ϕ) . For a different choice of foliation chart centered at x , the map dh is changed by conjugating with some element of $GL(n, \mathbb{R})$. Thus the conjugacy class of Γ in $GL(n, \mathbb{R})$ is an invariant of the germ of \mathcal{F} along L .

The leaf classes of L are obtained by considering the pullback via dh of the continuous cohomology of $GL(n, \mathbb{R})$. Recall from Haefliger [4] or Stasheff [18] that the continuous cohomology $H_c^*(G)$ of a topological group G is the cohomology of the cochain complex of real valued group cochains on the discrete group G^δ which are continuous with respect to the topology on G . The basic result is:

THEOREM 2.2 (van Est [4]). *Let G be a Lie group, and let $K \subset G$ be a maximal compact subgroup with G/K contractible. Then there is a natural isomorphism*

$$H^*(\mathfrak{g}, K) \cong H_c^*(G)$$

where \mathfrak{g} is the Lie algebra of G and $H^*(\mathfrak{g}, K)$ is the relative Lie algebra cohomology.

For $G = GL(n, \mathbb{R})$, it is well known that

$$H_c^*(GL(n, \mathbb{R})) \cong H^*(\mathfrak{gl}_n, O_n) \cong \Lambda(y_1, y_3, \dots, y_{n'}) \tag{2.3}$$

where y_i is a closed O_n -basic form on \mathfrak{gl}_n of degree $2i - 1$, and n' is the largest odd integer less than $(n + 1)$, (cf. Chapter 5 of [13].) Given an index $I = (i_1, \dots, i_r)$ with $1 \leq i_1 < \dots < i_r \leq n'$ we write $y_I = y_{i_1} \wedge \dots \wedge y_{i_r}$. The proof of Theorem 3 will depend upon the identification in (2.3) of $H_c^*(GL(n, \mathbb{R}))$, and the naturality in the conclusion of van Est's theorem.

Define the characteristic map χ_L as the composition

$$\chi_L : H^*(\mathfrak{gl}_n, O_n) \cong H_c^*(GL(n, \mathbb{R})) \xrightarrow{dh^*} H^*(\pi_1(L, x)) \longrightarrow H^*(L)$$

where we use that $\pi_1(L, x)$ is discrete so that

$$H_c^*(\pi_1(L, x)) \cong H^*(B\pi_1(L, x)) \rightarrow H^*(L),$$

where the second map is induced from the natural map $L \rightarrow B\pi_1(L, x)$. For a more detailed discussion of the leaf classes, see Kamber–Tondeur [12], Chapter 6 of [13] or Shulman–Tischler [17].

§3. Structure of the linear holonomy group

In this section we analyze how the structure of a countable subgroup $\Gamma \subset GL(n, \mathbb{R})$ is related to the map $H_c^*(GL(N, \mathbb{R})) \rightarrow H^*(\Gamma)$. Theorem 3 will follow from this, and we also establish some preliminary results needed for the proof of Theorem 2.

Consider $GL(n, \mathbb{R})$ as the real points of $GL(n, \mathbb{C})$ and let G denote the algebraic closure of Γ in $GL(n, \mathbb{C})$. The identity component G_0 of G has finite index, and passing to the subgroup $\Gamma \cap G_0$ does not affect the statements or conclusions of Theorems 2 and 3. Thus, we can assume G is connected.

Let $G^1 = [G, G]$ be the commutator subgroup of G , and set $G^{k+1} = [G^k, G^k]$. Similarly define $\Gamma^{k+1} = [\Gamma^k, \Gamma^k]$.

LEMMA 3.1. G^k is closed and connected for all k .

Proof. See §17.2 of [8], for example. \square

We denote the algebraic closure of a group $H \subset GL(n, \mathbb{C})$ by \bar{H} .

LEMMA 3.2. The algebraic closure $\bar{\Gamma^k} = G^k$.

Proof. The inclusion $\bar{\Gamma^k} \subset G^k$ is immediate, so it suffices to show $G^k \subseteq \bar{\Gamma^k}$. By definition $\bar{\Gamma} = G$, and we proceed by induction: assume $\bar{\Gamma^l} = G^l$ for $l < k$. Consider the commutator map

$$c: GL(n, \mathbb{C}) \times GL(n, \mathbb{C}) \rightarrow GL(n, \mathbb{C})$$

with $c(g, h) = [g, h]$. This is algebraic, so $H = c^{-1}(\bar{\Gamma^k})$ is algebraically closed. Clearly, $\Gamma^{k-1} \times \Gamma^{k-1} \subseteq H$ so $\overline{\Gamma^{k-1} \times \Gamma^{k-1}} \subset H$. Now $\overline{\Gamma^{k-1} \times \Gamma^{k-1}}$ is a group containing $\overline{\Gamma^{k-1}} \times e$ and $e \times \overline{\Gamma^{k-1}}$, so by induction $G^{k-1} \times G^{k-1} \subset \overline{\Gamma^{k-1} \times \Gamma^{k-1}} \subset \overline{\Gamma^{k-1} \times \Gamma^{k-1}} \subset H$. Since G^k is generated as a group by the image $c(G^{k-1} \times G^{k-1})$, we are done. \square

