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On the evolution of harmonic mappings of Riemannian surfaces

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Let $(\mathcal{M}, \gamma)$ be a Riemann surface with metric tensor $\gamma = (\gamma_{\alpha\beta})_{1\le\alpha,\beta\le 2}$ and $(\mathcal{N}, g)$ an n-manifold with metric tensor $g = (g_{ij})_{1\le i,j\le n}$. For differentiable mappings $u: \mathcal{M} \to \mathcal{N}$ an energy density $e(u)$ is defined, which in local coordinates $x = (x^1, x^2)$, $u = (u^1, \dots, u^n)$ on $\mathcal{M}, \mathcal{N}$ is given by (with $(\gamma^{\alpha\beta}) = (\gamma_{\alpha\beta})^{-1}$)

$$e(u) = \tfrac{1}{2}\gamma^{\alpha\beta}(x) g_{ij}(u) \frac{\partial}{\partial x^\alpha} u^i \frac{\partial}{\partial x^\beta} u^j. \tag{1.1}$$

Here and in the sequel we adopt the usual summation convention. The energy $E(u)$ of u then equals the integral

$$E(u) = \int_{\mathcal{M}} e(u)\, d\mathcal{M}. \tag{1.2}$$

By definition harmonic mappings from $\mathcal{M}$ onto $\mathcal{N}$ are the (regular) stationary points of E. They necessarily satisfy the Euler–Lagrange equations

$$\frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^\beta}\left(\sqrt{\gamma}\,\gamma^{\alpha\beta} g_{ik}(u) \frac{\partial}{\partial x^\alpha} u^i\right) = \tfrac{1}{2}\gamma^{\alpha\beta} g_{ij,k}(u) \frac{\partial}{\partial x^\alpha} u^i \frac{\partial}{\partial x^\beta} u^j, \qquad 1 \le k \le n, \tag{1.3}$$

where $\gamma = \det(\gamma^{\alpha\beta})$ and $g_{ij,k} = (\partial/\partial u^k) g_{ij}$ as usual. If we carry out the differentiation in the first term and denote

$$\Delta_{\mathcal{M}} = \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^\beta}\left(\sqrt{\gamma}\,\gamma^{\alpha\beta} \frac{\partial}{\partial x^\alpha}\right)$$

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the Laplace–Beltrami operator we obtain after multiplying by $g^{lk}(u)$:

$$\Delta_{\mathcal{M}} u^l = \tfrac{1}{2}\gamma^{\alpha\beta} g^{lk}(u) g_{ij,k}(u) \frac{\partial}{\partial x^\alpha} u^i \frac{\partial}{\partial x^\beta} u^j$$
$$-\gamma^{\alpha\beta} g^{lk}(u) g_{ik,j} \frac{\partial}{\partial x^\alpha} u^i \frac{\partial}{\partial x^\beta} u^j.$$

Using the symmetry of $\gamma^{\alpha\beta} = \gamma^{\beta\alpha}$ and the definition of the Christoffel symbols of the metric g:

$$\Gamma^l_{ij} = \tfrac{1}{2} g^{lk}(g_{ik,j} - g_{ij,k} + g_{jk,i})$$

the right hand side simplifies and we obtain the well-known equation for harmonic maps

$$-\Delta_{\mathcal{M}} u^l = \gamma^{\alpha\beta}\Gamma^l_{ij}(u) \frac{\partial}{\partial x^\alpha} u^i \frac{\partial}{\partial x^\beta} u^j. \tag{1.4}$$

In the following we will use the short-hand notation

$$-\Delta_{\mathcal{M}} u = \Gamma(u)(\nabla u, \nabla u)_{\mathcal{M}}. \tag{1.5}$$

Note that formally for any function φ on $\mathcal{M}$

$$\int_{\mathcal{M}} -\Delta_{\mathcal{M}} u \cdot \varphi - \Gamma(u)(\nabla u, \nabla u)_{\mathcal{M}} \varphi \, d\mathcal{M} = \frac{d}{d\varepsilon} E(u^k + \varepsilon g^{kl}(u)\varphi^l)|_{\varepsilon=0}$$
$$= \int_{\mathcal{M}} (dE(u)^k g^{kl}(u)\varphi^l) \, d\mathcal{M}, \tag{1.6}$$

i.e. the expression (1.5) is the L^2-gradient of the functional E with respect to the scalar product induced by the metric $(g^{lk}(u))$.

In this way the solutions of the evolution problem associated with (1.4) or (1.5)

$$\partial_t u - \Delta_{\mathcal{M}} u = \Gamma(u)(\nabla u, \nabla u)_{\mathcal{M}}; \qquad u(\cdot, 0) = u_0 \tag{1.7}$$

may be naturally interpreted as the trajectories of a gradient-like vector field related to the functional E. The study of these solution curves "in the large" hence will provide a complete description of the set of harmonic mappings from $\mathcal{M}$ to $\mathcal{N}$ as in Morse theory.

Indeed, the first existence results for harmonic mappings of manifolds by Eells and Sampson [2] were obtained from asymptotic estimates on the solutions of (1.7). Eells and Sampson however had to assume that the sectional curvature of the target manifold $\mathcal{N}$ was non-positive. Similar results later were given in [5], [10] under the assumption that the range of u_0 was small in terms of a bound for the sectional curvature of $\mathcal{N}$. These methods were not able to produce e.g. non-constant minimal immersions of the sphere S^2 into another manifold. The existence of such mappings was established in 1981 by Sacks and Uhlenbeck [8] using Ljusternik–Schnirelman theory for a suitable sequence of functionals approximating E. The approximation was necessary because the functional E does not satisfy the Palais–Smale condition (globally).

