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Homomorphisms of Fuchsian groups to PSL $(2, \mathbb{R})$

MARK JANKINS and WALTER NEUMANN[1]

Introduction

Let $\Gamma = \Gamma(g : \alpha_1, \ldots, \alpha_n)$ be a cocompact Fuchsian group of genus g with branch indices $\alpha_1, \ldots, \alpha_n$. In this paper we determine the set of components of Hom $(\Gamma, \mathrm{PSL}(2, \mathbb{R}))$. This was done by the first author in [J] for $g > 0$ so we only prove the genus zero case.

The main results are in terms of an *euler class*

$$e : \mathrm{Hom}(\Gamma, \mathrm{PSL}(2, \mathbb{R})) \to H^2(\Gamma; \mathbb{Z})$$

defined as follows: $e(f) = f^*(c)$ where $f^* : H^2(B\,\mathrm{PSL}(2, \mathbb{R}), \mathbb{Z}) \to H^2(B\Gamma; \mathbb{Z}) = H^2(\Gamma; \mathbb{Z})$ is the map induced by f and $c \in H^2(B\,\mathrm{PSL}(2, \mathbb{R}), \mathbb{Z}) \cong \mathbb{Z}$ is a generator. In [JN] it is shown that $H^2(\Gamma, \mathbb{Z})$ is the abelian group:

$$H^2(\Gamma; \mathbb{Z}) = ab\langle x_0, \ldots, x_n \mid \alpha_i x_i = x_0;\ i = 1, \ldots, n\rangle,$$

(this can easily be seen by constructing an explicit $B\Gamma$: replace each of n discs in a surface of genus g by a $B(\mathbb{Z}/\alpha_i)$). Thus any $x \in H^2(\Gamma, \mathbb{Z})$ has a unique representation

$$x = bx_0 + \beta_1 x_1 + \cdots + \beta_n x_n, \qquad 0 \le \beta_i < \alpha_i,$$

called the *normal form* representation.

THEOREM 1. *The above x equals $e(f)$ for some f if and only if the following is true:*

(i) *If* $g > 0$ *then* $2 - 2g - n \le b \le 2g - 2$;
(ii) *If* $g = 0$ *then either*
 (a) $2 - n \le b \le -2$

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or

(b) $b=-1$ *and* $\sum_{i=1}^{n}(\beta_i/\alpha_i)\le 1$

or

(c) $b=1-n$ *and* $\sum_{i=1}^{n}(\beta_i/\alpha_i)\ge n-1$.

THEOREM 2. *The fibers of e are the components of* Hom (Γ, PSL (2, ℝ)).

If $T(\Gamma)=\mathrm{Hom}(\Gamma, \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R})$, where the action is conjugation, then our results apply also to $T(\Gamma)$. One component of $T(\Gamma)$ is Teichmüller space; W. Goldman, in his 1981 Berkeley thesis, showed that this component is characterized by its euler class. The general case $g>0$ of Theorem 1 was proved in [EHN] in the context of transverse foliations of Seifert fibrations (this means that it was assumed that each pair (α_i, β_i) is coprime; this was however not needed in the proof). Both theorems are proved for $g>0$ in [J]. For Γ a surface group (that is $n=0$) Goldman [G] has an independent proof of Theorem 2 and Theorem 1 was proved earlier by Milnor [M].

In [JN2] we apply these results and other methods to discuss the question of which Seifert circle bundles over S^2 admit a transverse foliation. For Seifert circle bundles over higher genus surfaces this was answered in [EHN]. The genus 0 case turns out to be much more subtle – in particular we show in the final section of this paper that the answer conjectured in [EHN] and claimed in [Ga] is false.

§1. Fundamentals

In the following we shall denote PSL (2, ℝ) by G and its universal cover by $\tilde{G}$. G acts by projective automorphisms on $\mathbb{R}P^1\cong S^1$ and this action lifts to an action of $\tilde{G}$ on $(S^1)^\sim=\mathbb{R}$. Following [EHN] we let $\mathcal{D}$ denote the group of homeomorphisms $f:\mathbb{R}\to\mathbb{R}$ which are lifts of homeomorphisms of the circle $S^1=\mathbb{R}/\mathbb{Z}$. Contained in $\mathcal{D}$, as a subgroup of index 2, is the group

$$\mathcal{D}^+=\{g:\mathbb{R}\to\mathbb{R}\mid g \text{ monotone and } g(r+1)=g(r)+1 \text{ for all } r\in\mathbb{R}\}.$$

Therefore we can consider $\tilde{G}$ as a subgroup of $\mathcal{D}^+$. For each real number γ we define $\mathrm{sh}(\gamma)\in\tilde{G}$ by $\mathrm{sh}(\gamma): r\mapsto r+\gamma$ for $r\in\mathbb{R}$. The center of $\tilde{G}$ is

$$Z(\tilde{G})=\{\mathrm{sh}(n)\mid n\in\mathbb{Z}\}\cong\mathbb{Z}$$

and $\tilde{G}/Z(\tilde{G})=G$.