As each G^k is connected, there exists a least integer N such that $G^k = G^{k+1}$ for all $k \geq N$. The key to the proof of Theorem 2 is to understand the properties of Γ^N , which we now study.

DEFINITION 3.3 [20]. A topological group H is *amenable* if every continuous affine action of H on a compact convex separable set has a fixed point.

A connected amenable Lie group is a compact extension of a solvable group. For $H \subset GL(n, \mathbb{C})$ amenable, Moore proves in [15] that H is conjugate to a subgroup of one of 2^n standard maximal amenable algebraic subgroups.

DEFINITION 3.4. A subgroup $H \subset GL(n, \mathbb{C})$ is *distal* if for each $g \in H$, all eigenvalues of g have unit length.

PROPOSITION 3.4 (Conze–Guivarc’h [1]). *A distal subgroup of $GL(n, \mathbb{C})$ is amenable.*

For the linear group Γ we now observe:

LEMMA 3.6. *If Γ^k is distal for any $k > 0$, then G is amenable.*

Proof. Suppose that Γ^k is distal. Then Γ^k is amenable, so by Moore [15] its algebraic closure G^k is also amenable. This implies G is amenable, for G is obtained from G^k by a finite number of abelian extensions. \square

COROLLARY 3.7. *If Γ is not amenable, then G^N is not trivial, and for all $k > 0$ the group Γ^k is not distal.*

This corollary is the starting point for the proof of Theorem 2 in the next section. We now prove Theorem 3. First, note that the inclusion induced map $H_c^*(GL(n, \mathbb{C})) \rightarrow H_c^*(GL(n, \mathbb{R}))$ is onto, since $H^*(\mathfrak{gl}_n \mathbb{C}, U_n) \rightarrow H^*(\mathfrak{gl}_n, O_n)$ is onto (e.g., see Chapter 7 of [13]). By the remarks of §2, Theorem 3 then follows from:

PROPOSITION 3.8. *Let $i : \Gamma \rightarrow GL(n, \mathbb{C})$ be the inclusion, and suppose that Γ is amenable. Then*

$$i^* : H_c^m(GL(n, \mathbb{C})) \rightarrow H^m(\Gamma)$$

is zero for all $m > 1$.

Proof. Let $\Lambda \subset GL(n, \mathbb{C})$ be a maximal amenable subgroup containing Γ . From [15] we know there is a basis $\{v_1, \dots, v_n\}$ of \mathbb{C}^n and integers $\{n_1, \dots, n_d\}$ with $n_1 + \dots + n_d = n$ such that with respect to this basis, Λ has the form:

$$\begin{bmatrix} R^+U_{n_1} & * & \cdot & \cdot & * & * \\ 0 & * & \cdot & \cdot & * & * \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 0 & R^+U_{n_d} \end{bmatrix}$$

Here, R^+U_n denotes the positive reals product with the unitary group of dimension n_i . Let $U_n \subset GL(n, \mathbb{C})$ be the unitary subgroup with respect the basis $\{v_1, \dots, v_n\}$.

The map i^* factors through the map $H_c^*(GL(n, \mathbb{C})) \rightarrow H_c^*(\Lambda)$, so it will suffice to show this latter map is trivial in degrees greater than one. Let λ be the Lie algebra of Λ , and let $U \subset \Lambda$ be a maximal compact subgroup with $U = \Lambda \cap U_n = U_{n_1} \times \dots \times U_{n_a}$. By the van Est Theorem, it suffices to show that for Lie algebra cohomology,

$$j^* : H^m(\mathfrak{gl}_n \mathbb{C}, U_n) \rightarrow H^m(\lambda, U)$$

is zero when $m > 1$.