In this paper our aim is twofold. First we establish the existence of a unique global solution to (1.7) for finite initial energy $E(u_0)<\infty$, which is regular with exception of at most finitely many singular points where non-constant harmonic mappings of $S^2 \cong \mathbb{R}^2$ into $\mathcal{N}$ separate (Theorems 4.2, 4.3). No restriction on the range of u_0 is needed. For small initial energies the solutions to (1.7) are globally regular and asymptotically converge to constant mappings as $t \to \infty$. Hence the flow (1.7) induces a retraction of the space of mappings $u: \mathcal{M} \to \mathcal{N}$ with small energy onto the space of constant mappings.

Although the evolution problem (1.7) is our main point of interest we also present a local Palais–Smale type compactness result (Proposition 5.1) for the functional E which permits a direct proof of the Sacks–Uhlenbeck results. It may be interesting to note that instead of working in the "natural" $H^{1,2}(\mathcal{M}, \mathcal{N})$-topology for this compactness result it is essential to consider E on the dense subspace $H^{2,2}(\mathcal{M}, \mathcal{N})$ and evaluate its L^2-gradient (cp. the next section for notations).

For technical reasons in the following we assume that $\mathcal{M}$ and $\mathcal{N}$ are compact and that $\mathcal{N}$ is isometrically embedded in $\mathbb{R}^N$ for some $N \in \mathbb{N}$. (If $\mathcal{N}$ is compact this can always be achieved by the Nash embedding theorem.)

It seems that the Dirichlet problem for (1.7) on a manifold with boundary and prescribed (regular) boundary data may be handled in the same way, using e.g. the estimates of Ladyženskaya–Solonnikov–Ural'ceva [7, IV. Theorem 9.1, p. 341f] for the Cauchy–Dirichlet problem. Moreover, our methods carry over to evolution problems for general second order differential equations on plane domains with a variational structure. For such problems a result similar to our Theorem 4.1 has independently been obtained by Wieser [11]. For further references on harmonic mappings of manifolds we refer to the survey papers by Eells–Lemaire [1], Hildebrandt [4], and Jost [6].

2. Notation

L^p, $H^{m,p}$, $C^{m,\alpha}$, etc. denote the usual Lebesgue–, Sobolev–, and Hölder spaces. Domain and range may be specified like $L^p(\Omega; \mathbb{R}^n)$. For manifolds $\mathcal{M}$, $\mathcal{N} \subset \mathbb{R}^N$ e.g. the space $H^{m,p}(\mathcal{M}, \mathcal{N})$ may be introduced as the space of functions $u : \mathcal{M} \to \mathcal{N}$ such that $u|_\Omega \in H^{m,p}(\Omega; \mathbb{R}^N)$ for any coordinate chart Ω on $\mathcal{M}$.

Finally

$$V(\mathcal{M}_\tau^T; \mathcal{N}) := \Big\{ u : \mathcal{M} \times [\tau, T] \to \mathcal{N} \mid u \text{ measurable},$$

$$\operatorname*{ess\,sup}_{\tau \le t \le T} \int_{\mathcal{M}} |\nabla u(\cdot, t)|^2 \, d\mathcal{M} + \int_\tau^T \int_{\mathcal{M}} |\nabla^2 u|^2 + |\partial_t u|^2 \, d\mathcal{M} \, dt < \infty \Big\}.$$

Let $|\cdot|_{\mathcal{M}}$ denote geodesic distance on $\mathcal{M}$, and for $R < i_{\mathcal{M}}$, a global lower bound for the injectivity radius of the exponential map on $\mathcal{M}$, let

$$B_R^{\mathcal{M}}(x) = \{ y \in \mathcal{M} \mid |x - y|_{\mathcal{M}} < R \}.$$

For brevity $B_R^{\mathbb{R}^n}(x) = B_R^n(x) = B_R(x)$, $B_R(0) = B_R$. Also denote $U_R(\Omega = \bigcup_{x \in \Omega} B_R(x)$. For a domain Ω and $-\infty < s < t < \infty$ let $\Omega_s^t = \Omega \times [s, t]$. If s or $t = 0$, $\Omega_0^t = \Omega^t$, $\Omega_s^0 = \Omega_s$. In addition to the notation

$$E(u) = \int_{\mathcal{M}} e(u) \, d\mathcal{M}$$

introduced earlier we also define local energies

$$E_R(u; x) = \int_{B_R^{\mathcal{M}}(x)} e(u) \, d\mathcal{M},$$

$$E(u; \mathcal{M}') = \int_{\mathcal{M}'} e(u) \, d\mathcal{M}.$$

Occasionally, a superscript will indicate the space of departure like $E^{\mathcal{M}}$.

The letter c will designate a generic constant possibly depending on $\mathcal{M}$, $\mathcal{N}$, and other data but independent of a particular solution of (1.7) unless specified explicitly like $c(E(u_0))$, etc. For clarity individual constants may be numbered.

3. Estimates

The right class of functions in which to consider system (1.7) turns out to be the space $V(\mathcal{M}^T; \mathcal{N})$. To motivate this choice we drive a-priori estimates and uniqueness for solutions belonging to this class.

First we need an estimate of the L^4-norm of the spatial gradient of a function $u \in V(\mathcal{M}^T; \mathcal{N})$ in terms of its norm in V which is based upon a Sobolev inequality taken from [7; II. Theorem 2.2 and Remark 2.1, p. 63f].

