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Some remarks on compactifications of commutative algebraic groups

F. Knop and H. Lange

Introduction

In the theory of transcendental numbers commutative algebraic groups play an important role. In fact if such a group E is embedded in some projective space, its exponential map can be described by holomorphic functions. The values of these functions at algebraic points are good candidates for transcendency (cf. [8], [9]). In order to embed E into a projective space, it is convenient to compactify E and embed the compactification $\bar{E}$. The main method to compactify E (due to Serre [7]) is the following:

Let E be a connected commutative algebraic group over an algebraically closed field k. Then there is a canonical exact sequence

$$0 \to G \to E \xrightarrow{p} A \to 0 \tag{1}$$

with G a connected linear group and A an abelian variety over k. Given a projective G-variety P over $\mathbb{P}^n$, every open G-equivariant immersion $G \to P$ induces in a natural way a compactification $\bar{E}$ of E, namely the fibre bundle $E(P) = E \times^G P$ with fibre P over A, associated to the G-principal bundle $E \to A$ with respect to the given G-action of P (cf. [4]).

In this note we shall study the following questions: Are there other compactifications (section 1)? Which are the G-equivariant embeddings of $\bar{E}$ into $\mathbb{P}^n$ (section 2)? How many compactifications of the above type are there (section 3)? And by forms of which degrees is the homogeneous ideal of $\bar{E}$ in $\mathbb{P}^n$ generated (section 4)?

To be more precise, in section 1 we shall prove that if the linear part G of E is a torus or one dimensional, then there are no other normal compactifications of E then those described above (Theorem 1.1) and we give a counterexample to this statement in the non-normal case. In section 2 we investigate the group $\mathrm{Pic}_G(E(P))$ of G-line bundles on the compactification $E(P)$ and show (Theorem 2.1) that it is isomorphic to $\mathrm{Pic}_G(P) \times \mathrm{Pic}(A)$. Hence every G-embedding of

$E(P)$ into projective space comes from a pair consisting of a projective G-embedding of P and a projective embedding of A.

In section 3 we show that Serre's original compactifications of E depend on the splitting of G into one dimensional groups. Finally in section 4 we show that in the most important cases (with respect to applications in transcendence theory) the homogeneous ideal of $E(P)$ in $\mathbb{P}^n$ is generated by forms of degree $\leq \dim A+3$.

1. Compactifications of E

Let E denote a connected commutative algebraic group over a field k (algebraically closed for simplicity) with canonical exact sequence (1). Let X be an E-variety (not necessarily projective) and $i:E\to X$ an equivariant open immersion. Let $\bar{G}=\overline{Gi(0)}$ be the closure of the orbit of $i(0)$ under G in X. Then there is a canonical E-equivariant morphism

$$\psi: E(\bar{G})\to X$$

defined by $\psi(h,x)=h\cdot x$ for $h\in E$ and $x\in\bar{G}\subset X$. Here $E(G)$ denotes as usual the fibre bundle $E\times^G\bar{G}$ associated to the principal bundle. It is the compactification of E corresponding to the embedding $G\to\bar{G}$ (if $\bar{G}$ is proper over k, c.f. [4]). The following theorem in particular gives conditions under which the compactifications of type $E(G)$ are the only E-equivariant compactifications of E.

THEOREM 1.1. (a) *ψ is birational and proper, and in particular surjective.*

(b) *Suppose* (i) *X is normal and* (ii) *$\bar{G}$ consists of finitely many G-orbits. Then ψ is an isomorphism.*

Condition (ii) is always fulfilled if G is a torus or one dimensional (cf. [5]).

Proof. 1. Since ψ restricted to the open subset $E\times^G G$ of $E(\bar{G})$ is an isomorphism onto the open subset E of X, ψ is birational.

2. Let $\pi: E(\bar{G})\to A$ denote the canonical projection map. We claim that $(\pi,\psi): E(\bar{G})\to A\times X$ is a closed immersion.