Let $\tilde{\mathfrak{t}}$ be the solvable radical of λ , let \mathfrak{n} be the nilradical and $\tilde{\mathfrak{d}}$ the subspace of the complex diagonal matrices with $\tilde{\mathfrak{t}} = \mathfrak{n} \oplus \tilde{\mathfrak{d}}$. The intersection $\tilde{\mathfrak{t}} \cap \mathfrak{u}$ consists of purely imaginary diagonal matrices, so we consider $\mathfrak{t} = \tilde{\mathfrak{t}}/(\tilde{\mathfrak{t}} \cap \mathfrak{u})$ as those matrices in \mathfrak{t} with real diagonal entries. Similarly define $\mathfrak{d} = \tilde{\mathfrak{d}}/(\tilde{\mathfrak{t}} \cap \mathfrak{u})$ so that $\mathfrak{t} = \mathfrak{n} \oplus \mathfrak{d}$. As \mathfrak{t} is normal in λ , it follows from the definition of relative Lie algebra cohomology that

$$H^*(\lambda, U) \cong H^*(\mathfrak{t}^U) \cong H^*(\mathfrak{t})^U,$$

where superscript U means the $Ad(U)$ -invariant subspace. The adjoint action of U on \mathfrak{t} , λ and $\mathfrak{gl}_n \mathbb{C}$ are all compatible, so we get:

$$\begin{array}{ccc} H^m(\mathfrak{gl}_n \mathbb{C}, U_n) & \xrightarrow{j^*} & H^m(\lambda, U) \\ \downarrow \cong & & \downarrow \cong \\ H^m(\mathfrak{gl}_n \mathbb{C})^{U_n} & \xrightarrow{r^*} & H^m(\mathfrak{t})^U \end{array}$$

We will show $r^* \equiv 0$ for $m > 1$.

Recall from (p. 116 of [13]) that the generator $y_i \in H^{2i-1}(\mathfrak{gl}_n \mathbb{C})$ is represented by the $ad GL(n, \mathbb{C})$ -invariant form on $\mathfrak{gl}_n \mathbb{C}$,

$$y_i = k_i \operatorname{tr}(\underbrace{\Theta \wedge [\Theta, \Theta] \wedge \dots \wedge [\Theta, \Theta]}_{(i-1)\text{-factors}})$$

where Θ is the Maurer–Cartan form and k_i is a scalar. The algebra \mathfrak{n} is an ideal in \mathfrak{t} as an associative algebra, and $[\mathfrak{t}, \mathfrak{t}] \subset \mathfrak{n}$ so for $i > 1$ the form $r^*(y_i)$ on \mathfrak{t} is obtained by taking the traces of elements of \mathfrak{n} , which all have trace zero. Thus, $r^*(y_i) = 0$. As the $\{y_i\}$ generate the algebra $H^*(\mathfrak{gl}_n \mathbb{C})$, we are done. \square

As a corollary of the above proof, we have the general fact about Lie algebra cohomology which is useful in other contexts as well.

PROPOSITION 3.9. *Let G be an amenable subgroup of $GL(n, \mathbb{R})$ with Lie algebra \mathfrak{g} and maximal compact subgroup $K = G \cap O_n$. Then $H^m(\mathfrak{g}, O_n) \rightarrow H^m(\mathfrak{g}, K)$ is the zero map for all $m > 1$. In particular, the restriction of the forms y_i to \mathfrak{g} are exact for all $i > 1$ and odd.*

§4. Action of Γ on an attracting subspace

Let $\Gamma \subset GL(n, \mathbb{R})$ be a non-amenable countable subgroup, $G \subset GL(n, \mathbb{C})$ its connected algebraic closure and N the integer defined in §3 for which $G^N = G^{N+1}$. By Corollary 3.7 the group Γ^{N+1} is not distal, so there exists $f \in \Gamma^{N+1}$ with an eigenvalue of modulus greater than one. Let μ_1, \dots, μ_s be the eigenvalues of f and set

$$\mu = \max\{|\mu_1|, |\mu_1^{-1}|, \dots, |\mu_s|, |\mu_s^{-1}|\}.$$

By reordering the μ_i and replacing f with f^{-1} if necessary, one can assume

$$\mu = |\mu_1| = \dots = |\mu_r| > \lambda = |\mu_{r+1}| \geq \dots \geq |\mu_s|.$$

Let $\{v(i, j) \mid 1 \leq i \leq s; 1 \leq j \leq r(i)\}$ be a basis of \mathbb{C}^n in which f has Jordan form:

$$\begin{aligned} f v(i, 1) &= \mu_i \cdot v(i, 1) \\ f v(i, j) &= \mu_i [v(i, j) + v(i, j-1)] \quad \text{for } 1 \leq j \leq r(i). \end{aligned} \tag{4.1}$$

