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Semisimple group actions on the three dimensional affine space are linear

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Introduction

It is an interesting question in the theory of algebraic transformation groups over a field of characteristic zero whether an action of a reductive group on the affine space $\mathbb{A}^n$ is equivalent to a linear representation. In his paper [Ka] Kambayashi conjectures that this is always the case. (His conjecture more generally concerns actions of linearly reductive groups on $\mathbb{A}^n$ in any characteristic.) In the case of the affine plane $\mathbb{A}^2$ we have a fairly complete picture (cf. [Ka] §4). In particular every action of a reductive group on $\mathbb{A}^2$ is linearizable, but in general even the weaker problem of existence of fixed points is unsettled (cf. [Kr2] for some results in this direction). The best results so far known have been obtained for tori (cf. [BB]): Here there are always fixed points, and an effective action of an n-dimensional or an $(n-1)$-dimensional torus on $\mathbb{A}^n$ is equivalent to a linear one.

Another general result is a consequence of Luna's slice theorem (cf. [Lu]): If any G-invariant function on $\mathbb{A}^n$ is a constant (e.g. if G has a dense orbit in $\mathbb{A}^n$) then the action is equivalent to a linear one (cf. Proposition 5.1).

Our main goal here will be to settle the case of a semisimple group acting on $\mathbb{A}^3$ (Theorem 5.2):

THEOREM. *Let G be a connected semisimple group over a field of characteristic zero. Then any regular action of G on $\mathbb{A}^3$ is equivalent to a linear one.*

It is known that the group Aut $\mathbb{A}^n$ has the structure of a infinite dimensional algebraic group ([Sh], [Ka]). One can interpret our result as an analogue of the classical Levi–Maltzev theorem, i.e. *any two maximal connected semisimple (finite dimensional) subgroups of* Aut $\mathbb{A}^3$ *are conjugate.*

Our main tools come from geometric invariant theory and representation theory of algebraic groups, in particular of $SL_2(\mathbb{C})$. At several places we use some

Euler characteristic arguments in the case of algebraic varieties; in an appendix we develop the required results.*

It is obvious how to generalize some of our results, but we have not attempted to give the most general statements, since we wanted to keep the arguments as simple as possible.

Remark. Using similar methods D. I. Panyushev [Pa] has recently extended our result to the 4-dimensional affine space.

1. Some notations and general facts

1.1. We always work in the category of algebraic varieties over the field $\mathbb{C}$ of complex numbers. Of course we could replace $\mathbb{C}$ by any other algebraically closed field of characteristic zero.

For any variety X we denote by $\mathcal{O}(X)$ the algebra of (global) regular functions on X.

1.2. Given an algebraic group G and a regular action of G on an affine variety X (shortly an affine G-variety) a morphism $\pi: X \to Y$ is called an *algebraic quotient* if Y is affine and π induces an isomorphism of $\mathcal{O}(Y)$ with the algebra $\mathcal{O}(X)^G$ of G-invariant functions on X:

$$\pi^*: \mathcal{O}(Y) \xrightarrow{\sim} \mathcal{O}(X)^G.$$

For a reductive group G such a quotient exists and is unique up to isomorphism: In this case $\mathcal{O}(X)^G$ is finitely generated and we can take $Y = X/G := \operatorname{spec} \mathcal{O}(X)^G$ the maximal spectrum of $\mathcal{O}(X)^G$. *In addition π is surjective and each fibre of π contains exactly one closed orbit. Moreover the image of a closed G-stable subset of X is closed in X/G.*

If X is a linear representation of G the fibre $\pi^{-1}(\pi(0))$ is a cone and is usually called *zero fibre* or *nilpotent cone* (since in case of the adjoint representation on Lie G it is exactly the set of nilpotent elements). For these results and other general properties of algebraic quotients we refer to the literature ([Kr], [MF]).

An important result from the theory of algebraic transformation groups which plays a central role in our analysis is the *slice theorem* of Luna, analog to the well known slice theorem in the theory of compact transformation groups ([Lu], cf.

* We thank the referee for his suggestions concerning this appendix.

[Lu2]). One consequence is the existence of a *finite stratification* of the quotient X/G of a smooth variety X into locally closed smooth subvarieties $(X/G)_\lambda$ such that $\pi^{-1}((X/G)_\lambda) \to (X/G)_\lambda$ is a G-fibration (in the étale topology). As usual the strata $(X/G)_\lambda$ are determined by the slice representation associated to the closed orbit in the fibre.