Since $E\to A$ is faithfully flat, it suffices to show that $id_E\times_A(\pi,\psi): E\times_A E(\bar{G})\to E\times_A(A\times X)$ is a closed immersion. Now $\phi_1: E\times\bar{G}\to E\times_A(E\times^G\bar{G}): (h,x)\to(h,(h,x))$ is an isomorphism (with inverse map $\phi_1^{-1}(h,(h',x))=(h,(h^{-1}h')x)$; note that $h^{-1}h'\in G$). If $\phi_2: E\times_A(A\times X)\to E\times X$ denotes the canonical isomorphism and $\phi_3: E\times X\to E\times X$ the isomorphism

$(h, x) \to (h, h^{-1}x)$, then it suffices to show that the composed map

$$\phi_3\phi_2(id_E \times_A (\pi, \psi))\phi_1 : E \times \bar{G} \to E \times X$$

is a closed immersion. But

$$\begin{aligned}\phi_3\phi_2(id_E \times_A (\pi, \psi))\phi_1(h, x) &= \phi_3\phi_2(id_E \times_A (\pi, \psi))(h, (h, x))\\ &= \phi_3\phi_2(h, (\pi(h), hx))\\ &= \phi_3(h, hx) = (h, x)\end{aligned}$$

which proves the assertion. As composition of the closed immersion $E(\bar{G}) \to A \times X$ and the projection $A \times X \to X$ the map ψ is proper. (Note that if X is itself proper over k, assertion 2 follows immediately from EGA II, 5.4.3).

3. To complete the proof of the theorem, by Zariski's Theorem (EGA III 4.4.9) it suffices to show that ψ has finite fibres under the assumptions (i) and (ii).

Suppose $x \in X$. Since ψ is surjective there are points $h \in E$ and $x_0 \in \bar{G}$ with $x = hx_0$. Since ψ is E-equivariant, it suffices to show that $\psi^{-1}(x_0)$ is finite. We have

$$\begin{aligned}\psi^{-1}(x_0) &= \{(h, x) \in E(\bar{G}) \mid hx = x_0\}\\ &= \{(h, h^{-1}x_0) \in E(\bar{G}) \mid h^{-1}x_0 \in \bar{G}\}.\end{aligned}$$

Denote $E_0 = \{h \in E \mid h^{-1}x_0 \in \bar{G}\}$. E_0 in general is not a subgroup of E, however it is G-stable. Obviously we have a bijection $E_0/G \xrightarrow{\sim} \psi^{-1}(x_0)$. Let E_{x_0} denote the stabilizer of x_0 in E. E_0 is also E_{x_0}-stable and we claim:

Suppose $h, \bar{h} \in E_0$. Then $h^{-1}x_0$ and $\bar{h}^{-1}x_0$ are in the same G-orbit of $\bar{G}$ if and only if $h\bar{h}^{-1} \in GE_{x_0}$. In fact for $g \in G$ we have

$$g\bar{h}^{-1}x_0 = h^{-1}x_0 \Leftrightarrow gh\bar{h}^{-1}x_0 = x_0 \Leftrightarrow gh\bar{h}^{-1} \in E_{x_0} \Leftrightarrow h\bar{h}^{-1} \in GE_{x_0}.$$

Hence there is an injective map of E_0/GE_{x_0} into the set of G-orbits of $\bar{G}$. It follows from assumption (ii) that E_0/GE_{x_0} is a finite set. It remains to show that $GE_{x_0}/G = p(E_{x_0}) \subseteq A$ ($p : E \to A$ the canonical map) is finite. But by the following lemma E_{x_0} is a linear group. Hence $p(E_{x_0})$ is finite as a linear subgroup of an abelian variety. This completes the proof of Theorem 1.1. It remains to show:

LEMMA 1.2. *Let E be an algebraic group acting effectively on a variety X (i.e. $E \to \mathrm{Aut}(X)$ is injective). Then for every $x \in X$ the stabilizer E_x of x in E is linear.*