We also require that $v(i, 1) = \overline{v(j, 1)}$ if $\mu_i = \overline{\mu_j}$, where $\overline{}$ denotes complex conjugate. Set

$$V(i) = \text{Span} \{v(i, j) \mid 1 \leq j \leq r(i)\}.$$

Note that $V(i)$ is stable under f , and there is a superdiagonal nilpotent matrix $N(i)$ so that

$$f|_{V(i)} = \mu_i [Id + N(i)].$$

Let $V_{\mathbb{C}} = \bigoplus_{i=1}^s V(i)$ and $W_{\mathbb{C}} = \bigoplus_{i=r+1}^s V(i)$ so that $\mathbb{C}^n = V_{\mathbb{C}} \oplus W_{\mathbb{C}}$. Since f is real, both $V_{\mathbb{C}}$ and $W_{\mathbb{C}}$ are the complexifications of the real subspaces $V = V_{\mathbb{C}} \cap \mathbb{R}^n$ and $W = W_{\mathbb{C}} \cap \mathbb{R}^n$.

Endow \mathbb{C}^n with the Hermitian metric for which the vectors $\{v(i, j)\}$ are orthonormal. Let $|v|$ denote the length of $v \in \mathbb{C}^n$, and for $A \in GL(n, \mathbb{C})$ we set

$$|A| = \sup_{|v|=1} |Av|.$$

Define $\pi : \mathbb{R}^n - \{0\} \rightarrow S^{n-1}$ by $\pi(v) = v/|v|$. For a subspace $Z \subset \mathbb{R}^n$, let Z^1 denote the set of unit vectors in Z .

Note that (4.1) implies for all $k > 0$ and $1 \leq i \leq s$,

$$[f|V(i)]^k = \mu_i^k \left[Id + \binom{k}{1} N(i) + \cdots + \binom{k}{n} N(i)^n \right] \quad (4.2)$$

where $N(i)^j = 0$ for $j \geq r(i)$. Let $q(k) = \sum_{j=0}^{n-1} \binom{k}{j}$, a polynomial of degree $(n-1)$ in k . Then (4.2) and our choice of metric yields:

LEMMA 4.3

a) For $v \in V$,

$$|v| \mu^k \leq |f^k(v)| \leq \mu^k q(k) |v|.$$

b) For $w \in W$,

$$|w| |\mu_s|^k \leq |f^k(w)| \leq \lambda^k q(k) |w|. \quad \square$$

Define the arctangent function $a : \mathbb{R}^n - W \rightarrow \mathbb{R}^+$ between V and W by the rule:

For $y \in \mathbb{R}^n$ with $y = v + w$, $v \in V$, $w \in W$, $0 \neq v$,

$$a(y) = \frac{|w|}{|v|}.$$

LEMMA 4.4. For all $y \in \mathbb{R}^n - W$ and $k > 0$,

$$\left(\frac{|\mu_s|}{\mu} \right)^k \frac{1}{q(k)} a(y) \leq a(f^k(y)) \leq \left(\frac{\lambda}{\mu} \right)^k q(k) a(y).$$

Proof. For $y = v + w$, $f^k(y) = v_k + w_k$ where $w_k = f^k w \in W$ and $v_k = f^k v \in V$. Thus, $a(f^k(y)) = |w_k|/|v_k|$ and Lemma 4.3 yields the estimate. \square

The last result needed for the constructions in §5 asserts that Γ^N contains enough elements to map all of the strong expanding manifold V of f into the domain $\mathbb{R}^n - (V \cup W)$. We remark that if it were possible to find a single $g \in \Gamma$ for which $gV \cap V = \{0\}$ and $gV \cap W = \{0\}$, then a much simplified proof of Theorem 2 would be possible along the lines of [5]. As it is, we make do with the following:

PROPOSITION 4.5. *Let $v \in V$ be a non-zero vector. Then there exists $g \in \Gamma^N$ such that*

$$gv \notin V \text{ and } v \cdot gv \neq 0 \tag{4.6}$$

and hence $gv \notin W$.

Proof. Suppose to the contrary that for all $g \in \Gamma^N$, either $gv \in V$ or $v \cdot gv = 0$. These are algebraic conditions on Γ , so by Lemma 3.2 they also hold for all $g \in G^N$. Now G^N is irreducible as it is a connected algebraic group, so either $gv \in V$ for all $g \in G^N$, or $v \cdot gv = 0$ for all $g \in G^N$. Clearly we must have the first case, so $G^N \cdot v \subset V$. Let \tilde{V} denote the span of $G^N v$. Then \tilde{V} is a subspace of V stable under Γ^N , hence $f|_{\tilde{V}}$ is in the commutator group of $\Gamma^N|_{\tilde{V}}$. But the determinant of $f|_{\tilde{V}}$ is $\mu^{\dim \tilde{V}} > 1$, which contradicts $f|_{\tilde{V}}$ being a product of commutators. \square