Since SL (2, ℝ) is a connected 2-fold cover of $G=\mathrm{PSL}(2,\mathbb{R})$, $\tilde{G}$ is also the

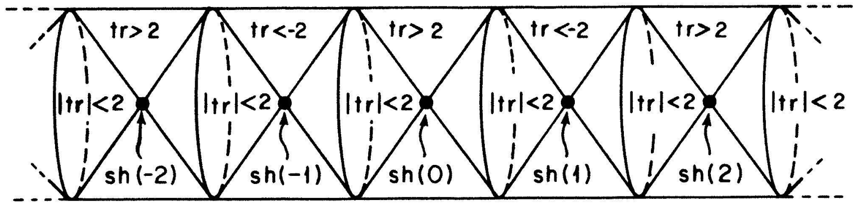


Fig. 1a

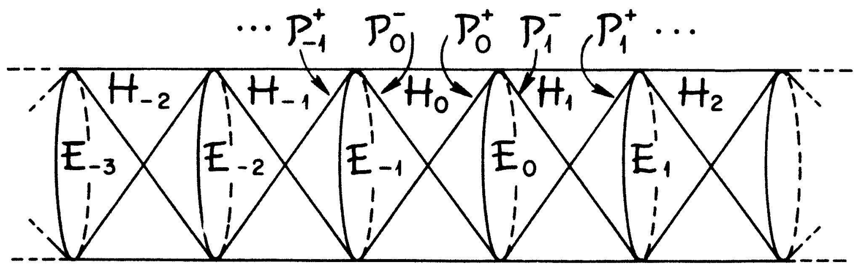


Fig. 1b

universal cover of $SL(2,\mathbb{R})$. For $A \in \tilde{G}$ we define $\operatorname{tr}(A)$ to be the trace of the image of A in $SL(2,\mathbb{R})$. We say $A \notin Z(\tilde{G})$ is *elliptic*, *parabolic*, or *hyperbolic* according as $|\operatorname{tr}(A)|<2$, $|\operatorname{tr}(A)|=2$, $|\operatorname{tr}(A)|>2$ respectively. $\tilde{G}$ is homeomorphic to $\mathbb{R}\times \mathring{D}^2$ and is subdivided according to trace as in Fig. 1a (an explicit diffeomorphism is given in an appendix to this paper). We label the components of the elliptic and hyperbolic regions as in Fig. 1b. The cones $\mathcal{P}_i^{\pm}$ which separate the elliptic from the hyperbolic regions consist of parabolic elements and the cone points are the central elements. A convenient way of distinguishing all these regions is by the following invariants of $A \in \tilde{G}$ (see [EHN].):

$$\underline{m}A = \min\{A(r) - r \mid r \in \mathbb{R}\},$$

$$\bar{m}A = \max\{A(r) - r \mid r \in \mathbb{R}\}.$$

Namely,

$$A \in \mathcal{E}_i \Leftrightarrow [\underline{m}A, \bar{m}A] \subset (i, i+1),$$
$$A \in \mathcal{H}_i \Leftrightarrow i \in (\underline{m}A, \bar{m}A),$$
$$A \in \mathcal{P}_i^{+} \Leftrightarrow \underline{m}A = i < \bar{m}A,$$
$$A \in \mathcal{P}_i^{-} \Leftrightarrow \underline{m}A < i = \bar{m}A.$$

Any element of $SL(2,\mathbb{R})$ has a unique lift into the region

$$\mathcal{P} = \{A \in \tilde{G} \mid -1 \le \underline{m}a < 1, -1 \le \bar{m}A < 2\}.$$

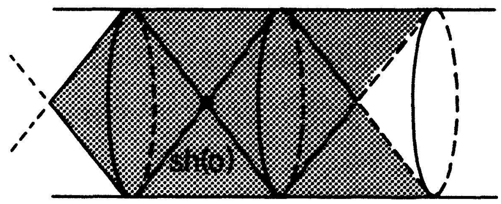


Fig. 2

We call this lift the *principal lift* and we call $\mathscr{P}$ the *principal region* (Fig. 2). We denote the principal lift of A by $A^{\sim}$.

PROPOSITION 1.1. *Elements* $A, B \in \tilde{G} - Z(\tilde{G})$ *are conjugate if and only if* $\operatorname{tr} A = \operatorname{tr} B$ *and* A *and* B *lie in the same region* ($\mathscr{E}_i$, $\mathscr{H}_i$, $\mathscr{P}_i^+$ or $\mathscr{P}_i^-$).

This is immediate from the corresponding fact for conjugacy classes in SL (2, ℝ) plus the easy observation that inner automorphisms of $\tilde{G}$ preserve regions.

For a hyperbolic element $A \in \mathrm{SL}(2, \mathbb{R})$, the region ($\mathscr{H}_0$ or $\mathscr{H}_1$) in which its principal lift lies is determined by tr A. For elliptic and parabolic elements trace is insufficient and we need the following lemma.

LEMMA 1.2. *Given* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}) - \{\pm I\}$ *with* $|\operatorname{tr} A| \leq 2$, *the principal lift* $A^{\sim}$ *satisfies*

$$A^{\sim} \in \bar{\mathscr{E}}_{-1} \Leftrightarrow c - b < 0.$$

$$A^{\sim} \in \bar{\mathscr{E}}_0 \Leftrightarrow c - b > 0.$$

Proof. We first consider the case $|\operatorname{tr} A| < 2$. Since the set of $B \in \mathrm{SL}(2, \mathbb{R})$ with $\operatorname{tr} B = \operatorname{tr} A$ has two components, we only need show that these components are distinguished by $\operatorname{sgn}(c - b)$. Thus we consider $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ subject to the constraints

$$\begin{aligned} &a + d = t \qquad (|t| < 2) \\ &ad - bc = 1. \end{aligned} \tag{1}$$

Letting $a = x + y$, $d = x - y$, $b = z - w$, $c = z + w$, equations (1) combine to give

$$w^2 - y^2 - z^2 = 1 - \frac{t^2}{4} \tag{2}$$

and $c-b$ is $2w$. Equation (2) describes a hyperboloid of two sheets with the sheets distinguished by the sign of w. This proves the case $|\mathrm{tr}\, A|<2$ of the lemma and the general case follows by continuity.