Proof. We have the following canonical inclusions

$$E_x \hookrightarrow \operatorname{Aut} \mathcal{O}_{X,x} \hookrightarrow \operatorname{Aut} \hat{\mathcal{O}}_{X,x} = \varprojlim_n \operatorname{Aut} (\mathcal{O}_{X,x}/m_x^n).$$

Hence there is an integer n, such that the canonical map $E \to \operatorname{Aut} (\mathcal{O}_{X,x}/m_X^n)$ is injective. Since $\operatorname{Aut} (\mathcal{O}_{X,x}/m_x^n)$ as a subgroup of the group of automorphisms of a finite dimensional vector space is linear, the same is true for E_x.

EXAMPLE 1.3. We want to give an example showing that Theorem 1.1 is not correct without the assumption that X is normal.

Let $l \neq \operatorname{char} k$ be a prime number, k algebraically closed, E as above with canonical exact sequence (1) with the additional assumption that $G = \mathbb{G}_m$. Let $\bar{G} = \mathbb{P}^1$ be the canonical compactification and $\Gamma \subseteq A$ be a subgroup with $\Gamma \simeq \mathbb{Z}/l\mathbb{Z}$. The exact sequence

$$0 \to G \to p^{-1}(\Gamma) \to \Gamma \to 0$$

splits (cf. [2] Theorem 16.2). Since $\operatorname{Hom}(\Gamma, G) \neq 0$, there are 2 sections $s_1, s_2 : \Gamma \to p^{-1}(\Gamma) \subseteq E$ such that $s_1(\Gamma) \cap s_2(\Gamma) = \{0\}$ and $s_1(\Gamma) \cap G = s_2(\Gamma) \cap G = \{0\}$.

Define $E_1 = E/s_1(\Gamma)$ and $E_2 = E/s_2(\Gamma)$. Then the sequences

$$0 \to G \to E_i \to A/\Gamma \to 0$$

are exact. Moreover for $i = 1, 2$ there is a natural E-equivariant morphism $\phi_i : E(\bar{G}) \to E_i(\bar{G})$.
Let

$$X := \operatorname{Im} (\phi_1, \phi_2) \subseteq E_1(\bar{G}) \times E_2(\bar{G}).$$

Then $i : E \to E(\bar{G}) \to X$ is an open imbedding, since $s_1(\Gamma) \cap s_2(\Gamma) = \{0\}$. On the other hand $i \mid \bar{G}$ is an isomorphism, since $s_1(\Gamma) \cap G = s_2(\Gamma) \cap G = \{0\}$. But $A \simeq E(\infty) \subseteq E(\bar{G})$ is mapped onto A/Γ. It follows $p(E_{x_0}) = \Gamma \neq \{0\}$, where x_0 denotes the image of $(0, \infty) \in E(\bar{G})$ in X, which means that $\psi : E(\bar{G}) \to X$ is not an isomorphism.

2. $\mathrm{Pic}_G\,(EP))$

Let E be as in section 1 with canonical exact sequence (1). Let P be a complete G-variety over k and $i:G\to P$ an equivariant open immersion. Then $E(P)=E\times^G P$ is a compactification of E. Moreover if L is a G-linearized line bundle in P, then $E(L)=E\times^G L$ is a G-linearized (even E-linearized) line bundle on $E(P)$ (cf. [4], Lemma 1.2). If as usual Pic ( ) (resp. Pic_G ( )) denotes the group of line bundles (resp. G-linearized line bundles), there is a canonical map

$$\Phi:\begin{cases}\mathrm{Pic}_G\,(P)\times\mathrm{Pic}\,(A)\to\mathrm{Pic}_G\,(E(P))\\ \qquad\qquad (L,M)\mapsto E(L)\otimes\pi^*M.\end{cases}$$

(Note that, whereas $E(L)$ is E-linearized, π^*M is only G-linearized on $E(P)$). The aim of this section is to prove

THEOREM 2.1. *Given an exact sequence* (1) *and let P denote a complete G-variety over k. Then the canonical map*

$$\Phi:\mathrm{Pic}_G\,(P)\times\mathrm{Pic}\,(A)\to\mathrm{Pic}_G\,(E(P))$$

is an isomorphism of groups.