1.3. The group SL_2 is of special interest for us. Let us fix the following notations for some subgroups of SL_2.

$$T := \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \,\middle|\, t \in \mathbb{C} \right\},$$

$$N := \mathrm{Norm}_{SL_2} T = T \cup \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} T \quad \text{the normalizer of } T,$$

$$U_n := \left\{ \begin{pmatrix} \zeta & r \\ 0 & \zeta^{-1} \end{pmatrix} \,\middle|\, \zeta, r \in \mathbb{C}, \zeta^n = 1 \right\} \quad \text{for} \quad n = 1, 2, \ldots,$$

$$B := \left\{ \begin{pmatrix} t & r \\ 0 & t^{-1} \end{pmatrix} \,\middle|\, t \in \mathbb{C}^*, r \in \mathbb{C} \right\}.$$

It is well known that any closed subgroup $H \subset SL_2$ is finite or conjugate to T, B, N or some U_n. In particular a one dimensional orbit under SL_2 is of the form $SL_2/B \simeq \mathbb{P}^1$ and can therefore not appear in an affine SL_2-variety.

1.4. The representation theory for SL_2 is rather simple: For any integer i there is up to isomorphism exactly one simple SL_2-module V_i of dimension $i+1$, e.g. the binary forms of degree i with the usual action of SL_2 by substitution of variables. In particular V_2 is isomorphic to the adjoint representation on the Lie algebra $\mathfrak{sl}_2$.

Given an affine SL_2-variety X the induced representation of SL_2 on the coordinate ring $\mathcal{O}(X)$ is locally finite, and we may consider the decomposition into isotypic components:

$$\mathcal{O}(X) = \bigoplus_{i=0}^{\infty} \mathcal{O}(X)_i, \qquad \mathcal{O}(X)_i := \sum_{\substack{W \subset \mathcal{O}(X) \\ W \simeq V_i}} W.$$

Each isotypic component $\mathcal{O}(X)_i$ is a finitely generated module over the ring $\mathcal{O}(X)^{SL_2} = \mathcal{O}(X)_0$ of invariants. In particular if $\mathcal{O}(X)^{SL_2} = \mathbb{C}$ the isotypic components $\mathcal{O}(X)_i$ are finite dimensional and the *multiplicities* $m_i(X)$ of V_i in $\mathcal{O}(X)$ are given by

$$m_i(X)=\frac{\dim \mathcal{O}(X)_i}{i+1}.$$

1.5. LEMMA. *Consider an* SL_2*-action on* $\mathbb{A}^3$ *and a two dimensional orbit* $O\subset\mathbb{A}^3$.

(a) *If* O *is closed then* O *is isomorphic to* SL_2/T *or to* SL_2/N.

(b) *If* O *is not closed then* $\bar{O}=O\cup\{z\}$ *with a fixed point* z*, and either* (i) $O \xrightarrow{\sim} SL_2/U_1$ *and* $\bar{O}\xrightarrow{\sim} V_1$ *or* (ii) $O\xrightarrow{\sim} SL_2/U_2$ *and* $\bar{O}\xrightarrow{\sim} C$*, where* C *is the nilpotent cone in* $V_2\cong\mathfrak{sl}_2$.

((a) follows from the fact that an orbit O of a point x in an affine G-variety can be closed only if the stabilizer G_x is reductive, and it is closed if the stabilizer G_x contains a maximal torus (cf. [Kr] III.4.1 Lemma 3). (b) is an easy consequence of Luna's slice theorem, since the tangent representation in a fixed point is isomorphic to V_2 or $V_0\oplus V_1$.)

Remark. For the multiplicities in the different cases of the lemma one finds:

$$m_i(SL_2/T)=m_i(C)=\begin{cases}1 & \text{for } i \text{ even},\\ 0 & \text{for } i \text{ odd};\end{cases}$$

$$m_i(SL_2/N)=\begin{cases}1 & \text{for } i\equiv 0 \bmod 4,\\ 0 & \text{otherwise};\end{cases}$$

$m_i(V_1)=1$ for all i.

1.6. We repeatedly use the following result due to R. W. Richardson (cf. [Kr] II.3.4 Lemma).

LEMMA. *Let* $\varphi: X\to Y$ *be a birational morphism between irreducible affine varieties. Assume that* Y *is normal and that* $\operatorname{codim}_Y \overline{(Y-\varphi(X))}\geqslant 2$. *Then* φ *is an isomorphism.*

As a consequence we see that for any affine variety X a non empty affine open subset $U\neq X$ has a complement of codimension 1, and this complement is even a hypersurface if X is factorial. (In fact the union of the irreducible components of $X-U$ of codimension 1 is the zero set of a function f on X. Hence U is open in the affine variety $X_f:=\{x\in X \mid f(x)\neq 0\}$ with a complement of codimension $\geqslant 2$, and so $U=X_f$.)

1.7. In some arguments we will use the *Euler–Poincaré characteristic* of an algebraic variety, calculated in ordinary cohomology (with respect to the $\mathbb{C}$-topology) or what amounts to the same thing in cohomology with compact support. The following facts seem to be well known, but we couldn't find a suitable reference. In the appendix we outline the proofs of these results.