LEMMA 3.1. *There exist constants c, $R_0 > 0$ such that for any $u \in V(M^T; \mathcal{N})$, any $R \in]0, R_0]$ there holds the estimate*

$$\int_{\mathcal{M}^T} |\nabla u|^4 \, d\mathcal{M}\, dt \le c \cdot \operatorname*{ess\,sup}_{(x,t)\in\mathcal{M}^T} \int_{B_R^{\mathcal{M}}(x)} |\nabla u(\cdot, t)|^2 \, d\mathcal{M} \cdot \left(\int_{\mathcal{M}^T} |\nabla^2 u|^2 \, d\mathcal{M} dt + R^{-2} \int_{\mathcal{M}^T} |\nabla u|^2 \, d\mathcal{M}\, dt \right).$$

This lemma will be a consequence of a more refined local result:

LEMMA 3.2. *There exist constants c, $R_0 > 0$ such that for any $u \in V(\mathcal{M}^T; \mathcal{N})$, any $R \in]0, R_0]$, any $x \in \mathcal{M}$, any function $\varphi \in L^\infty(B_R^{\mathcal{M}}(x))$ depending only on the distance from $x : \varphi(y) \equiv \varphi(|y - x|_{\mathcal{M}})$ and non-increasing as a function of this distance there holds the estimate*

$$\int_{\mathcal{M}^T} |\nabla u|^4 \varphi \, d\mathcal{M}\, dt \le c \cdot \operatorname*{ess\,sup}_{0\le t\le T} \int_{B_R^{\mathcal{M}}(x)} |\nabla u(\cdot, t)|^2 \, d\mathcal{M} \cdot \left(\int_{\mathcal{M}^T} |\nabla^2 u|^2 \varphi \, d\mathcal{M}\, dt + R^{-2} \int_{\mathcal{M}^T} |\nabla u|^2 \varphi \, d\mathcal{M}\, dt \right).$$

Proof of Lemma 3.2: i) Suppose $\varphi \equiv \text{const} \equiv 1$, and let $\overline{\nabla u}(t) = (\text{meas } B_R^{\mathcal{M}}(x))^{-1} \cdot \int_{B_R^{\mathcal{M}}(x)} \nabla u(\cdot, t) \, d\mathcal{M}$ be the mean value of ∇u. By [7, II. Theorem 2.2, Remark 2.1,

p. 63f, and (3.2), p. 74]

$$\int_0^T\int_{B_R^{\mathcal{M}}(x)} |\nabla u|^4 \, d\mathcal{M}\, dt \le c\int_0^T\int_{B_R^{\mathcal{M}}(x)} |\nabla u-\overline{\nabla u}|^4 d\mathcal{M}\, dt + c\int_0^T\int_{B_R^{\mathcal{M}}(x)} |\overline{\nabla u}|^4 \, d\mathcal{M}\, dt$$

$$\le c \operatorname*{ess\,sup}_{0\le t\le T} \int_{B_R^{\mathcal{M}}(x)} |\nabla u(\cdot, t)-\overline{\nabla u}(t)|^2 \, d\mathcal{M} \cdot$$

$$\cdot \int_0^T\int_{B_R^{\mathcal{M}}(x)} |\nabla^2 u|^2 \, d\mathcal{M}\, dt + c\int_0^T (\text{meas}\,(B_R^{\mathcal{M}}(x)))^{-3} \cdot$$

$$\cdot \left|\int_{B_R^{\mathcal{M}}(x)} \nabla u(\cdot, t)\, d\mathcal{M}\right|^4 dt$$

with a constant c independent of T, R, and u. Since $q=\overline{\nabla u}(\cdot, t)$ minimizes $\int_{B_R^{\mathcal{M}}(x)} |\nabla u(\cdot, t)-q|^2 \, d\mathcal{M}$ we have for a.e. $t\in[0, T]$

$$\int_{B_R^{\mathcal{M}}(x)} |\nabla u(\cdot, t)-\overline{\nabla u}(t)|^2 \, d\mathcal{M} \le \int_{B_R^{\mathcal{M}}(x)} |\nabla u(\cdot, t)|^2 \, d\mathcal{M}$$

Moreover, by Hölder's inequality

$$\left|\int_{B_R^{\mathcal{M}}(x)} \nabla u(\cdot, t)\, d\mathcal{M}\right|^4 \le (\text{meas}\,(B_R^{\mathcal{M}}(x)))^2 \left(\int_{B_R^{\mathcal{M}}(x)} |\nabla u(\cdot, t)|^2 \, d\mathcal{M}\right)^2$$

$$\le (\text{meas}\,(B_R^{\mathcal{M}}(x)))^2 \operatorname*{ess\,sup}_{0\le t\le T} \int_{B_R^{\mathcal{M}}(x)} |\nabla u(\cdot, t)|^2 \, d\mathcal{M} \cdot$$

$$\cdot \int_{B_R^{\mathcal{M}}(x)} |\nabla u(\cdot, t)|^2 \, d\mathcal{M},$$

which concludes the proof in case $\varphi\equiv$ const.

ii) By linearity and i) the assertion remains true for step functions φ which are non-increasing in the radial distance. Finally, the general case follows by density of step functions in $L^\infty(B_R^{\mathcal{M}}(x))$ in measure. q.e.d.

Lemma 3.1 may now be derived from Lemma 3.2 via the following covering argument.