Proof. We shall construct an inverse map $\Psi:\mathrm{Pic}_G\,(E(P))\to\mathrm{Pic}_G\,(P)\times\mathrm{Pic}\,(A)$. Suppose $N\in\mathrm{Pic}_G\,(E(P))$. Let $\pi:E(P)\to A$ denote the projection map. $\pi^{-1}(0)$ is canonically isomorphic to P. We identify both and consider the closed embedding $j:P\hookrightarrow E(P)$. Define $L=j^*N$ with the induced G-action. We claim that $E(L)^{-1}\otimes N\mid\pi^{-1}(a)\simeq\mathcal{O}_{\pi^{-1}(a)}$ (without G-action) for every point $a\in A$.

First of all $E(L)\mid\pi^{-1}(0)=L$ (even as G-line bundles) that is $E(L)^{-1}\otimes N\mid\pi^{-1}(0)\simeq\mathcal{O}_{\pi^{-1}(0)}$ with trivial G-action. Since $E(L)^{-1}\otimes N$ may be considered as a family of line bundles on P parametrized by A, it suffices to show that the Picard variety $\mathrm{Pic}^0\,(P)$ of P is zerodimensional, since then every deformation of a line bundle on P is trivial. But P is as compactification of a connected linear group *a* rational variety which implies $\dim\mathrm{Pic}^0\,(P)\le\dim H^1(P,G_P)=0$.

Applying Grauert's theorem (cf. [1], III, 12.9), we get that $M:=\pi_*(E(L)^{-1}\otimes N)$ is a line bundle on A. We claim that the natural map

$$\sigma:\pi^*M=\pi^*\pi_*(E(L)^{-1}\otimes N)\to E(L)^{-1}\otimes N$$

is an isomorphism of line bundles. Since σ is a homomorphism of line bundles we have only to show that σ is surjective.

For this is sufficient to show that for every point $a \in A$ there is a neighbourhood U in A such that $E(L)^{-1} \otimes N \mid \pi^{-1}(U)$ admits a nowhere vanishing section. But since $\pi_*(E(L)^{-1} \otimes N)$ is a line bundle on A, we may take for U any trivializing open set in A. Hence $1 \times \sigma : E(L) \otimes \pi^* M \to N$ is an isomorphism of line bundles and it remains to show that it is compatible with the G-actions. Since any 2 G-linearizations of a given line bundle on $E(P)$ differ by a character on G, it suffices to check this on the restrictions to the fibre $\pi^{-1}(0)$. But we noted already that $E(L) \mid \pi^{-1}(0) \simeq N \mid \pi^{-1}(0)$ as G-line bundles.

Now define $\Psi : \mathrm{Pic}_G(E(P)) \to \mathrm{Pic}_G(P) \times \mathrm{Pic}(A)$ by $\Psi(N) = (L, M)$ with L and M as above. It is easy to see that the maps Φ and Ψ are inverse to each other.

3. Serre-compactifications of E

In this section let k denote the field of complex numbers. The group G in the exact sequence (1) then is of the form

$$G = \prod_{i=1}^{r} \mathbb{G}_m \times \prod_{j=1}^{s} \mathbb{G}_a. \tag{2}$$

In [7] Serre constructed a compactification of E as follows: Consider for each factor $\mathbb{G}_m$ and $\mathbb{G}_a$ of G in (2) the natural embedding into $\mathbb{P}^1$. This gives a G-equivariant embedding of G into $(\mathbb{P}^1)^{r+s}$. The compactification of E is defined to be the associated bundle $X = E((\mathbb{P}^1)^{r+s})$. We want to show by an example that this compactification heavily depends on the splitting (2) of G.