(a) *If $Y \subset X$ is a locally closed subvariety then $\chi(X) = \chi(X - Y) + \chi(Y)$*

(b) *If $\varphi : X \to Y$ is a fibration of algebraic varieties* (in the étale topology) *with fibre F then $\chi(X) = \chi(Y) \cdot \chi(F)$.*

As a consequence one easily gets the following: If G is a connected reductive group, $G \neq \{e\}$, $T \subset G$ a maximal torus and $B \subset G$ a Borel subgroup then $\chi(G) = 0$ and $\chi(G/T) = \chi(G/B)$ is the order $|W|$ of the Weyl group W of G. (Use the fibrations $G \to G/T$ and $G/T \to G/B$, and the Bruhat decomposition of G/B.) In particular $\chi(\mathrm{SL}_2/T) = 2$ and $\chi(\mathrm{SL}_2/N) = 1$.

2. Actions with one dimensional quotient and multiplicities

2.1. LEMMA. *Consider a non trivial action of* SL_2 *on* $\mathbb{A}^3$. *Then* $\mathbb{A}^3/\mathrm{SL}_2 \cong \mathbb{A}^1$ *and the quotient map is flat with reduced and irreducible fibres. In addition the orbits in* $\mathbb{A}^3$ *are two dimensional or fixed points.*

Proof. We first remark that $\mathbb{A}^3$ does not contain a 3-dimensional orbit O. Such an orbit would be affine, since $O \cong \mathrm{SL}_2/\text{finite}$. Hence $O = \mathbb{A}^3$ or $\mathbb{A}^3 - O$ is a hypersurface (1.6). The first case contradicts the fact that $\chi(\mathbb{A}^3) = 1$ and $\chi(O) = 0$ (1.7). In the second case the hypersurface $\mathbb{A}^3 - O$ is the zero variety of an SL_2-invariant function (since every character $\mathrm{SL}_2 \to \mathbb{C}^*$ is trivial). But any invariant function is a constant because O is a dense orbit in $\mathbb{A}^3$.

So we see that all orbits in $\mathbb{A}^3$ are 2-dimensional or fixed points (cf. 1.3), and in particular $\dim \mathbb{A}^3/\mathrm{SL}_2 \leq 1$. Since the closure of a 2-dimensional orbit is the zero variety of an invariant function we have $\dim \mathbb{A}^3/\mathrm{SL}_2 = 1$. It follows that $\mathbb{A}^3/\mathrm{SL}_2$ is a unirational normal curve which clearly implies $\mathbb{A}^3/\mathrm{SL}_2 \cong \mathbb{A}^1$. Hence there is an invariant function $f \in \mathcal{O}(\mathbb{A}^3)$ generating the ring of invariants and thus the quotient map is flat. By a degree argument we see that for all $\lambda \in \mathbb{C}$ the polynomial $f - \lambda$ is irreducible which proves the remaining claims. Q.E.D.

2.2. *Remark. All fibres of the quotient map* $\pi : \mathbb{A}^3 \to \mathbb{A}^3/\mathrm{SL}_2$ *have the same multiplicities.* In fact the isotypic component R_i of the coordinate ring $R := \mathcal{O}(\mathbb{A}^3)$ is a finitely generated torsion free module over the polynomial ring $R_0 = R^{\mathrm{SL}_2}$ (1.4), hence a free module of finite rank. This rank is the multiplicity $m_i(F)$ of any fibre, since F is reduced. Using Remark 1.5 we see that we are in one of the following cases:

(1) *All fibres are isomorphic to* SL_2/T *or* to C;

(2) *All fibres are isomorphic to* V_1;

(3) *All fibres are isomorphic to* SL_2/N.

It will turn out that the last case cannot occur (Lemma 2.1).

2.3. *Remark.* The lemma above can be generalized in two ways:

(a) For any non trivial action of SL_2 on an affine variety Z with $\chi(Z) \neq 0$ there exists a 2-dimensional orbit. (We have seen above that a 3-dimensional orbit under SL_2 is affine. It has to be closed since otherwise it contains a 2-dimensional orbit (cf. 1.6 and 1.3.) It follows that the fibres of the quotient map $\pi : Z \to Z/SL_2$ are precisely the orbits (1.2), hence have Euler-characteristic $\chi(F) = 0$. This implies $(Z) = 0$ (cf. 1.7) in contrast to the assumption.)

(b) Let G be a connected semisimple group acting on $\mathbb{A}^n$. Assume that there exists a $(n-1)$-dimensional orbit. Then $\mathbb{A}^n/G \cong \mathbb{A}^1$ and the quotient map $\pi : \mathbb{A}^n \to \mathbb{A}^n/G$ is flat with irreducible and reduced fibres.