LEMMA 3.3. *There exist constants K, $R_0>0$ depending only on $\mathcal{M}$ such that for any $R\in\,]0, R_0]$ there exists a cover of $\mathcal{M}$ by balls $B_{R/2}^{\mathcal{M}}(x_i)$ with the property that at any point $x\in\mathcal{M}$ at most K of the balls $B_R^{\mathcal{M}}(x_i)$ meet.*

Proof. We may assume that $\mathcal{M} \subset \mathbb{R}^M$ is isometrically embedded. There exists a neighborhood $U_\delta(\mathcal{M})$ in $\mathbb{R}^M$ such that any $x \in U_\delta(\mathcal{M})$ has a unique nearest neighbor $\underline{x} \in \mathcal{M}$ satisfying $|x - \underline{x}| = \inf\{|x-y| \mid y \in \mathcal{M}\}$. Choose $R_0 \leq \min\{\delta, i_{\mathcal{M}}/2\}$ and such that for $R \leq R_0$ any (Euclidean) ball $B_R^M(x_0) \cap \mathcal{M}$ with center at $x_0 \in \mathcal{M}$ satisfies

$$B_{R/2}^{\mathcal{M}}(x_0) \subset (B_R^M(x_0) \cap \mathcal{M}) \subset B_{2R}^{\mathcal{M}}(x_0).$$

Now for any $R > 0$ there is a cover of $\mathbb{R}^M$ by balls $B_{R/4}^M(x_i)$ with at most $K = K(M)$ of the balls $B_{8R}^M(x_i)$ intersecting at any given point. Suppose $0 < R \leq R_0$, and let $\{x_i\}_{1 \leq i \leq I}$ be those points belonging to the cover corresponding to R that lie within $U_{R/4}(\mathcal{M})$. Projecting x_i to $\mathcal{M}$ we obtain the cover $\{B_R^{\mathcal{M}}(x_i)\}_{1 \leq i \leq I}$ of $\mathcal{M}$ that we seek. Indeed,

$$\mathcal{M} \subset \bigcup_{i=1}^{I} B_{R/4}^M(x_i) \cap \mathcal{M} \subset \bigcup_{i=1}^{I} B_{R/2}^M(\underline{x}_i) \cap \mathcal{M} \subset \bigcup_{i=1}^{I} B_R^{\mathcal{M}}(x_i),$$

while for any $x \in \mathcal{M}$

$$x \in B_{2R}^{\mathcal{M}}(\underline{x}_i) \subset B_{4R}^M(\underline{x}_i) \subset B_{8R}^M(x_i)$$

for at most K indices i. q.e.d.

Proof of Lemma 3.1. Choose a cover $\{B_R^{\mathcal{M}}(x_i)\}$ of $\mathcal{M}$ with the properties in Lemma 3.3. Apply Lemma 3.2 with $\varphi \equiv 1$ on each $B_R^{\mathcal{M}}(x_i)$ and add, using the finite intersection property of the cover. q.e.d.

We may now state the following simple a-priori estimate:

LEMMA 3.4. *There exists a constant $c = c(\mathcal{N})$ such that for any solution $u \in V(\mathcal{M}^T; \mathcal{N})$ of* (1.7) *there holds the estimate*

$$\int_{\mathcal{M}^T} |\partial_t u|^2 \, d\mathcal{M} \, dt \leq cE(u_0).$$

Moreover, $E(u(\cdot, t))$ is absolutely continuous on $[0, T]$ and non-increasing.

Proof. By Lemma 3.1 we may multiply (1.7) by $g_{ij}(u)\, \partial_t u^j$ and integrate. On account of (1.6) this gives for any $s, t \in [0, T]$:

$$\int_{\mathcal{M}_s^t} g_{ij}(u)\, \partial_t u^i\, \partial_t u^j\, d\mathcal{M}\, dt + \int_s^t \frac{d}{dt} E(u(\cdot, t))\, dt = 0,$$

and the claim follows. q.e.d.

Remark 3.5. Combining Lemmata 3.1, 3.4 we obtain the estimate

$$\int_{\mathcal{M}^T} |\nabla u|^4 d\mathcal{M}\, dt \le c \cdot \sup_{(x,t)\in\mathcal{M}^T} E_R(u(\cdot, t); x) \cdot \left(\int_{\mathcal{M}^T} |\nabla^2 u|^2\, d\mathcal{M}\, dt + \frac{T}{R^2} \cdot E(u_0) \right)$$

for any solution $u \in V(\mathcal{M}^T; \mathcal{N})$ of (1.7) and any $R \in]0, R_0]$. This makes it important to control energy *locally*.

LEMMA 3.6. *There exists a constant* $c_1 = c_1(\mathcal{M}, \mathcal{N})$ *such that for any solution* $u \in V(\mathcal{M}^T; \mathcal{N})$ *of* (1.7), *any* $R \in]0, R_0]$, *and any* $(x, t) \in \mathcal{M}^T$ *there holds the estimate*

$$E_R(u(\cdot, t); x) \le E_{2R}(u(\cdot, 0), x) + c_1 \frac{t}{R^2} E(u_0).$$

Proof. Let $\varphi \in C_0^\infty(B_{2R}^{\mathcal{M}}(x))$ satisfy $0 \le \varphi \le 1$, $\varphi \equiv 1$ on $B_R^{\mathcal{M}}(x)$, $|\nabla\varphi| \le c/R$. Multiply equation (1.7) by $g_{ij}(u)\, \partial_t u^j \varphi^2$ and use Young's inequality to obtain

$$\int_{\mathcal{M}^t} g_{ij}(u)\, \partial_t u^i\, \partial_t u^j \varphi^2\, d\mathcal{M}\, dt + \int_{\mathcal{M}^t} \frac{d}{dt}(e(u)\varphi^2)\, d\mathcal{M}\, dt$$

$$\le c \int_{\mathcal{M}^t} |\nabla u|\, |\partial_t u|\, |\nabla\varphi|\, \varphi\, d\mathcal{M}\, dt$$

$$\le \int_{\mathcal{M}^t} g_{ij}(u)\, \partial_t u^i\, \partial_t u^j \varphi^2\, d\mathcal{M}\, dt + cR^{-2} \int_{\mathcal{M}^t} |\nabla u|^2\, d\mathcal{M}\, dt.$$