Start with an abelian variety A and $G = \mathbb{G}_m^2$. Let $G = \mathbb{G}_m \times \mathbb{G}_m$ be a given decomposition. The isomorphism $\Phi : G \to G$, $\Phi(a_1, a_2) = (a_1^{-1} a_2, a_2)$ yields another decomposition of G.

Now let $E_1 = \mathbb{G}_m \times A$ be the trivial bundle and $E_2 \to A$ be an arbitrary principal $\mathbb{G}_m$-bundle over A, whole associated line bundle M_2 on A is algebraically equivalent to zero. By the theorem of Weil–Rosenlicht–Serre (cf. [6], p. 184 Théorème 6) $E = E_1 \times_A E_2$ is a group extension of A by G.

If $(U_i, \alpha_{ij})_{i,j}$ is a description of E_2 by open sets U_i in A and transition morphisms α_{ij}, then $(U_i, (1, \alpha_{ij}))_{ij}$ is a description of the principal G-bundle $E = E_1 \times_A E_2$ over A. On the other hand any principal G-bundle over A may be considered as an element of $H^1(A, G)$. The element of $H^1(A, G)$ corresponding to E over A does not reflect the decomposition of G which however the description $(U_i, (1, \alpha_{ij}))_{i,j}$ does. Applying the isomorphism $\Phi : G \to G$ we get

$$E \,\hat{=}\, (U_i, (1, \alpha_{ij}))_{ij} \,\hat{=}\, (U_i, \Phi((1, \alpha_{ij})))_{i,j} = (U_i, (\alpha_{ij}, \alpha_{ij}))_{i,j}$$

which means

$$E \simeq E_1 \times_A E_2 \simeq E_2 \times_A E_2.$$

If X (resp. $\bar{X}$) denotes the compactification of E corresponding to the given decomposition of G (resp. the decomposition of G given by applying Φ), then we have according to the definitions

$$X \simeq P(\mathcal{O}_A^2) \times_A P(\mathcal{O}_A \oplus M_2)$$

and

$$\bar{X} \simeq P(\mathcal{O}_A \oplus M_2) \times_A P(\mathcal{O}_A \oplus M_2)$$

where $P(\cdot)$ denotes the projective bundle associated to the vector bundle $(\cdot)$.

We claim that X and $\bar{X}$ are not isomorphic in general. For this it suffices to compute the canonical line bundles K_X and $K_{\bar{X}}$. We get

$$K_X = M_2 \otimes_A \mathcal{O}_{P(\mathcal{O}_A^2)}(-2) \otimes_A \mathcal{O}_{P(\mathcal{O}_A \oplus M_2)}(-2)$$

and

$$K_{\bar{X}} = M_2^2 \otimes_A \mathcal{O}_{P(\mathcal{O}_A \oplus M_2)}(-2) \otimes_A \mathcal{O}_{P(\mathcal{O}_A \oplus M_2)}(-2)$$

which obviously are nonisomorphic for $M_2 \neq \mathcal{O}_A$. (Consider X and $\bar{X}$ as projective bundles over $P(\mathcal{O}_A \oplus M_2)$).

4. Projective embeddings

We want to study the projective embeddings of the compactifications $\bar{E}$ of E, and in particular the question: by forms of which degrees is the homogeneous ideal of $\bar{E}$ in $\mathbb{P}^N$ generated? For this we need a slight generalization of a criterium of Mumford (cf. [5], pp. 39–40) which we shall prove first.