3. Existence of fixed points

3.1. LEMMA. *Any action of* SL_2 *on* $\mathbb{A}^3$ *has fixed points. If the action is non trivial then* $\dim (\mathbb{A}^3)^{SL_2} \leq 1$.

Proof. If there are no fixed points then by Lemma 2.1 all orbits are closed and of dimension 2. Hence all fibres of the quotient map $\pi : \mathbb{A}^3 \to \mathbb{A}^3/SL_2 \cong \mathbb{A}^1$ are isomorphic (Remark 2.2). Therefore there is only one stratum in the Luna-stratification (1.2) and so π is a fibration with fibre isomorphic to SL_2/T or SL_2/N. The first case is not possible since $\chi(SL_2/T) = 2$ which would imply $\chi(\mathbb{A}^3) = 2$ (cf. 1.7). In the second case we get an isomorphism $SL_2/N \times (\mathbb{A}^3)^N \xrightarrow{\sim} \mathbb{A}^3$ induced by $(g, x) \mapsto gx$. This is also a contradiction since SL_2/N is not simply connected. For the last assertion we remark that any fibre of π contains at most one fixed point (1.2) and so π restricted to the fixed point set $(\mathbb{A}^3)^{SL_2}$ is injective. Q.E.D.

3.2. The following generalisation of 3.1 due to Panyushev ([Pa] 3.2 Theorem 7) has been indicated to us by A. Borel (cf. [B] Chap XIV).

LEMMA. *An action of* SL_2 *on the affine space* $\mathbb{A}^n$ *without* 3-*dimensional orbits has fixed points.*

Proof. If there are no fixed points every orbit is closed and of the form SL_2/T

or SL_2/N (1.5). But then there is a unique SL_2-equivariant map $\varphi : \mathbb{A}^n \to SL_2/N$; it is given by $\varphi(x) = gN$, where $g \in SL_2$ is such that $gx \in (\mathbb{A}^n)^T$. (It follows from the slice theorem that φ is a morphism.) Since $\chi(SL_2/T) = 2$ not all orbits are of the form SL_2/T, hence φ has a section. Therefore we get an inclusion $\varphi^* : H^*(SL_2/N) \hookrightarrow H^*(\mathbb{A}^3)$ in singular cohomology, which contradicts the well known fact that $H^2(SL_2/N) \neq \{0\}$. Q.E.D.

4. Linearization of SL_2-actions on $\mathbb{A}^3$

4.1. LEMMA. *Consider an* SL_2*-action on* $\mathbb{A}^3$. *Then the coordinate ring* $A := \mathcal{O}(\mathbb{A}^3)$ *is a free module over the invariant ring* A^{SL_2}, *and there is a* SL_2-submodule $M \subset A$ *such that the canonical map*

$$A^{SL_2} \otimes M \to A, \qquad f \otimes m \mapsto fm,$$

is an isomorphism.

Proof. Each isotypic component A_i of A is a finitely generated torsionfree module over the polynomial ring A^{SL_2} (1.4), hence a free module. We claim that for all i there is a finite dimensional SL_2-submodule $M_i \subset A_i$ such that the canonical map $A^{SL_2} \otimes M_i \xrightarrow{\sim} A_i$ is an isomorphism. In fact consider the subspace $A_i^{U_1}$ of all U_1-invariants (i.e. the set of the highest weight vectors) which is also a free A^{SL_2}-module of finite rank, and let $f_1, f_2, \ldots, f_m$ be a basis over A^{SL_2}. Then each f_j generates an irreducible submodule $\langle SL_2 \cdot f_j \rangle \subset A_i$ isomorphic to V_i and the sum $M_i = \sum_{j=1}^m \langle SL_2 \cdot f_j \rangle$ is direct, since the f_j are linearly independent highest weight vectors. Furthermore the SL_2-homomorphism $A^{SL_2} \otimes M_i \to A_i$ is an isomorphism because it becomes an isomorphism if we restrict it to the U_1-invariants. Now the claim follows with $M := \bigoplus_i M_i$. Q.E.D.

4.2. If $V \subset \mathcal{O}(\mathbb{A}^3)$ is any finite dimensional submodule, we have a canonical SL_2-equivariant morphism

$$\varphi : \mathbb{A}^3 \to V^*$$

given by $(\varphi(x))(\alpha) := \alpha(x)$ for $x \in \mathbb{A}^3$, $\alpha \in V \subset \mathcal{O}(\mathbb{A}^3)$. As a consequence of the lemma above we get the following useful result:

LEMMA. *If* $M \subset \mathcal{O}(\mathbb{A}^3)$ *is as in Lemma* 4.1 *and* $V \subset M$ *a finite dimensional* SL_2*-submodule, the canonical map* $\varphi : \mathbb{A}^3 \to V^*$ *is* SL_2*-equivariant and nonconstant on the fibres of the quotient map* $\pi : \mathbb{A}^3 \to \mathbb{A}^3/SL_2$.