Hence by Lemma 3.4

$$E_R(u(\cdot, t); x) \le \int_{\mathcal{M}} e(u(\cdot, t))\varphi^2\, d\mathcal{M} \le \int_{\mathcal{M}} e(u_0)\varphi^2\, d\mathcal{M} + cR^{-2} \int_0^t E(u(\cdot, t))\, dt$$

$$\le E_{2R}(u_0, x) + c \frac{t}{R^2} E(u_0) \quad \text{q.e.d.}$$

For a solution $u \in V(\mathcal{M}^T; \mathcal{N})$ of (1.7) and $R \in]0, R_0]$ let

$$\varepsilon(R) = \varepsilon(R; u, T) = \sup_{(x,t)\in\mathcal{M}^T} E_R(u(\cdot, t); x).$$

In the sequel we give a-priori bounds for the V-norm and Hölder norms of u in terms of initial energy $E(u_0)$, T, and the number

$$R = \sup\{R > 0 \mid \varepsilon(R; u, T) \leq \varepsilon_1\}$$

measuring distribution of energy along the flow. Here $\varepsilon_1 > 0$ is a parameter depending only on $\mathcal{M}$ and $\mathcal{N}$ which will be determined in Lemmata 3.7, 3.7′, 3.10, 3.10′. (We agree to let ε_1 equal the smallest of the numbers ε_1 occurring in these lemmata.)

LEMMA 3.7. *There exists a constant $\varepsilon_1 > 0$ such that for any solution $u \in V(\mathcal{M}^T; \mathcal{N})$ of* (1.7) *and any number $R \in]0, R_0]$ there holds the estimate*

$$\int_{\mathcal{M}^T} |\nabla^2 u|^2 \, d\mathcal{M}\, dt \leq cE(u_0)(1 + TR^{-2}),$$

provided $\varepsilon(R) \leq \varepsilon_1$.

Proof. Multiply (1.7) by $\Delta_{\mathcal{M}} u$ and integrate over $\mathcal{M}^T$ to obtain

$$\begin{aligned} \frac{1}{2}\int_{\mathcal{M}^T} \frac{d}{dt}(\nabla u, \nabla u)_{\mathcal{M}} d\mathcal{M}\, dt + \int_{\mathcal{M}^T} |\Delta_{\mathcal{M}} u|^2 \, d\mathcal{M}\, dt \\ \leq c\int_T |\Delta_{\mathcal{M}} u|\, |\nabla u|^2 \, d\mathcal{M}\, dt \\ \leq \frac{1}{2}\int_{\mathcal{M}^T} |\Delta_{\mathcal{M}} u|^2 \, d\mathcal{M}\, dt + c\int_{\mathcal{M}^T} |\nabla u|^4 \, d\mathcal{M}\, dt, \end{aligned}$$

by Young's inequality. Here, of course $(\nabla u, \nabla u)_{\mathcal{M}} = \gamma^{\alpha\beta}\, \partial_\alpha u^i\, \partial_\beta u^i$. By Remark 3.5 and definition of $\varepsilon(R)$ we may estimate

$$\int_{\mathcal{M}^T} |\nabla u|^4 \, d\mathcal{M}\, dt \leq c\varepsilon(R)\left(\int_{\mathcal{M}^T} |\nabla^2 u|^2 \, d\mathcal{M}\, dt + \frac{T}{R^2} E(u_0)\right).$$

Moreover, integration by parts yields the estimate

$$\int_{\mathcal{M}^T} |\Delta_{\mathcal{M}} u|^2 \, d\mathcal{M}\, dt \geq \int_{\mathcal{M}^T} |\nabla^2 u|^2 \, d\mathcal{M}\, dt - c\int_{\mathcal{M}^T} |\nabla u|^2 \, d\mathcal{M}\, dt, \tag{3.1}$$

the second term on the right resulting from differentiating the coefficients of the metric γ.

Therefore, if $\varepsilon(R) \leq \varepsilon_1$ for some $\varepsilon_1 = \varepsilon_1(\mathcal{M}; \mathcal{N}) > 0$ we obtain that

$$\int_{\mathcal{M}^T} |\nabla^2 u|^2 \, d\mathcal{M}\, dt \leq cE(u_0)(1 + TR^{-2})$$

as claimed. q.e.d.

In order to be able to state pointwise a-priori estimates we now derive uniform *local* estimates for $\int |\nabla u|^4 \, d\mathcal{M}\, dt$ in terms of the data.

LEMMA 3.8. *For any numbers* ε, τ, $E_0 > 0$, $R_1 \in \,]0, R_0]$ *there exists a number* $\delta > 0$ *such that for any solution* $u \in V(\mathcal{M}^T; \mathcal{N})$ *of* (1.7) *and any* $I \subset [\tau, T]$ *with measure* $|I| < \delta$ *there holds the estimate*

$$\int_I \left(\int_{\mathcal{M}} |\nabla u|^4 \, d\mathcal{M} \right) dt < \varepsilon,$$

provided $\varepsilon(R_1) \leq \varepsilon_1$, $E(u_0) \leq E_0$.

Proof. For any solution $u \in V(\mathcal{M}^T; \mathcal{N})$ by Lemma 3.1 $|\nabla u| \in L^4(\mathcal{M}^T)$ and $\int (\int_{\mathcal{M}} |\nabla u|^4 \, d\mathcal{M})\, dt$ is absolutely continuous.