The condition (4.6) is open for $v \in V$, so given any $v \in V^1$ and $g_v \in \Gamma^N$ satisfying (4.6), there is a $\delta(v) > 0$ so that for the closed $2\delta(v)$ -ball $B(v, 2\delta(v))$ in \mathbb{R}^n centered at v , we have (4.6) is satisfied for g_v and all $y \in B(v, 2\delta(v))$. Since V^1 is compact, we can choose a finite set $\{g_1, \dots, g_d\} \subset \Gamma^N$ and radii $\{\delta_1, \dots, \delta_d\}$ so that the balls $B_i = B(v_i, \delta_i) \cap V^1$ cover V^1 , and (4.6) is satisfied for each g_i with $y \in B(v_i, 2\delta_i)$. Note this implies that for $1 \leq i \leq d$, the arctangent a is defined and bounded away from zero on the set $g_i B(v_i, 2\delta_i)$.

Finally, replacing f with a positive multiple if necessary, we can assume that $\mu > 3$, and for all $1 \leq i \leq d$ both $\mu > |g_i|$ and $\mu > |g_i^{-1}|$. By our choice of metric on \mathbb{C}^n and (4.1), we also have both $|f| < 2\mu$ and $|f^{-1}| < 2\mu$.

§5. Exponential growth on the expanding manifold.

Let \mathcal{G} be the groupoid given in Theorem 2 and Γ the linear group of Jacobians at 0. Assume that Γ is not amenable. Let $f \in \Gamma^{N+1}$ and $\{g_1, \dots, g_d\} \subset \Gamma^N$ be chosen as in §4. Choose $\gamma \in \mathcal{G}$ with $J_0 \gamma = f$, and for each $1 \leq i \leq d$ choose $\gamma_i \in \mathcal{G}$ with $J_0 \gamma_i = g_i$. For notational convenience, set $\gamma_0 = \gamma$. Let $D \subset \mathbb{R}^n$ be an open neighborhood of 0 on which all of the γ_i are defined. Let \mathcal{G}_0 denote the

subgroupoid of \mathcal{G} generated by the set $\{\gamma_0, \dots, \gamma_d\}$. We will show \mathcal{G}_0 has a continuum of orbits with exponential growth.

By the stable manifold theorem (cf. [11]) applied to γ^{-1} , there is a connected submanifold $S \subset D$ with $0 \in S$, the tangent space T_0S at 0 is equal V , and γ^{-1} is uniformly contracting on S . In particular, $\gamma^{-1}S \subset S$. By a change of coordinates on \mathbb{R}^n , we can assume S is an open neighborhood of 0 in V .

Before entering into the details of the proof of Theorem 2, a brief overview of the argument may help the reader. We first define an open cone $C \subset S$ whose points satisfy $\lim_{k \rightarrow \infty} \pi(\gamma^{-k}y) \in V^1$ and $|\gamma^{-k}y| < \mu^{-k/2}$. For an appropriate constant e_0 , we set $y_p = \gamma^{-pe_0}y$ for a given $y \in C$. For each $p > 0$ we construct a subset $\mathcal{R}_p \subset \mathcal{G}_0$ consisting of 2^p words of length $\leq m_0 \cdot p$, such that the linear parts of the words in \mathcal{R}_p move y_p to 2^p distinct points. We furthermore obtain an exponentially decreasing lower bound on the distance between these 2^p points. Using Taylor’s theorem for C^2 -maps, and for e_0 sufficiently large so that y_p is sufficiently small, we conclude that $\mathcal{R}_p \cdot y_p$ consists of 2^p distinct points. The last remark is that in constructing \mathcal{R}_p , we use a version of the “ping-pong” lemma of [5]. In our version, the orbits are repeatedly returned to the attractor V by applying high powers of f , and are then scattered back into $\mathbb{R}^n - (V \cup W)$ by the elements of $\{g_1, \dots, g_d\}$. Thus, all of the orbits we build concentrate on the subspace V , and one does not have the bilateral symmetry inherent in the method of Tits. Instead of 2 players, one can think of this as an instructor with many students.