Proof. In fact for any fibre F of π the restriction map $A \to \mathcal{O}(F)$ induces an isomorphism $M \xrightarrow{\sim} \mathcal{O}(F)$ of SL_2-modules, and so V cannot be in the kernel of the restriction map. Q.E.D.

4.3. We now consider the case where the fixed point set $(\mathbb{A}^3)^{\mathrm{SL}_2}$ is of dimension 1.

PROPOSITION. *Consider an* SL_2*-action on* $\mathbb{A}^3$ *with one dimensional fixed point set. Then this action is equivalent to the linear action of* SL_2 *on* $V_0 \oplus V_1$ (1.4).

Proof. We have already seen that the restriction of $\pi: \mathbb{A}^3 \to \mathbb{A}^3/\mathrm{SL}_2$ to the fixed point set $X := (\mathbb{A}^3)^{\mathrm{SL}_2}$ is injective (proof of Lemma 3.1). Hence by assumption $\overline{\pi(X)} = \mathbb{A}^3/\mathrm{SL}_2$. But $\pi(X)$ is closed, since X is closed and SL_2-stable (1.2) and so π induces an isomorphism

$$(\mathbb{A}^3)^{\mathrm{SL}_2} \xrightarrow{\sim} \mathbb{A}^3/\mathrm{SL}_2 \cong \mathbb{C}.$$

Now it follows from the slice theorem that the tangent representation in any fixed point is isomorphic to $V_0 \oplus V_1$ and that any fibre of π is isomorphic to V_1.

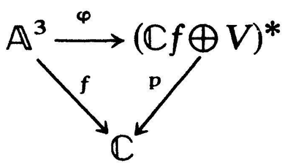
Let $f \in A := \mathcal{O}(\mathbb{A}^3)$ be a generator for the invariant ring A^{SL_2} (Lemma 2.1) and let z be the fixed point in the fibre $F := f^{-1}(0)$, i.e. $f \in \mathfrak{m}_z$ where $\mathfrak{m}_z$ denotes the maximal ideal in A corresponding to z. Now

$$\mathfrak{m}_z/\mathfrak{m}_z^2 \cong T_z(\mathbb{A}^3)^* \cong V_0 \oplus V_1$$

and so $f \notin \mathfrak{m}_z^2$. Since for the submodule $M \subset A$ of Lemma 4.1 the canonical map $M \xrightarrow{\sim} A/fA$ is an SL_2-isomorphism, there is a submodule $V \subset M$ isomorphic to V_1, such that $\mathfrak{m}_z = (\mathbb{C}f \oplus V) \oplus \mathfrak{m}_z^2$. It follows that the induced SL_2-equivariant map

$$\varphi: \mathbb{A}^3 \to (\mathbb{C}f \oplus V)^* \cong V_0 \oplus V_1$$

is étale at z and non constant on the fibres of π (4.2). This implies that φ maps each fibre isomorphically onto its image, since a non constant SL_2-equivariant map $V_1 \to V_1$ is an isomorphism. It is easy to check from the definition of φ that the following diagram commutes:



(p is defined by $p(\alpha) := \alpha(f)$). Since f and p are quotient maps it follows from the above that φ is bijective and hence an isomorphism (1.6). Q.E.D.

4.4. PROPOSITION. *Any* SL_2*-action on* $\mathbb{A}^3$ *with finitely many fixed points is equivalent to the linear action on* V_2*. In particular there is exactly one fixed point.*

Proof. The slice theorem shows that the tangent representation in any isolated fixed point of $\mathbb{A}^3$ is isomorphic to V_2. In particular the fibre of $\pi : \mathbb{A}^3 \to \mathbb{A}^3/\mathrm{SL}_2$ through any fixed point is isomorphic to C, the nilpotent cone in V_2, and by remark 2.2 the other fibres are isomorphic to SL_2/T. We first show that there is exactly one fixed point. Put $\pi((\mathbb{A}^3)^{\mathrm{SL}_2}) = \{y_1, y_2, \ldots, y_s\}$ and $Y' := \mathbb{A}^3/\mathrm{SL}_2 - \{y_1, y_2, \ldots, y_s\}$. Since the induced map $\pi^{-1}(Y') \to Y'$ is a fibration (in the étale topology; 1.2) the Euler characteristic is given by

$$\chi(\pi^{-1}(Y')) = \chi(Y') \cdot \chi(\mathrm{SL}_2/T) = (1-s) \cdot 2$$

(cf. 1.6). Furthermore $\pi^{-1}(y_i) \cong C$ is contractible, hence $\chi(\pi^{-1}(y_i)) = 1$. It follows

$$1 = \chi(\mathbb{A}^3) = \sum_{i=1}^{s} \chi(\pi^{-1}(y_i)) + \chi(\pi^{-1}(Y')) = s + (1-s) \cdot 2 = 2 - s$$