COROLLARY 1.3. *The conjugacy class of* $A\in \mathrm{SL}(2,\mathbb{R})-\{\pm I\}$ *(and hence of its principal lift* $A^{\sim}$*) is determined by* $\mathrm{tr}\, A$ *if* $|\mathrm{tr}\, A|>2$ *and by* $\mathrm{tr}\, A$ *together with* $\mathrm{sgn}\,(c-b)$ *if* $|\mathrm{tr}\, A|\leq 2$.

§2. Products of conjugates

The proofs of Theorems 1 and 2 depend on understanding what elements of $\tilde{G}$ can occur as products of conjugates of given elements. The general answer easily follows from Lemma 2.1 and Proposition 2.2 below; we describe only the case we need, which is the most subtle case, in Corollary 2.3.

For $A, B\in\tilde{G}$ write $A\sim B$ for "A is conjugate to B." For fixed $A_1,\ldots,A_n\in\tilde{G}$ define

$$S(A_1,\ldots,A_n)=\{B_1\cdots B_n \mid A_i\sim B_i,\ i=1,\ldots,n\}.$$

In particular, $S(A)$ is the conjugacy class of A.

LEMMA 2.1.

(1) $S(A_1,\ldots,A_n)=S(A_1,\ldots,A_k)S(A_{k+1},\ldots,A_n)$

(2) $S(A_1,\ldots,A_n)=S(A_{\sigma(1)},\ldots,A_{\sigma(n)})$ *for any permutation* σ *of* $\{1,\ldots,n\}$.

(3) $S(\mathrm{sh}(k_1)A_1,\ldots,\mathrm{sh}\,(k_n)A_n)=\mathrm{sh}\,(\sum_{i=1}^n k_i)S(A_1,\ldots,A_n)$ *for* $k_1,\ldots,k_n\in\mathbb{Z}$.

Proof. (1) and (3) are trivial. Statement (2) follows for $n=2$ from the identity $A_1A_2=A_2(A_2^{-1}A_1A_2)$ and then follows for general n by induction using statement (1).

In view of this lemma, the following Proposition gives all possible $S(A_1, A_2)$ up to integral shifts.

PROPOSITION 2.2. *The following regions in* $\tilde{G}$ *are* $S(A_1, A_2)$ *for the indicated* A_1 *and* A_2. *Here*

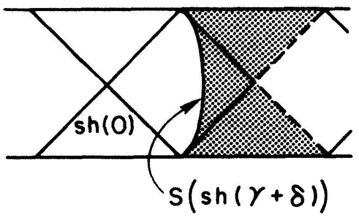
$\gamma, \delta\in(0, 1)$,

$\rho, \lambda\in(0,\infty)$.

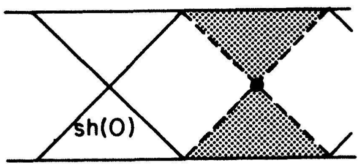
In the pictures we just draw a slice through the picture of Fig. 1; dotted lines indicate *open boundary and* sh (k) *is in* $S(A_1, A_2)$ *for* $k \in \mathbb{Z}$ *only if explicitly indicated by a dot*

(a) $A_1 = \text{sh}(\gamma) \qquad A_2 = \text{sh}(\delta)$

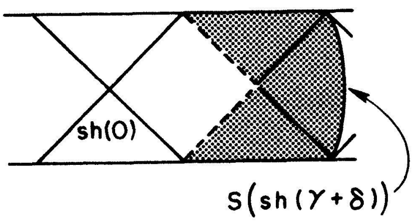
(i) $0 < \gamma + \delta < 1$



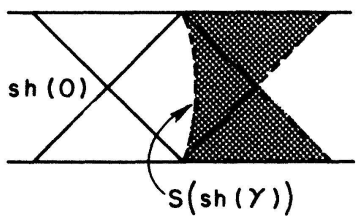
(ii) $\gamma + \delta = 1$



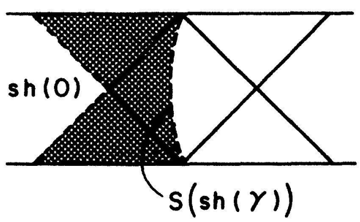
(iii) $\gamma + \delta > 1$



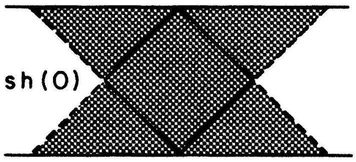
(b) (i) $A_1 = \text{sh}(\gamma) \qquad A_2 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^{\sim}$