Recall that for a C^2 -diffeomorphism ϕ with $\phi(0) = 0$, Taylor’s Theorem gives an estimate on the spherical error between ϕ and $J_0\phi$, and the estimate is linear in y :

For all $\epsilon > 0$ sufficiently small, there exists $k(\phi, \epsilon) > 0$ so that

$$\frac{|\phi y - J_0\phi y|}{|y|} < k(\phi, \epsilon) \cdot |y| \quad \text{for all } |y| < \epsilon. \tag{5.1}$$

As an immediate consequence we have:

LEMMA 5.2. *Let $\mathcal{R} = \{\phi_1, \dots, \phi_p\}$ be a set of local C^2 -diffeomorphisms of an open neighborhood U of $0 \in \mathbb{R}^n$ into \mathbb{R}^n with $\phi_i(0) = 0$ for all i . Let $\epsilon > 0$ be sufficiently small so that there exists constants $k(\phi_i, \epsilon)$ for which (5.1) holds. Then for $K = \max_{1 \leq i \leq p} k(\phi_i, \epsilon)$ and $y \in U$ with $|y| < \epsilon$, suppose that*

$$|J_0\phi_i y - J_0\phi_j y| > 2 \cdot K \cdot |y|^2 \quad \text{for all } i \neq j.$$

Then the set $\mathcal{R} \cdot y = \{\phi_i y \mid 1 \leq i \leq p\}$ consist of p distinct points. \square

LEMMA 5.3. *There exists $\delta > 0$ and an integer $b > 0$ such that $|\gamma^{-b}y| < \mu^{-b/2} |y|$ for all $y \in S$ with $|y| < \delta$.*

Proof. By Lemma 4.3 there exists an integer $b > 0$ for which $|f^{-b} | V| < \mu^{-3b/4}$. Choose $\delta > 0$ sufficiently small so that

$$\delta \cdot k(\epsilon, \gamma^{-b}) < \{\mu^{-b/2} - \mu^{-3b/4}\}$$

where ϵ is such that (5.1) holds for γ^{-b} , and $\delta < \epsilon$. Then

$$\begin{aligned} |\gamma^{-b}y| &\leq |\gamma^{-b}y - f^{-b}y| + |f^{-b}y| \\ &\leq |y|^2 \cdot k(\epsilon, \gamma^{-b}) + \mu^{-3b/4} |y| \\ &\leq \mu^{-b/2} |y|. \quad \square \end{aligned}$$

For b, δ as in (5.3) we replace f, γ and μ with f^b, γ^b and μ^b , so we can assume:

$$|\gamma^{-p}y| < \mu^{-p/2} |y| \quad \text{for all } p > 0, y \in S, |y| < \delta \tag{5.4}$$

Choose $\epsilon > 0$ to satisfy $\epsilon < \delta, \epsilon < \mu^{-1}$ and there exists a constant K_0 so that for all $\phi \in \{\gamma, \gamma^{-1}, \gamma_1, \dots, \gamma_d\}$, condition (5.1) holds for all $|y| < \epsilon$ and $k(\phi, \epsilon) = K_0$. Then set

$$C = \{y \in S \mid 0 < |y| < \epsilon\}$$

These remarks are then summarized by

COROLLARY 5.5. $\gamma^{-1}C \subset C$, and for all $p > 0$ and $y \in C$,

$$|\gamma^{-p}y| < \mu^{-p/2} \cdot \epsilon. \quad \square$$

Set $K = \max\{K_0, 2\mu\}$ and $\epsilon_p = \min\{\epsilon, K^{-p}\}$. For a word $\phi = \phi_1 \circ \dots \circ \phi_p$ of length $\leq p$ with each $\phi_i \in \{\gamma_0, \dots, \gamma_d\}$, we estimate the constant $k(\phi, \epsilon_p)$ required for (5.1):

LEMMA 5.6. For ϕ, K and ϵ_p as above

$$|\phi y - J_0 \phi y| < K^{2p} |y|^2 \quad \text{for } |y| < \epsilon_p \tag{5.7}$$

Thus, $K(\phi, \epsilon_p) \leq K^{2p}$.

Proof. For $p = 1$, (5.7) follows from the definition of K . Assume (5.7) holds for

ϕ of length $(p-1)$, and set $\tilde{\phi}_2 = \phi_2 \circ \dots \circ \phi_p$. Then

$$\begin{aligned} |\phi y - J_0 \phi y| &= |\phi_1 \circ \tilde{\phi}_2 y - J_0 \phi_1 \circ J_0 \tilde{\phi}_2 y| \\ &\leq |y|^2 \cdot \{|J_0 \phi_1| \cdot K^{2p-2} + |J_0 \tilde{\phi}_2|^2 \cdot K \\ &\quad + 2 |J_0 \tilde{\phi}_2| |y| K^{2p-1} + |y|^2 K^{4p-3}\}. \end{aligned}$$