which implies the claim. By lemma 2.1 we have $A^{\mathrm{SL}_2} = \mathbb{C}[f]$, $A := \mathcal{O}(\mathbb{A}^3)$. We may assume that $f(z) = 0$ for the unique fixed point $z \in \mathbb{A}^3$. We have already seen that $\mathfrak{m}_z/\mathfrak{m}_z^2 = T_z(\mathbb{A}^3) = V_2$ as SL_2-modules, and so $f \in \mathfrak{m}_z^2$. With the notations of lemma 4.1 this implies $f \cdot \mathbb{C}[f] \cdot M \subset \mathfrak{m}_z^2$, hence $\mathfrak{m}_z^2 + M = A$. Therefore there is a submodule $V \subset M$ isomorphic to V_2 such that $\mathfrak{m}_z = V \oplus \mathfrak{m}_z^2$. The induced map

$$\varphi : \mathbb{A}^3 \to V^*$$

is SL_2-equivariant, étale in z and non constant on the fibres of the quotient map $\pi : \mathbb{A}^3 \to \mathbb{A}^3/\mathrm{SL}_2$ (cf. 4.2). Again this implies that φ maps each fibre of π isomorphically onto its image, and in particular $\varphi^{-1}(0) = \{z\}$. In fact this is clear for the fibres of the form SL_2/T. Furthermore any non constant map $\psi : C \to C$ is bijective on the dense orbit and maps the fixed point onto itself, hence it is an isomorphism since C is normal (1.6). By construction φ is étale in z and so the induced map $\bar{\varphi} : \mathbb{A}^3/\mathrm{SL}_2 \to V^*/\mathrm{SL}_2$ is étale in $\pi(z)$ ([Lu2] 1.3 lemme fondamental). Furthermore $\bar{\varphi}^{-1}(\pi(0)) = \{\pi(z)\}$ since $\varphi^{-1}(0) = \{z\}$. But this implies that $\bar{\varphi}$ is an isomorphism because $\mathbb{A}^3/\mathrm{SL}_2$ and V^*/SL_2 are both isomorphic to $\mathbb{A}^1$. As a consequence φ is bijective hence an isomorphism (1.6). Q.E.D.

4.5. *Remark.* We want to give a variant of the proof above which has been used by Panyushev [Pa]. The map $\varphi : \mathbb{A}^3 \to V^*$ constructed as above induces a commutative diagram:

$$\begin{array}{ccc} \mathbb{A}^3 & \xrightarrow{\varphi} & V^* \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ \mathbb{A}^3/\mathrm{SL}_2 & \xrightarrow{\bar{\varphi}} & V^*/\mathrm{SL}_2 \end{array}$$

Since φ maps each fibre of π isomorphically onto its image (see above) it is enough to prove that $\bar{\varphi}$ is an isomorphism. We can identify $\mathbb{A}^3/\mathrm{SL}_2$ and V^*/SL_2 with $\mathbb{C}$ and V^* with $\mathbb{C}^3$ in such a way that π' is given by $(x, y, z) \mapsto x^2 + y^2 + z^2$ and $\bar{\varphi}$ by a polynomial $h(t)$. Now consider the fibre product

$$Z = \mathbb{A}^3/\mathrm{SL}_2 \times_{V^*/\mathrm{SL}_2} V^* := \{(t, x, y, z) \in \mathbb{C}^4 \mid h(t) = x^2 + y^2 + z^2\}.$$

It is easy to see that φ induces a bijection $\tilde{\varphi} : \mathbb{A}^3 \to Z$. Since Z is normal (by Serre's criterion; cf. [Kr] AI.6.2) $\tilde{\varphi}$ is an isomorphism (1.6); in particular $h(t)$ has no multiple roots. Now we can apply [Ko] to our situation and obtain $\chi(Z) = 2 - \deg h$. Since $Z \cong \mathbb{A}^3$ we get $\deg h = 1$ which finishes the proof.

5. The general case

5.1. For the proof of our main theorem we need the following result.

PROPOSITION. *Let G be a reductive group operating on $\mathbb{A}^n$. If every G-invariant function on $\mathbb{A}^n$ is a constant then the action is equivalent to a linear one.*

Proof. We have only to show that there is a fixed point in $\mathbb{A}^n$, since in this case the claim is a consequence of the slice theorem ([Lu] III, corollaire 2). By assumption we have a unique closed orbit Gx with reductive stabilizer $H := G_x$, and hence again by the slide theorem (loc. cit.) we have a G-isomorphism

$$G *^H U \xrightarrow{\sim} \mathbb{A}^n,$$

where U is the slice representation of H in x and $G *^H U$ is a vector-bundle over G/H. In particular G/H is connected and simply connected and $\chi(G/H) = \chi(\mathbb{A}^n) = 1$. So we may assume that G is connected (replace G by G^0). Then H has to be connected too. If T is a maximal torus of H we get

$$1 = \chi(G/H) = \frac{\chi(G/T)}{\chi(H/T)}.$$

Considering the fibration $G/T \to G/T'$, where $T' \supset T$ is a maximal torus of G, it follows that $T' = T$ and that the two groups have the same Weyl group (cf. 1.7). Since both are connected this is possible only for $H = G$. Q.E.D.