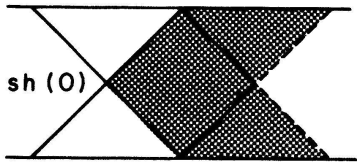
(ii) $A_1 = \text{sh}(\gamma) \qquad A_2 = \begin{pmatrix} 1 & +1 \\ 0 & 1 \end{pmatrix}^{\sim}$



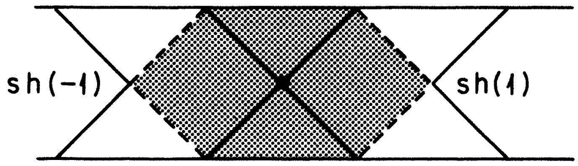
(c) $A_1 = \mathrm{sh}\,(\gamma) \qquad A_2 = \begin{pmatrix} e^l & 0 \\ 0 & e^{-l} \end{pmatrix}^{\sim}$



(d) (i) $A_1 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^{\sim} \qquad A_2 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^{\sim}$

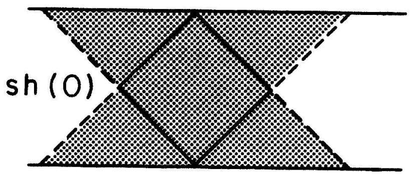


(ii) $A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{\sim} \qquad A_2 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^{\sim}$



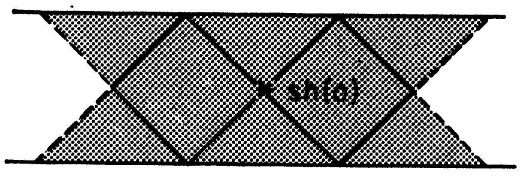
(iii) $A_1 = \begin{pmatrix} 1 & +1 \\ 0 & 1 \end{pmatrix}^{\sim} \qquad A_2 = \begin{pmatrix} 1 & +1 \\ 0 & 1 \end{pmatrix}^{\sim}$: *mirror image of* (d)(i).

(e) (i) $A_1 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^{\sim} \qquad A_2 = \begin{pmatrix} e^l & 0 \\ 0 & e^{-l} \end{pmatrix}^{\sim}$



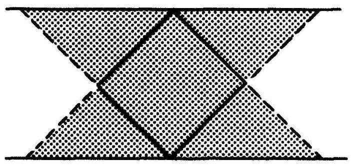
(ii) $A_1 = \begin{pmatrix} 1 & +1 \\ 0 & 1 \end{pmatrix}^{\sim} \qquad A_2 = \begin{pmatrix} e^l & 0 \\ 0 & e^{-l} \end{pmatrix}^{\sim}$: *mirror image of* (e)(i).

(f) (i) $A_1 = \begin{pmatrix} e^l & 0 \\ 0 & e^{-l} \end{pmatrix}^{\sim} \qquad A_2 = \begin{pmatrix} e^\lambda & 0 \\ 0 & e^{-\lambda} \end{pmatrix}^{\sim}$



(*) $\mathrm{sh}(o) \in S(A_1, A_2) \Leftrightarrow l = \lambda$

Proof. The general technique in each case is that an elementary computation using $\underline{m}$ and $\bar{m}$ gives a rough bound on $S(A_1, A_2)$ and then the exact $S(A_1, A_2)$ is found by computing the possible values of $a+d$ (and, if necessary $c-b$) for $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ a product of conjugates of the images of A_1 and A_2 in SL (2, ℝ). We do the first case, (i), which is typical. In this case a trivial computation with $\underline{m}$ and $\bar{m}$ shows $S(A_1, A_2)$ is contained in the following region



Thus we see it suffices to calculate the possible traces.

A general element $A \in S(A_1, A_2)$ is conjugate to the principal lift of

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix}$$

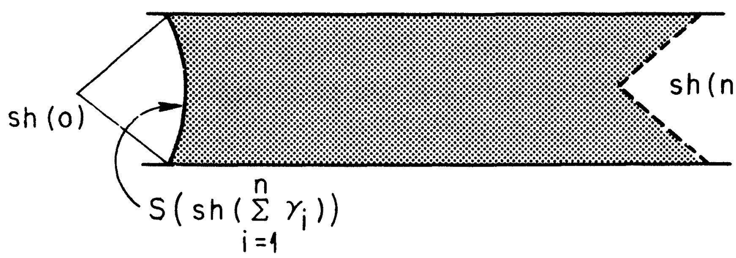
and $\psi = \pi\gamma$, $\theta = \pi\delta$. In particular $a+d = 2\cos\psi$ and $c-b>0$. Thus

$$\operatorname{tr} A = \operatorname{tr}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\right) = 2\cos\psi\cos\theta - (c-b)\sin\theta.$$

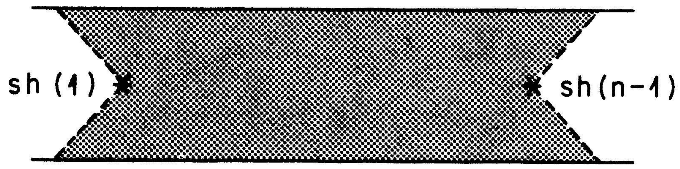
By equation (2) in the proof of Lemma 1.2 we see that $c-b$ ($=2w$ in the notation of that proof) lies in the interval $[2\sqrt{(1-t^2/4)}, \infty)$ with $t = 2\cos\psi$. It is easy to see that $c-b$ takes on all these values. Therefore, since $\sqrt{(1-t^2/4)} = 2\sin\psi$, we see that $\operatorname{tr} A$ takes on all values in the interval $(-\infty, 2\cos(\theta+\psi)]$. This completes the proof of case a(i); the other cases are similar calculations.