If X is any projective variety embedded in $\mathbb{P}^N$ by the complete linear system of a very ample line bundle L, denote by $I := \oplus_{k \geq 0} I_k$ its homogenous ideal in $\mathbb{P}^N$. For any $i \geq 1$ define

$$\mathcal{R}(L^i, L) = \text{Ker}\,(H^0(L^i) \otimes H^0(L) \to H^0(L^{i+1}))$$

and

$$\mathcal{R}_i(L) = \operatorname{Ker}(H^0(L)^i \to H^0(L^i)).$$

If moreover for a vector space V we denote by $S^n(V)$ its n-th symmetric product, we have

LEMMA 4.1. *If L is normally generated on X, then for any $k \geq 1$ the following conditions are equivalent*

(1) *The canonical map $H^0(L) \otimes \mathcal{R}(L^i, L) \to \mathcal{R}(L^{i+1}, L)$ is surjective for every $i \geq k$.*

(2) *The canonical map $I_{k+1} \otimes S^{i-k} H^0(L) \to I_{i+1}$ is surjective for every $i \geq k$.*

In other words: condition (1) is equivalent to the fact that the ideal I is generated by forms of degree $\leq k+1$. For $k=1$ this is just Mumford's result.

Proof. We shall prove only the implication (1)$\Rightarrow$(2), since we do not need the converse. Consider the condition (2′): The canonical map

$$\phi = \sum_v \phi_v : \bigoplus_v \mathcal{R}_{k+1}(L) \otimes H^0(L)^{i-k} \to \mathcal{R}_{i+1}(L)$$

is surjective, where the direct sum is to be taken over all $v = (v_1, \ldots, v_{k+1})$ with $1 \leq v_1 < \cdots < v_{k+1} \leq i+1$. Here the map ϕ_v on the direct summand with index $v = (v_1, \ldots, v_{k+1})$ is given by

$$a_1 \otimes \cdots \otimes a_{k+1} \otimes b_1 \otimes \cdots \otimes b_{i-k}$$

$$\mapsto b_1 \otimes \cdots \otimes \underset{\overset{\vee}{v_1}}{a_1} \otimes \cdots \otimes \underset{\overset{\vee}{v_2}}{a_2} \otimes \cdots \otimes \underset{\overset{\vee}{v_{k+1}}}{a_{k+1}} \otimes \cdots \otimes b_{i-k}$$

that is a_i is inserted in the v_i-th place. It is easy to see that (2′) is just the desymmetrization of (2). Hence it suffices to show that (1) implies (2′).

Consider the commutative diagram

$$\bigoplus_v (\mathcal{R}_{k+1}(L) \otimes H^0(L)^{i-k}) \xrightarrow{\alpha_0,\ \alpha_1,\ \alpha_2\ \cdots,\ \alpha_{i-k}}$$

$$H^0(L)^{i+1} \xrightarrow{\beta_0} H^0(L^{k+1}) \otimes H^0(L)^{i-k} \xrightarrow{\beta_1} H^0(L^{k+2}) \otimes H^0(L^{i-k-1}) \xrightarrow{\beta_2} \cdots \to H^0(L^i) \otimes H^0(L) \xrightarrow{\beta_{i-k}} H^0(L^{i+1})$$

with $\alpha_0 = \phi$, considered as a map into $H^0(L)^{i+1}$, with β_j the canonical maps, and

$\alpha_j = \beta_j \cdot \alpha_{j-1}$. We have to show:

$$\operatorname{Ker}(\beta_{i-k} \cdot \cdots \cdot \beta_1 \cdot \beta_0) \subseteq \operatorname{Im} \alpha_0.$$

Since L is normally generated, β_j is surjective for every $j = 0, \ldots, i-k$ (c.f. [5]). Hence it suffices to show

$$\operatorname{Ker}(\beta_j) \subseteq \operatorname{Im} \alpha_j \quad \text{for} \quad j = 0, \ldots, i-k.$$

This is true for $j = 0$ by definition of the maps. For $j = 1, \ldots, i-k$ we have

$$\operatorname{Ker}(\beta_j) = \mathcal{R}(L^{k+j}, L) \otimes H^0(L^{i-k-j}).$$