From $|J_0 \phi_1| \leq K$, $|J_0 \tilde{\phi}_2| < K^{p-1}$ and $|y| < K^{-p}$ we conclude

$$\begin{aligned} |\phi y - J_0 \phi y| &< |y|^2 \{K^{2p-1} + K^{2p-1} + 2K^{2p-2} + K^{2p-3}\} \\ &= |y|^2 \cdot K^{2p-1} \left\{ 2 + \frac{2}{K^2} + \frac{1}{K^3} \right\} \\ &\leq |y|^2 \cdot K^{2p} \end{aligned}$$

since $K > \mu > 3$. \square

LEMMA 5.8. For $g \in \{f, f^{-1}, g_1, \dots, g_d\}$ and all $u_1, u_2 \in \mathbb{R}^n$, $|gu_1| > \frac{1}{2\mu} |u_1|$ and $|gu_1 - gu_2| > \frac{1}{2\mu} |u_1 - u_2|$.

Proof. $|g| < 2\mu$, so $|gw| < 2\mu \cdot |w|$ and hence for $w = g^{-1}u_1$ or $w = g^{-1}(u_1 - u_2)$ we get the estimate. \square

Recall that $\{B_i = B(v_i, \delta_i) \mid 1 \leq i \leq d\}$ is the covering of V^1 by closed balls in V defined at the end of §4. By compactness of the sets $g_i B_i(v_i, 2\delta_i)$ and the continuity of the arctangent function a on them, there exists constants $0 < c_1 < c_2$ for which $c_1 < a(g_i y) < c_2$ for all $1 \leq i \leq d$ and $y \in B(v_i, 2\delta_i)$.

Set $X = \{x \in \mathbb{R}^n \mid |x| = 1 \text{ and } c_1 \leq a(x) \leq c_2\}$.

For $\delta > 0$, set

$$A(\delta) = \{x \in \mathbb{R}^n \mid |x| = 1 \text{ and } a(x) < \delta\}$$

$$A_i(\delta) = \{x \in A(\delta) \mid x = v + w, v \in B_i, w \in W\}$$

Note the sets $\{A_1(\delta), \dots, A_d(\delta)\}$ cover $A(\delta)$. Choose $\delta_0 > 0$ sufficiently small so that for all $1 \leq i \leq d$, $g_i A_i(2\delta_0) \subset X$. Lemma 4.4 implies there exists an integer e for which $f^p(X) \subset A(\delta_0)$ for all $p \geq e$. Set $m_0 = 2d \cdot e + 1$, and define

$$c_1 = \infimum_{\substack{y, z \in X \\ 1 \leq i < j \leq 2d}} |\pi f^{i \cdot e} y - \pi f^{j \cdot e} z|.$$

Choose $e_0 > 1$ so that for all $p \geq 1$,

$$\mu^{(2m_0 - e_0/2)p} < \frac{c_1}{2K^{2p} \cdot \epsilon \cdot 2^{2 \cdot p \cdot m_0}} \quad (5.9)$$

and

$$\mu^{e_0} > K^4. \quad (5.10)$$

For all non-zero $y \in C$ we now show the groupoid \mathcal{R}_0 has exponential orbit growth on y . Fix a choice of $0 \neq y \in C$. For $p > 0$ set $y_p = \gamma^{-p \cdot e_0} y$. By Lemma 5.5 and (5.10) we have $|y_p| < K^{-2p} \epsilon < \epsilon_p$, and then (5.9) yields

$$\begin{aligned} 2|y_p|^2 K^{2p} &\leq 2K^{2p} \cdot |y_p| \cdot \mu^{-p \cdot e_0/2} \cdot \epsilon \\ &\leq \frac{c_1}{(2\mu)^{2p \cdot m_0}} |y_p|. \end{aligned}$$

We can now define the set \mathcal{R}_p , which consists of 2^p words of length $\leq p \cdot m_0$ in \mathcal{R}_0 . The set \mathcal{R}_p will be chosen so that for all $\phi \neq \psi \in \mathcal{R}_p$,

$$|J_0 \phi y_p - J_0 \psi y_p| > \frac{c_1}{(2\mu)^{2p \cdot m_0}} |y_p|. \quad (5.12)$$

By Lemma 5.2 and (5.11), the set $\mathcal{R}_p y_p = \{\phi y_p \mid \phi \in \mathcal{R}_p\}$ consists of 2^p distinct points. Thus, $\mathcal{R}_p \cdot \gamma^{-p e_0}$ consists of words of length $\leq (m_0 + e_0)p$, and applied to y yields 2^p distinct orbits. Since $y_p \rightarrow 0$, this will finish the proof of Theorem 2.