COROLLARY 2.3. *The following regions are* $S = S(\operatorname{sh}(\gamma_1), \ldots, \operatorname{sh}(\gamma_n))$ *for the indicated* γ_i *with* $0<\gamma_i<1$ *and* $n>1$.

(a) $0 < \sum_{i=1}^{n} \gamma_i < 1$.



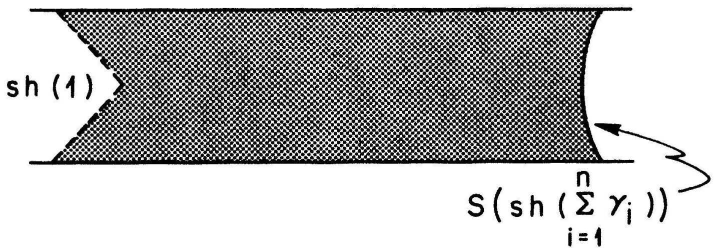
(b) $1 \le \sum_{i=1}^{n} \gamma_i \le n-1$



$$\text{sh}\,(1) \in S \Leftrightarrow \sum_{i=1}^{n} \gamma_i = 1$$

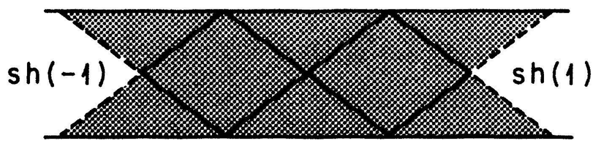
$$\text{sh}\,(n-1) \in S \Leftrightarrow \sum_{i=1}^{n} \gamma_i = n-1$$

(c) $n-1 < \sum_{i=1}^{n} \gamma_i$

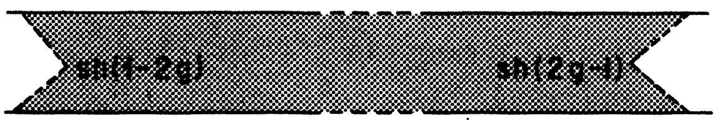


The proof is an easy induction using Lemma 2.2.

Remark. Since a commutator $[A, B]$ is in $S(A, A^{-1})$, Lemma 2.2 easily gives a result proven in [EHN], namely that the set of commutators in $\tilde{G}$ is given by



By induction then, the set of products of $g > 1$ commutators is



§3. Proofs of Theorems 1 and 2

We use the notation of the introduction. The case $g > 0$ is done in [EHN] (it also follows easily with our present approach); we thus assume now $g = 0$.

$\Gamma = \Gamma(0; \alpha_1, \ldots, \alpha_n)$ has presentation

$$\langle q_1, \ldots, q_n \mid q_i^{\alpha_i} = 1, q_1 \cdots q_n = 1 \rangle.$$

Given a homomorphism $f:\Gamma\to G=\mathrm{PSL}(2,\mathbb{R})$, an equivalent way to define the euler class $e(f)$ is as follows. Take the pull-back central extension from

$$\begin{array}{ccccccccc} & & & & & & \Gamma & & \\ & & & & & & \downarrow f & & \\ 0 & \to & \mathbb{Z} & \to & \tilde{G} & \to & G & \to & 1. \end{array}$$

This gives a diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{Z} & \to & \tilde{\Gamma} & \to & \Gamma & \to & 1 \\ & & \| & & \downarrow \tilde{f} & & \downarrow f & & \\ 0 & \to & \mathbb{Z} & \to & \tilde{G} & \to & G & \to & 1 \end{array} \tag{1}$$

where $\tilde{\Gamma}$ has a presentation (by lifting $q_i\in\Gamma$ to an element, which we also call q_i, in $\tilde{\Gamma}$)

$$\langle q_1,\ldots,q_n,z \mid q_i^{\alpha_i}=z^{\beta_i}, q_1\cdots q_n=z^{-b}, z \text{ central}\rangle$$

with $0\le\beta_i<\alpha_i$. In the notation of the introduction we then have

$$e(f)=bx_0+\beta x_1+\cdots+\beta_n x_n.$$

This is shown for instance in [JN] (see also [J]; it is essentially the statement that $H^2(\Gamma;\mathbb{Z})\cong\mathrm{Ext}(\Gamma;\mathbb{Z})$ which is the group of central extensions of Γ by $\mathbb{Z}$). Thus $bx_0+\beta_1x_1+\cdots+\beta_nx_n$ is $e(f)$ for some f if and only if diagram (1) above exists. This is so if and only if $\tilde{f}:\tilde{\Gamma}\to\tilde{G}$ exists with $\tilde{f}(z)=\mathrm{sh}(1)$. Equivalently, there must exist $Q_1,\ldots,Q_n\in\tilde{G}$ with $Q_i^{\alpha_i}=\mathrm{sh}(\beta_i)$ and $Q_1\cdots Q_n=\mathrm{sh}(-b)$. But $Q^\alpha=\mathrm{sh}(\beta)$ if and only if $Q\sim\mathrm{sh}(\beta/\alpha)$ so case (ii) of Theorem 1 reduces to showing that

$$S\left(\mathrm{sh}\left(\frac{\beta_i}{\alpha_i}\right),\ldots,\mathrm{sh}\left(\frac{\beta_n}{\alpha_n}\right)\right)$$

contains $\mathrm{sh}(-b)$ if and only if (a), (b), or (c) of Theorem 1 holds. This is immediate from Corollary 2.3.