By restriction to a suitable direct summand of $\oplus_v (\mathcal{R}_{k+1}(L) \otimes H^0(L)^{i-k})$ and omission of some tensor factors $H^0(L)$ we see, that it suffices to show that the canonical map

$$\tilde{\alpha}_j : H^0(L)^j \otimes \mathcal{R}_{k+1}(L) \to \mathcal{R}(L^{k+j}, L)$$

is surjective for $j = 1, \ldots, k-i$. But $\tilde{\alpha}_j$ factorizes canonically as follows

$$\begin{array}{ccc} H^0(L)^j \otimes \mathcal{R}_{k+1}(L) & \xrightarrow{\tilde{\alpha}_j} & \mathcal{R}(L^{k+j}, L) \\ & {\scriptstyle 1\otimes\gamma_j} \searrow \quad \nearrow {\scriptstyle \delta_j} & \\ & H^0(L)^j \otimes \mathcal{R}(L^k, L) & \end{array}$$

δ_j is surjective according to the assumption (1). To show the surjectivity of γ_j consider the diagram

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ & & & & 0 & & \mathcal{R}(L^k, L) & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{Ker} \phi & \longrightarrow & H^0(L)^k \otimes H^0(L) & \xrightarrow{\phi} & H^0(L^k) \otimes H^0(L) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi} & & \| & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{R}_{k+1}(L) & \longrightarrow & H^0(L)^{k+1} & \longrightarrow & H^0(L^{k+1}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi} & & \downarrow & & & & \\ & & \operatorname{Coker} \psi & & 0 & & & & \\ & & \downarrow & & & & & & \\ & & 0 & & & & & & \end{array}$$

According to the serpent lemma $\mathcal{R}(L^k, L) \simeq \text{Coker } \psi$ canonically, and under this isomorphism π identifies with γ_j which completes the proof of the lemma.

In order to apply Lemma 4.1 suppose we are given an exact sequence (1). Let P be a G-equivariant compactification of G. It induces a compactification $E(P)$ of E. Denote by $\pi: E(P) \to A$ the natural projection. In [4] the following result was proved (cf. [4], pp. 564–567).

THEOREM 4.2. *Let $L \in \text{Pic}(P)$ be normally presented, G-linearized, $M \in \text{Pic}(A)$ ample, generated by its global sections, and $F = E(L) \otimes \pi^* M$. Then for every $i \geq \dim A + 2$ the canonical map*

$$H^0(F) \otimes \mathcal{R}(F^i, F) \to \mathcal{R}(F^{i+1}, F)$$

is surjective.

If moreover M is normally generated on A, the methods of [4], section 3 show, that $F = E(L) \otimes \pi^* M$ is normally generated on $E(P)$ and we may apply Lemma 4.1 to get:

COROLLARY 4.3. *Let $L \in \text{Pic}(P)$ be normally presented, G-linearized, and $M \in \text{Pic}(A)$ normally generated. Then $F = E(L) \otimes \pi^* M$ is normally generated on $E(P)$ and the homogeneous ideal of the corresponding projective embedding $E(P) \hookrightarrow \mathbb{P}^N$ is generated by forms of degree $\leq \dim A + 3$.*

The most important compactifications of E are those where P is a multiprojective space. Since for such a P a line bundle is normally presented if and only if it is very ample (or even ample) (cf. [4] section 6) we get

COROLLARY 4.4. *Let P be a multiprojective G-equivariant compactification of G, $L \in \text{Pic}(P)$ very ample, G-linearized and $M \in \text{Pic}(A)$ normally generated. Let $E(P) \hookrightarrow \mathbb{P}^N$ be the projective embedding associated to the line bundle $F = E(L) \otimes \pi^* M$. Then the homogeneous ideal of $E(P)$ in $\mathbb{P}^N$ is generated by its forms of degree $\leq \dim A + 3$.*

Since for any ample line bundle M an A the third power M^3 is normally generated, one can even give a bound in case M is only very ample. We omit this.

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