To show uniformity let $u_m \in V(\mathcal{M}^T; \mathcal{N})$ be solutions of (1.7) satisfying the conditions of the lemma. By Lemmata 3.4, 3.7 we have

$$\int_{\mathcal{M}^T} |\partial_t u_m|^2 + |\nabla^2 u_m|^2 \, d\mathcal{M}\, dt \leq c(E_0, R_1, T),$$

provided $\varepsilon(R_1; u_m, T) \leq \varepsilon_1, \forall m \in \mathbb{N}$. Moreover

$$E(u_m(\cdot, t)) \leq E_0, \qquad \forall t \in [0, T], \qquad \forall m \in \mathbb{N}.$$

Hence there exists a subsequence $\{u_m\}$ (relabelled) such that $u_m \to u$ a.e., $\partial_t u_m \rightharpoonup \partial_t u$, $\nabla^2 u_m \rightharpoonup \nabla^2 u$ weakly in $L^2(\mathcal{M}^T)$ and $\nabla u_m \to \nabla u$ strongly in $L^2(\mathcal{M}^T)$. This permits passing to the limit $m\to\infty$ in (1.7) and we find that also $u\in V(\mathcal{M}^T;\mathcal{N})$ is a solution to the equation (1.7). The results of Lemma 3.4 therefore apply to u. In particular, $\nabla u(\cdot,t)$ is continuous in $t\in[0,T]$ with respect to the L^2-norm. By compactness of $[0,T]$ for any given $\varepsilon>0$ we can thus find a number $R=R(\varepsilon)>0$ such that

$$\varepsilon(2R;u,T)<\varepsilon.$$

Moreover, we may assume that $\nabla u_m(\cdot,t)\to\nabla u(\cdot,t)$ in $L^2(\mathcal{M})$ for a.e. $t\in[0,T]$. Hence we may determine numbers $0\le t_1\le\tau<t_2<\cdots<t_L\le T=:t_{L+1}$ having distance

$$|t_{\ell+1}-t_\ell|\le\frac{\varepsilon R^2}{c_1E_0},\qquad \ell=1,\dots,L,$$

and an index $m_0=m_0(\varepsilon,\tau)$ such that for $m\ge m_0$

$$\int_{\mathcal{M}^T}|\nabla u_m(\cdot,t_\ell)-\nabla u(\cdot,t_\ell)|^2\,d\mathcal{M}<\varepsilon,\qquad \ell=1,\dots,L. \tag{3.2}$$

Here, c_1 is the constant of Lemma 3.6. By Lemma 3.6 then we may estimate

$$\begin{aligned}E_R(u_m(\cdot,t);x)&\le E_{2R}(u_m(\cdot,t_\ell),x)+c_1\frac{t-t_\ell}{R^2}E_0\\&\le E_{2R}(u(\cdot,t_\ell),x)+2\varepsilon\le3\varepsilon,\end{aligned}$$

uniformly for $(x,t)\in\mathcal{M}_\tau^T$, $m\ge m_0$, where $t_\ell=\max\{t_j\mid t_j\le t\}$. Finally, Remark 3.5 gives the estimate

$$\int_I\int_{\mathcal{M}}|\nabla u_m|^4\,d\mathcal{M}\,dt\le c\cdot\varepsilon\left(\iint_{\mathcal{M}^T}|\nabla^2u_m|^2\,d\mathcal{M}\,dt+\frac{|I|}{R^2}E_0\right)\le c^*\varepsilon$$

with a constant $c^*=c^*(\mathcal{M},\mathcal{N},E_0,R_1,T)$, for any $I\subset[\tau,T]$, provided $|I|<R^2$, $m\ge m_0$. This proves uniformity and the lemma. q.e.d.

Remark 3.9. The preceding proof also shows compactness in $V(\mathcal{M}^T;\mathcal{N})$ of solutions $u_m\in V(\mathcal{M}^T;\mathcal{N})$ to equation (1.7) issuing from a set of initial data which is compact in $H^{1,2}(\mathcal{M},\mathcal{N})$, $u_m(0)\to u_0$ $(m\to\infty)$, provided $\varepsilon(R;u_m,T)\le\varepsilon_1$ for some $R\in\,]0,R_0]$. Indeed, in this case we may choose $t_1=0$ in (3.2). The proof of Lemma 3.8 then shows that $\int(\int_{\mathcal{M}}|\nabla u_m|^4\,d\mathcal{M})\,dt$ is uniformly absolutely continuous on $[0, T]$. We may suppose $u_m \to u$ a.e., and $\partial_t u_m \to \partial_t u$, $\nabla^2 u_m \rightharpoonup \nabla u$ weakly in $L^2(\mathcal{M}^T)$, $\nabla u_m \to \nabla u$ strongly in $L^2(\mathcal{M}^T)$. Let $v_m := u_m - u$; $|\nabla U_m| := |\nabla u_m| + |\nabla u|$. Then

$$|\partial_t v_m - \Delta_{\mathcal{M}} v_m| \le c(|v_m|\,|\nabla U_m|^2 + |\nabla v_m|\,|\nabla U_m|).$$

Hence, multiplying by $\Delta_{\mathcal{M}} v_m$ and integrating gives

$$\frac{1}{2}\int_{\mathcal{M}^t} \frac{d}{dt}(\nabla v_m, \nabla v_m)_{\mathcal{M}}\, d\mathcal{M}\, dt + \int_{\mathcal{M}^T} |\Delta_{\mathcal{M}} v_m|^2\, d\mathcal{M}\, dt$$
$$\le \frac{1}{2}\int_{\mathcal{M}^T} |\Delta_{\mathcal{M}} v_m|^2\, d\mathcal{M}\, dt + c\int_{\mathcal{M}^T} (|v_m|^2\,|\nabla U_m|^4 + |\nabla v_m|^2\,|\nabla U_m|^2)\, d\mathcal{M}\, dt.$$