For Theorem 2 we again assume $g=0$, since the case $g>0$ is proved in [J]. We introduce the following notation for $Q\in\tilde{G}$ and $1\le k\le n$:

$$\gamma_i=\frac{\alpha_i}{\beta_i}\qquad i=1,\ldots,n,$$

$$X_k(Q)=\{(Q_1,\ldots,Q_k)\mid Q_i\sim\mathrm{sh}(\gamma_i), Q_1\cdots Q_k=Q\}$$

$$Y_k(Q)=\{(Q_1,\ldots,Q_k)\mid Q_i\sim\mathrm{sh}(\gamma_i), Q_1\cdots Q_k\sim Q\}.$$

By the previous discussion, $e^{-1}(bx_0+\beta_1x_1+\cdots+\beta_nx_n)\subset \operatorname{Hom}(\Gamma, G)$ is a continuous image of $X_n(\operatorname{sh}(-b))$ (it is actually homeomorphic to $X_n(\operatorname{sh}(-b))$. Thus it suffices to show $X_n(\operatorname{sh}(-b))$ is connected. Notice that, since $\operatorname{sh}(-b)$ is central, $X_n(\operatorname{sh}(-b))=Y_n(\operatorname{sh}(-b))$. We shall show the more general result that $X_n(Q)$ and $Y_n(Q)$ are connected for any Q.

LEMMA 3.1. *$X_n(Q)$ is connected if and only if $Y_n(Q)$ is connected.*

Proof. For $A\in G$ and $Q\in\tilde{G}$ define $Q^A=A'Q(A')^{-1}$ where A' is a lift of A to $\tilde{G}$. Denote $Z_G(Q)=\{A\in G \mid Q^A=Q\}$. The map $\phi: G\times X_n(Q)\to Y_n(Q)$ given by $\phi(A, Q_1,\ldots,Q_n)=(Q_1^A,\ldots,Q_n^A)$ has fibers isomorphic to $Z_G(Q)$ which is connected. Since ϕ is onto, the lemma follows.

We prove the connectivity of $Y_n(Q)$ by induction. It is trivial for $n=1$. Consider the map

$$\psi: Y_k(Q)\to\tilde{G}$$
$$\psi(Q_1,\ldots,Q_k)=Q_1\cdots Q_{k-1}.$$

The image of ψ is

$$\begin{aligned}\operatorname{Im}(\psi)&=\{P\in\tilde{G}\mid P=Q_1\cdots Q_{k-1};\, PQ_k\sim Q;\, Q_i\sim \operatorname{sh}(\gamma_i)\ i=1,\ldots,k\}\\ &=S(\operatorname{sh}(\gamma_1),\ldots,\operatorname{sh}(\gamma_{k-1}))\cap S(Q,\operatorname{sh}(-\gamma_k)).\end{aligned}$$

This is an intersection of two regions of the type described in Proposition 2.2 and Corollary 2.3 and is thus connected. The proof is complete once we show $\psi^{-1}(P)$ is connected. The mapping $(Q_1,\ldots,Q_k)\mapsto(Q_1,\ldots,Q_{k-1})$ maps $\psi^{-1}(P)$ onto $X_{k-1}(P)$ with fiber

$$W=\{Q_k\mid Q_k\sim \operatorname{sh}(\gamma_k),\, PQ_k\sim Q\}.$$

Since $X_{k-1}(P)$ is connected by induction hypothesis and Lemma 3.1, we must just show that W is connected. Since W is contained in the principal region, we may work in $\mathrm{SL}(2,\mathbb{R})$ instead of $\tilde{G}$. If the image of P in $\mathrm{SL}(2,\mathbb{R})$ is $\pm I$ then W is trivially connected, so we may assume, by conjugating P if necessary, that $P=\begin{pmatrix}0&1\\-1&s\end{pmatrix}$ (for $|s|\le 2$ we should also consider $P=\begin{pmatrix}0&-1\\1&s\end{pmatrix}$; the argument is similar). Thus we consider, with $\theta=\pi\gamma_k$ and $T=\operatorname{tr} Q$,

$$W'=\left\{R\mid R\sim\begin{pmatrix}\cos\theta&-\sin\theta\\ \sin\theta&\cos\theta\end{pmatrix}, \operatorname{tr}\left(\begin{pmatrix}0&1\\-1&s\end{pmatrix}R\right)=T\right\}.$$

If $|T|>2$ then $W'=W$; otherwise W' consists of two disjoint (possibly empty) pieces of which one equals W. We show W' is connected. Write $R=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, so the conditions on a, b, c, d are $a+d=2\cos\theta$, $c-b>0$, $ad-bc=1$ and $c-b+sd=T$. With the change of coordinates used in the proof of 1.2 these become:

$$\begin{cases} w^2-y^2-z^2=1-\dfrac{t^2}{4}, & (t=2\cos\theta) \\ w>0 & \\ 2w+s(t-2y)=T. & \end{cases}$$

The first two conditions determine one sheet of a hyperboloid of two sheets. The third equation describes a plane which thus intersects this sheet in a connected set.