The following corollary has been pointed out to us by the referee.

COROLLARY. *For $n > 0$ the affine space $\mathbb{A}^n$ cannot be a homogeneous space under a reductive group.*

5.2. THEOREM. *Any action of a connected semisimple group G on $\mathbb{A}^3$ is equivalent to a linear action.*

Proof. The case $\operatorname{rk} G = 1$ has been settled in section 4. In view of the proposition above it is enough to show that any effective action of a semisimple group G of rank $\geqq 2$ on $\mathbb{A}^3$ has a dense orbit. This is a consequence of the results in [P] (§3, theorem 5). Q.E.D.

Remark. It is possible to avoid the use of [P] in the proof above by the following argument due to the referee: Assume $\operatorname{rk} G \geqq 2$ and choose a simple subgroup $H \subset G$ of rank 1. By the first part of the proof $\mathbb{A}^3$ is H-isomorphic to V_2 or to $V_1 \oplus V_0$. If $T_1 \subset G$ is a one dimensional torus stabilizing H we see that in the first case T_1 acts by homotheties on V_2. Hence G has a dense orbit which implies the claim by 5.1. In the second case we consider the H-stable subspace $V_1 \subset V_1 \oplus V_0$. If it is not stable under G then G has a dense orbit and we are done by 5.1. If it is stable under G then G acts linearly on V_1 (5.1), and the slice representation in the fixed point shows that G is isomorphic to a subgroup of GL_2 contradicting the assumption $\operatorname{rk} G \geqq 2$.

Appendix. On the Euler–Poincaré characteristic of complex algebraic varieties

For the convenience of the reader we sketch the proofs of some results on the Euler–Poincaré characteristic of complex algebraic varieties used in this paper. In the sequel cohomology is *Alexander–Spanier cohomology* with respect to a fixed field.

A1. We recall that if X is a locally compact space and $Y \subset X$ a closed subspace, then there is a long exact sequence

$$\cdots \to H_c^i(X-Y) \to H_c^i(X) \to H_c^i(Y) \to \cdots \qquad (i \in \mathbb{N}). \tag{1}$$

This implies that if two of the cohomology space $H_c^\cdot(Z)$, $Z = X, Y$ or $X - Y$, are finitely generated, then so is the third and

$$\chi_c(X) = \chi_c(Y) + \chi_c(X-Y). \tag{2}$$

By an easy induction this implies the following: *If $X = \bigcup_i X_i$ is a finite stratification of X* (i.e. the X_i are locally closed and disjoint and $\bar{X}_i$ is the union of some X_j's) *and if all $H_c^{\cdot}(X_i)$ are finitely generated, then $H_c^{\cdot}(X)$ is finitely generated and*

$$\chi_c(X) = \sum_i \chi_c(X_i).$$

A2. We now recall some well known properties of complex algebraic varieties.

PROPOSITION ([Gi] §5, Satz 4, Folgerung 1). *Let X be a complex algebraic variety and $Y \subset X$ a closed subvariety. Then there is a triangulation of X such that Y is a subcomplex.*

This has a number of important consequences for an algebraic variety X:

(i) *X is locally contractible, hence HCL. In particular $H^{\cdot}(X)$ is also the singular cohomology and it vanishes above $2 \cdot \dim_{\mathbb{C}} X$.*

(ii) *X has a finite open covering $X = \bigcup_{i \in I} U_i$ such that for any non-empty $J \subset I$ the open subset $U_J := \bigcap_{j \in J} U_j$ is contractible. In particular X has the homotopy type of a finite CW-complex, i.e. of the nerve of this cover.*

In fact using a Nagata compactification $Z \supset X$ by a compact algebraic variety Z [Na] and applying the proposition above to Z and $Y := Z - X$, we see that X is the union of the interiors of finitely many simplices from which the claim follows easily.

(iii) *$H^{\cdot}(X)$ and $H_c^{\cdot}(X)$ are finitely generated and vanish above $2 \cdot \dim_{\mathbb{C}} X$.*

For $H^{\cdot}(X)$ this follows from (ii) and (i). From this one can deduce it for $H_c^{\cdot}(X)$ using again a Nagata compactification [Na] and the long exact sequence (1) in A1.