Rearranging, and using (3.1)

$$\sup_{0\le t\le T} \int_{\mathcal{M}} |\nabla v_m(\cdot, t)|^2\, d\mathcal{M} + \int_{\mathcal{M}^T} |\nabla^2 v_m|^2\, d\mathcal{M}\, dt$$
$$\le c\int_{\mathcal{M}} |\nabla v_m(\cdot, 0)|^2\, d\mathcal{M} + c\int_{\mathcal{M}^T} (|v_m|^2\,|\nabla U_m|^4 + |\nabla v_m|^2\,|\nabla U_m|^2)\, d\mathcal{M}\, dt$$
$$\to 0 \qquad (m \to \infty),$$

by Vitali's theorem and since $\nabla v_m(\cdot, 0) \to 0$ in $L^2(\mathcal{M})$. Similarly,

$$\int_{\mathcal{M}^T} |\partial_t v_m|^2\, d\mathcal{M}\, dt \le c\int_{\mathcal{M}^T} (|\Delta_{\mathcal{M}} v_m|^2 + |v_m|^2\,|\nabla U_m|^4 + |\nabla v_m|^2\,|\nabla U_m|^2)\, d\mathcal{M}\, dt$$
$$\to 0 \qquad (m \to \infty),$$

i.e. $u_m \to u$ strongly in $V(\mathcal{M}^T; \mathcal{N})$.

LEMMA 3.10. *Let* $u \in V(\mathcal{M}^T; \mathcal{N}) \bigcap_{\tau>0} C^2(\mathcal{M}^T_\tau; \mathcal{N})$ *be a regular solution to* (1.7). *Then for any* $\tau > 0$ *the Hölder norms of* u *and its derivatives may be estimated uniformly on* $\mathcal{M}^T_\tau$ *by quantities involving only* $E(u_0)$, τ, T *and* R, *provided* $\varepsilon(R) \le \varepsilon_1$.

Remark. Replacing $\partial_t u$ by difference quotients our proof below may be modified to establish interior regularity of $V(\mathcal{M}^T; \mathcal{N})$—solutions to (1.7).

Proof. First note that (1.7) implies

$$\int_{\mathcal{M}} |\Delta_{\mathcal{M}} u(\cdot, t)|^2\, d\mathcal{M} \le c\int_{\mathcal{M}} |\nabla u(\cdot, t)|^4 + |\partial_t u(\cdot, t)|^2\, d\mathcal{M} \tag{3.3}$$

at a.e. time $t \in [0, T]$. To bound the right hand side differentiate (1.7) with respect to t, multiply with $\partial_t u$, and integrate over $\mathcal{M}_s^t$, $\tau \le s < t \le T$, to obtain

$$\begin{aligned}\frac{1}{2}\int_{\mathcal{M}_s^t} \partial_t |\partial_t u|^2 \, d\mathcal{M}\, dt + \int_{\mathcal{M}_s^t} |\nabla \partial_t u|^2 \, d\mathcal{M}\, dt \\ \le c\int_{\mathcal{M}_s^t} (|\partial_t u|^2 |\nabla u|^2 + |\partial_t u|\, |\nabla u|\, |\nabla \partial_t u|) \, d\mathcal{M}\, dt \\ \le \frac{1}{2}\int_{\mathcal{M}^t} |\nabla \partial_t u|^2 \, d\mathcal{M}\, dt + c\int_{\mathcal{M}^t} |\partial_t u|^2 |\nabla u|^2 \, d\mathcal{M}\, dt \qquad (3.4)\end{aligned}$$

Estimate like Lemma 3.1 for $|t-s| \le 1$:

$$\begin{aligned}\int_{\mathcal{M}_s^t} |\partial_t u|^2 |\nabla u|^2 \, d\mathcal{M}\, dt &\le \left(\int_{\mathcal{M}_s^t} |\partial_t u|^4 \, d\mathcal{M}\, dt \int_{\mathcal{M}_s^t} |\nabla u|^4 \, d\mathcal{M}\, dt\right)^{1/2} \\ &\le c\left(\int_{\mathcal{M}_s^t} |\nabla u|^4 \, d\mathcal{M}\, dt\right)^{1/2} \cdot \left(\operatorname*{ess\,sup}_{s \le \theta \le t} \int_{\mathcal{M}} |\partial_t (u(\cdot, \theta))|^2 \, d\mathcal{M} \right. \\ &\quad \left. + \int_{\mathcal{M}_s^t} |\nabla \partial_t u|^2 \, d\mathcal{M}\, dt\right)\end{aligned}$$