§4. Transverse foliations in Seifert fibrations

Let M be a Seifert fibered 3-manifold with normal form Seifert invariant $(g; b; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n))$. In [EHN] it was suggested that the numerical conditions of Theorem 1 might be necessary and sufficient for M to admit a codimension 1 foliation transverse to its fibers; this was proved for $g>0$ but only sufficiency of the conditions was proved for $g=0$. Here we give an example to show that the conditions are *not* necessary for $g=0$.

Recall that

$$\mathcal{D}^+=\{g:\mathbb{R}\to\mathbb{R} \mid g \text{ monotonic and } g(r+1)=\dot{g}(r)+1 \text{ for all } r\in\mathbb{R}\}.$$

Let $\tilde{\Gamma}=\tilde{\Gamma}(g; b; (a_1, \beta_1), \ldots, (\alpha_n, \beta_n))$ be the central extension of $\Gamma=\Gamma(g; \alpha_1, \ldots, \alpha_n)$ defined in Section 3 with center generated by z. In [EHN] it is shown that M has a transverse foliation if and only if there exists a homomorphism $\tilde{f}:\tilde{\Gamma}\to\mathcal{D}^+$ with $\tilde{f}(z)=\mathrm{sh}\,(1)$. (This is equivalent to the existence of a homomorphism $f:\Gamma\to\mathrm{Homeo}^+(S^1)$ with euler class $e(f)=bx_0+\beta_1x_1+\cdots+\beta_nx_n$.)

By our results we can find $R_1, R_2, R_3\in\tilde{G}$ with $R_i\sim\mathrm{sh}\,(\frac{4}{5})$ for $i=1,2,3$ and $R_1R_2R_3=\mathrm{sh}\,(2)$. Let $Q_i=P_2^{-1}R_iP_2$ for $i=1,2,3$ where $P_2:\mathbb{R}\to\mathbb{R}$ is map $P_2(r)=2r$. Clearly, $Q_i\in\mathcal{D}^+$ and $Q_i\sim\mathrm{sh}\,(\frac{2}{5})$ for each i. Moreover $Q_1Q_2Q_3=\mathrm{sh}\,(1)$. This gives a homomorphism $f:\tilde{\Gamma}(0; -1; (5,2), (5,2), (5,2))\to\mathcal{D}^+$ with $f(z)=\mathrm{sh}\,(1)$, showing the existence of a transverse foliation on $M(0; -1; (5, 2), (5, 2), (5, 2))$. This is the claimed counterexample.

Analogous constructions using $P_m : \mathbb{R} \to \mathbb{R}$, $P_m(r) = mr$ give many more examples. We have strong evidence that if $M(0; b; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n))$ admits a transverse foliation, then a transverse foliation can be constructed by this method; details will appear elsewhere (see [JN2]).

Appendix. A homeomorphism for Fig. 1

In the above paper we need only such topological information related to Fig. 1 as is easily available by inspection, so we did not give an explicit homeomorphism. This appendix supplies such a homeomorphism, with a sketch proof.

PROPOSITION. *Let D denote the open disk of radius $\frac{1}{2}$ and parametrize $\mathbb{R} \times D$ as $\mathbb{R} \times D = \{(z, r, \theta) \mid z \in R, 0 \leq r < \frac{1}{2}\}$ using polar coordinates. For $Q \in \tilde{G}$ let $t(Q) \in \mathbb{R}$ be any stable fixed point of* $\mathrm{sh}\,(-\frac{1}{2}(\bar{m}Q + \underline{m}Q))Q$. *Define*

$$\Phi : G \to \mathbb{R} \times D$$

$$\Phi : Q \mapsto (\tfrac{1}{2}(\bar{m}Q + \underline{m}Q), \tfrac{1}{2}(\bar{m}Q - \underline{m}Q), 2\pi t(Q))$$

and

$$\Psi : \mathbb{R} \times D \mapsto \tilde{G}$$

$$\Psi : (z, r, \theta) \mapsto \mathrm{sh}\,(z) \cdot \mathrm{sh}\,(\theta/2\pi) \cdot \begin{pmatrix} \sec(\pi r) + \tan(\pi r) & 0 \\ 0 & \sec(\pi r) - \tan(\pi r) \end{pmatrix}^{\sim} \cdot \mathrm{sh}\,(-\theta/2\pi).$$

Then Φ and Ψ are mutually inverse analytic diffeomorphisms which exhibit the structure of Fig. 1.

Remark. The projection of $\Psi(z, r, \theta)$ into $\mathrm{SL}\,(2, \mathbb{R})$ is

$$\tan(\pi r)\begin{pmatrix} \cos(\pi z + \theta) & \sin(\pi z + \theta) \\ \sin(\pi z + \theta) & -\cos(\pi z + \theta) \end{pmatrix} + \sec(\pi r)\begin{pmatrix} \cos(\pi z) & -\sin(\pi z) \\ \sin(\pi z) & \cos(\pi z) \end{pmatrix}.$$

Thus the trace of $\Psi(z, r, \theta)$ is $2\cos(\pi z)/\cos(\pi r)$. In particular the cones of parabolic elements in Fig. 1 are given by the equations $\cos(\pi z)/\cos(\pi r) = \pm 1$, i.e. $z = n \pm r$ for $n \in \mathbb{Z}$, while other surfaces of constant trace are given by smooth surfaces $2\cos(\pi z) = t\cos(\pi r)$ with $t \neq \pm 2$.