Fix p , choose i_0 with $\pi y_p \in B(i_0)$, and consider the $2d$ points

$$F_1 = \{\pi f^{e \cdot k} g_{i_0} y_p \mid 1 \leq k \leq 2d\} \subset A(\delta_0).$$

There exists an integer i_1 with $1 \leq i_1 \leq d$ for which $Q_1 = F_1 \cap A_{i_1}$ contains at least 2 points.

Now proceed inductively, and suppose i_{q-1} , F_{q-1} and Q_{q-1} have been chosen with $Q_{q-1} = F_{q-1} \cap A_{i_{q-1}}$ and $\#Q_{q-1} \geq 2^{q-1}$. The set

$$F_q = \{\pi f^{e \cdot k} g_{i_{q-1}} Q_{q-1} \mid 1 \leq k \leq 2d\} \subset A(\delta_0)$$

consists of at least $2d \cdot 2^{q-1}$ points, since

$$g_{i_{q-1}} A_{i_{q-1}} \subset X \quad \text{and} \quad f^{e \cdot k}(X) \cap f^{e \cdot j}(X) = \emptyset \quad \text{for} \quad j \neq k.$$

Therefore, there exists i_q with $Q_q = F_q \cap A_{i_q}$ containing at least 2^q points. This completes the inductive step.

Let F_p be the set obtained in this inductive fashion; let \mathcal{R}_p be the set of words in $\{\gamma_0, \dots, \gamma_d\}$ corresponding to the words in $\{f, g_1, \dots, g_d\}$ which are applied to y_p to obtain the points in F_p . A typical element of \mathcal{R}_p has the form

$$\phi = y^{e \cdot k_p} \circ \gamma_{i_{p-1}} \circ \gamma^{e \cdot k_{p-1}} \circ \gamma_{i_{p-2}} \circ \dots \circ \gamma_{i_0}$$

for some integers $1 \leq k_1, \dots, k_p \leq 2d$. The length of ϕ is at most $p \cdot m_0$ with respect to the set $\{\gamma_0, \dots, \gamma_p\}$, and $\mathcal{R}_p y_p$ consists of at least $2d \cdot 2^{p-1} \geq 2^p$ points, once we have established the estimate (5.12).

Let $\phi \neq \psi \in \mathcal{R}_p$ and let $g = J_0 \phi$, $h = J_0 \psi$ be their linear parts. There are integers $1 \leq k_1, \dots, k_p \leq 2d$ and $1 \leq j_1, \dots, j_p \leq 2d$ for which

$$\begin{aligned} g &= f^{e \cdot j_p} g_{i_{p-1}} \cdots f^{e \cdot j_1} g_{i_0} \\ h &= f^{e \cdot k_p} g_{i_{p-1}} \cdots f^{e \cdot k_1} g_{i_0} \end{aligned}$$

Let q be the largest integer such that $j_{q-1} \neq k_{q-1}$. Set

$$\begin{aligned} \xi &= f^{e \cdot k_p} g_{i_{p-1}} \cdots f^{e \cdot k_q} g_{i_{q-1}} \\ g' &= \xi^{-1} g, \quad h' = \xi^{-1} h \end{aligned}$$

Apply Lemma 5.8 at most $q \cdot m_0$ times to obtain

$$\begin{aligned} |gy_p - hy_p| &= |\xi(g'y_p - h'y_p)| \\ &\geq (2\mu)^{-qm_0} |g'y_p - h'y_p|. \end{aligned}$$

Next, g' and h' have length $\leq pm_0$, so Lemma 5.8 again yields

$$\min\{|g'y_p|, |h'y_p|\} \geq (2\mu)^{-pm_0} |y_p|.$$

Hence,

$$|g'y_p - h'y_p| \geq (2\mu)^{-pm_0} |y_p| \cdot |\pi g'y_p - \pi h'y_p| \geq (2\mu)^{-pm_0} \cdot |y_p| \cdot c_1$$

and so

$$|gy_p - hy_p| \geq (2\mu)^{-2pm_0} |y_p| \cdot c_1. \quad \square$$

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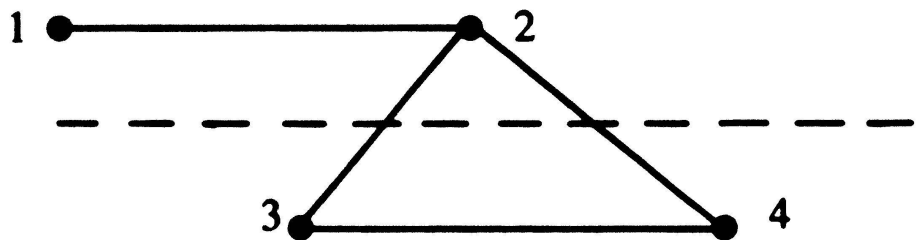
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