(iv) *For any finite open covering $X = \bigcup_{i \in I} U_i$ we have*

$$\chi(X) = -\sum (-1)^{|J|} \chi(U_J) \quad \text{and} \quad \chi_c(X) = -\sum_{J \neq 0} (-1)^{|J|} \chi_c(U_J),$$

where J runs through all non-empty subsets of I, $U_J := \bigcap_{j \in J} U_j$ and $|J|$ is the cardinality of J. (For χ this follows from the exact Mayer–Vietoris cohomology sequence, and for χ_c it is a consequence of A1(2).)

A3. Next we describe the behaviour of the Euler–Poincaré characteristic for a fibration of algebraic varieties (in the étale topology).

PROPOSITION. *If $\varphi: X \to Y$ is a fibration of algebraic varieties with fibre F then*

$$\chi(X) = \chi(Y) \cdot \chi(F) \quad \textit{and} \quad \chi_c(X) = \chi_c(Y) \cdot \chi_c(F).$$

For χ this follows from A2 (iv) using a finite covering of Y with contractible U_i as in A2 (ii) and applying the Künneth formula to $\varphi^{-1}(U_j) \cong U_j \times F$. For χ_c, using induction on $\dim_{\mathbf{C}} X$ and A1 (2), we may assume that Y and F are both smooth, hence $\chi_c(X) = \chi(X) = \chi(Y) \cdot \chi(F) = \chi_c(Y) \cdot \chi_c(F)$ by Poincaré duality.

A4. The last result completes the picture; it is not really needed in the paper, since we could work with χ_c instead of χ.

PROPOSITION. *For an algebraic variety X we have $\chi(X) = \chi_c(X)$.*

Clearly this holds for smooth X by Poincaré duality. Let $Y \subset X$ be the singular locus. By induction on dimension we may assume that $\chi(Y) = \chi_c(Y)$, and since $X - Y$ is smooth we also have $\chi(X - Y) = \chi_c(X - Y)$. Therefore by A1 (2) it is sufficient to show that $\chi(X) = \chi(Y) + \chi(X - Y)$. If U is an open neighbourhood of Y of which Y is a strong deformation retract this is equivalent to showing $\chi(U - Y) = 0$ (use A2 (iv)). Consider a resolution of singularities $\eta : \tilde{X} \to X$ and an open neighbourhood $\tilde{U}$ of $\tilde{Y} := \eta^{-1}(Y)$ of which $\tilde{Y}$ is a strong deformation retract. (The existence of such a $\tilde{U}$ follows again from proposition A2.) Since $\tilde{X}$ is smooth we have $\chi(\tilde{X}) = \chi_c(\tilde{X}) = \chi_c(\tilde{Y}) + \chi_c(\tilde{X} - \tilde{Y}) = \chi(\tilde{Y}) + \chi(\tilde{X} - \tilde{Y})$, hence $\chi(\tilde{U} - \tilde{Y}) = 0$. But $U := \eta(\tilde{U})$ is an open neighbourhood of Y of which Y is a strong deformation retract and η induces a homeomorphism $\tilde{U} - \tilde{Y} \xrightarrow{\sim} U - Y$, hence $\chi(U - Y) = 0$ and the claim follows.

Remark. There is another proof of the proposition above suggested by the referee which avoids the resolution of singularities. It uses a theorem of Sullivan asserting that for a Whitney-stratified space with only odd dimensional strata the Euler characteristic vanishes. For an l-adic version see [L].

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Note added in proof. We want to make some remarks on non-linearizable actions in positive characteristic.

EXAMPLE. Let k be a field of characteristic 2. Consider the following action of SL_2 on $\mathbb{A}_k^2$:

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a^2x + b^2y + ab \\ c^2x + d^2y + cd \end{pmatrix}.$$

Then $(\mathbb{A}_k^2)^{SL_2} = \varnothing$.

More general one has the following result.

PROPOSITION. *Let G be a connected reductive group over a field k of characteristic $p > 0$. Assume that G is not a torus. Then for sufficiently large n there is an action of G on $\mathbb{A}_k^n$ without fixed points.*

Proof. There exists a linear representation $G \to GL(V)$ which is reducible but not completely reducible (Nagata). Let $W \subset V$ be a G-stable linear subspace which has no G-stable complement in V. Consider the G-module $L := \text{Hom}(V/W, V)$ and choose an element $\rho \in L$ such that the composition with the projection $V \to V/W$ is the identity map on V/W. Define $M \subset L$ to be the linear span of $\{g\rho \mid g \in G\}$ and $N \subset M$ the linear span of $\{g\rho - \rho \mid g \in G\}$. Then N and M are G-stable and $\dim M = \dim N + 1$. It follows from the construction that $\mathbb{P}(M)^G \subset \mathbb{P}(N)$. Hence the action of G on the affine space $\mathbb{P}(M) - \mathbb{P}(N)$ has no fixed points and the claim follows. □ qed.