Proof. The action of any $q \in \mathrm{SL}(2, \mathbb{R})$ on $\mathbb{R}^2$ can be written in polar coordinates in the form

$$(r, \theta) \mapsto (rq_1(\theta), q_2(\theta)). \tag{1}$$

The action $\theta \mapsto q_2(\theta)$ on S^1 is the double cover of the action of $G = \mathrm{PSL}(2, \mathbb{R})$ on $\mathbb{R}P^1 = S^1$ of §1. The action of SL (2, $\mathbb{R}$) on $\mathbb{R}^2$ is area preserving. A map given as in (1) above is area preserving if and only if

$$\frac{dq_2}{d\theta} = q_1(\theta)^{-2}, \tag{2}$$

so this holds in the present situation. We deduce:

LEMMA. *Let $Q \in \tilde{G}$ be a lift of $q \in \mathrm{SL}(2, \mathbb{R})$ and, for $t \in \mathbb{R}$, let $e(t)$ denote* $\exp(\pi i t) \in \mathbb{C} = \mathbb{R}^2$. *Then one of $\bar{m}Q$ and $\underline{m}Q$ is attained at $t \in \mathbb{R}$ if and only if* $\|q \cdot e(t)\| = 1$.

Indeed, $Q(x) - x$ has derivative 0 at $x = t$ if and only if $Q'(t) = 1$, or equivalently, $q_2'(\theta) = 1$ with $\theta = \pi t$. By (2) this means $q_1(\theta) = 1$ and by (1) this means $\|q \cdot e(t)\| = 1$. We must show that such a point t represents an extremum of $Q(x) - x$ and not just an inflection point. But q^{-1} takes the unit circle in $\mathbb{R}^2$ to an ellipse of area π. Unless q is a strict rotation, this ellipse will intersect the unit circle in just two points $e(t)$ with $0 \le t < 1$, so these points must be where the two extrema of $Q(x) - x$ are attained.

Returning to the proof of the proposition, we first show that $\Phi \circ \Psi = id$. Suppose $\Phi(Q) = (z_0, r_0, \theta_0)$. Then clearly $\Phi(\mathrm{sh}(z) \cdot \mathrm{sh}(\theta/2\pi) \cdot Q \cdot \mathrm{sh}(-\theta/2\pi) = (z_0 + z, r_0, \theta_0 + \theta)$. It thus suffices to show for

$$Q = \begin{pmatrix} \sec(\pi r) + \tan(\pi r) & 0 \\ 0 & \sec(\pi r) - \tan(\pi r) \end{pmatrix}^{\sim}$$

that $\Phi(Q) = (0, r, 0)$ with $r = \frac{1}{2}(\bar{m}Q - \underline{m}Q)$. In other words, we must show that $\bar{m}Q = r$ and $\underline{m}Q = -r$ and $Q(0) = 0$. The latter is trivial; for the former we apply the lemma. Now

$$\left\| \begin{pmatrix} \sec(\pi r) + \tan(\pi r) & 0 \\ 0 & \sec(\pi r) - \tan(\pi r) \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \right\|$$

$$= \frac{1}{\cos^2(\pi r)} (1 + \sin^2(\pi r) + 2\sin(\pi r)\cos(2\theta)),$$

which equals 1 if and only if $2\sin(\pi r)\cos(2\theta) = -2\sin^2(\pi r)$. This holds if and only if $\theta/\pi = \pm(\frac{1}{2}r + \frac{1}{4})$ modulo 1. Symmetry properties of the graph of Q (or direct computation) now show that $Q(\frac{1}{2}r+\frac{1}{4}) = \frac{1}{2}r + \frac{1}{4} - r$ and $Q(-\frac{1}{2}r - \frac{1}{4}) = -\frac{1}{2}r - \frac{1}{4} + r$, showing that $\bar{m}Q = r$ and $\underline{m}Q = -r$.

To complete the proof that Φ and Ψ are inverse homeomorphisms it is enough to show surjectivity of Ψ. This follows easily from the following two observations, each of which is a simple application of the intermediate value theorem:

(i) For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$ there exists a $\begin{pmatrix} \cos \pi z & -\sin \pi z \\ \sin \pi z & \cos \pi z \end{pmatrix}$ such that $\begin{pmatrix} \cos \pi z & -\sin \pi z \\ \sin \pi z & \cos \pi z \end{pmatrix}^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is symmetric.

(ii) For symmetric $\begin{pmatrix} a & b \\ b & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$ there exists $\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ such that $\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ is diagonal.

We next show analyticity of the maps Φ and Ψ. Away from $r = 0$ this is clear. To see it at $r = 0$ we write Ψ in the form

$$(z, r, \theta) \mapsto \mathrm{sh}(z) \cdot \left(\tan(\pi r) \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix} + \sec(\pi r) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)^{\sim}.$$

But

$$\tan(\pi r)/r, \sec(\pi r), \text{ and } \begin{pmatrix} r\cos \theta & r\sin \theta \\ r\sin \theta & -r\cos \theta \end{pmatrix}$$

are analytic, so Ψ is analytic. Using this description, the jacobian of Ψ at any point with $r = 0$ is also easily seen to be nonsingular, so analyticity of Φ follows by the inverse function theorem.

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