By Lemma 3.8, if $t - s < \delta$ is sufficiently small, the right-hand-side of (3.4) may be absorbed in the left, yielding

$$\int_{\mathcal{M}} |\partial_t u(\cdot, t)|^2 d\mathcal{M} \le c \inf_{\substack{t-\delta \le s \le t \\ 0 \le s}} \int_{\mathcal{M}} |\partial_t u(\cdot, s)|^2 \, d\mathcal{M}$$

with a constant c depending only on the modulus of continuity of $\int(\int_{\mathcal{M}} |\nabla u|^4 \, d\mathcal{M})\, dt$, i.e. on $E(u_0)$, τ, T, R, providing $\varepsilon(R) \le \varepsilon_1$. Estimating the infimum by the mean value now gives

$$\begin{aligned}\operatorname*{ess\,sup}_{2\tau \le t \le T} \int_{\mathcal{M}} |\partial_t u(\cdot, t)|^2 \, d\mathcal{M} &\le c(1+\tau^{-1}) \int_{\mathcal{M}_s^t} |\partial_t u|^2 \, d\mathcal{M}\, dt \\ &\le c(1+\tau^{-1}) E(u_0) \qquad (3.5)\end{aligned}$$

with another such constant. In order to bound $\int_{\mathcal{M}} |\nabla u(\cdot, t)|^4 \, d\mathcal{M}$ apply Lemma 3.1

to the constant function $v\equiv\nabla u(\cdot,t)$ to obtain the estimate

$$\begin{aligned}\int_{\mathcal{M}}|\nabla u(\cdot,t)|^4\,d\mathcal{M}&\le c\cdot\operatorname*{ess\,sup}_{x\in\mathcal{M}}\int_{B_R^{\mathcal{M}}(x)}|\nabla u(\cdot,t)|^2\,d\mathcal{M}\cdot\left(\int_{\mathcal{M}}|\nabla^2u(\cdot,t)|^2\,d\mathcal{M}\right.\\&\quad\left.+R^{-2}\int_{\mathcal{M}}|\nabla u(\cdot,t)|^2\,d\mathcal{M}\right)\\&\le c\cdot\operatorname*{ess\,sup}_{(x,s)\in\mathcal{M}_\tau^T}E_R(u(\cdot,s),x)\left(\int_{\mathcal{M}}|\nabla^2u(\cdot,t)|^2\,d\mathcal{M}\right.\\&\quad\left.+R^{-2}E(u_0)\right).\end{aligned}\tag{3.6}$$

Hence, if $\varepsilon(R)\le\varepsilon_1$ is sufficiently small (3.1), (3.3), (3.5) and (3.6) yield the uniform estimate for $t\in[\tau,T]$:

$$\int_{\mathcal{M}}|\nabla^2u(\cdot,t)|^2\,d\mathcal{M}\le cE(u_0)(1+\tau^{-1}+R^{-2})\tag{3.7}$$

with a constant depending only on $\mathcal{M},\mathcal{N}$, $E(u_0)$, τ, T, and R, provided $\varepsilon(R)\le\varepsilon_1$. By the embedding $H^{2,2}(\mathcal{M})\to H^{1,p}(\mathcal{M})$ for any $p<\infty$ (3.7) now yields a bound for $|\partial_tu-\Delta_{\mathcal{M}}u|\in L^p(\mathcal{M}_\tau^T)$ of the same type. Using [7, IV. Theorem 9.1, p. 341f, and II. Lemma 3.3, p. 80] we thus obtain Hölder estimates for u in terms of the quantities listed in the assertion of the lemma. Higher regularity is standard. q.e.d.

Remark 3.11. If u_0 is regular we may improve estimate (3.5) using Lemma 3.1 and Remark 3.9 to obtain with a constant c depending on u_0, T, and R, where $\varepsilon(R)\le\varepsilon_1$:

$$\operatorname*{ess\,sup}_{0\le t\le T}\int_{\mathcal{M}}|\partial_tu(\cdot,t)|^2\,d\mathcal{M}\le cE(u_0)\left(1+\int_{\mathcal{M}}|\nabla^2u_0|^2\,d\mathcal{M}\right).\tag{3.5$'$}$$

Hence for regular u_0 we obtain global a-priori bounds of Hölder norms of u and its derivatives on $\mathcal{M}^T$ in terms of the data and any number $R\in\,]0,R_0]$ such that $\varepsilon(R)\le\varepsilon_1$.

We also need local versions of the preceding results. For any $\mathcal{M}'\subset\mathcal{M}$, $R\in\,]0,R_0]$, any $u\in V((U_R(\mathcal{M}'))^T;\mathcal{N})$ let

$$\varepsilon(R;\mathcal{M}')=\varepsilon(R;\mathcal{M}';u,T)=\sup_{(x,t)\in(\mathcal{M}')^T}E_R(u(\cdot,t),x).$$

LEMMA 3.7′. *There exists a constant* $\varepsilon_1 > 0$ *such that for any* $R \in \,]0, R_0]$, *any* $\mathcal{M}' \subset \mathcal{M}$, *and any solution* $u \in V((U_R(\mathcal{M}'))^T; \mathcal{N}) \bigcap_{T'<T} V(\mathcal{M}^{T'}; \mathcal{N})$ *there holds the estimate*

$$\int_{(\mathcal{M}')^t} |\nabla^2 u|^2 \, d\mathcal{M}\, dt \le cE(u_0)(1 + TR^{-2}),$$

provided $\varepsilon(R, \mathcal{M}') \le \varepsilon_1$.

Proof. Let $\{B^{\mathcal{M}}_{R/2}(x_i)\}$ be a cover of $\mathcal{M}$ as constructed in Lemma 3.3 and for each i let $\varphi \in C_0^\infty(B^{\mathcal{M}}_R(x_i))$ satisfying $0 \le \varphi \le 1$, $\varphi \equiv 1$ on $B^{\mathcal{M}}_{R/2}(x_i)$, $|\nabla\varphi| \le cR^{-1}$ be a non-increasing function of the distance from x_i. For each i we now multiply (1.7) by $\Delta_{\mathcal{M}} u \varphi^2$ and integrate. Note that e.g.

$$-\int_{\mathcal{M}^T} \partial_t u \cdot \Delta_{\mathcal{M}} u \varphi^2 \, d\mathcal{M}\, dt \ge \frac{1}{2}\int_{\mathcal{M}^T} \frac{d}{dt}[(\nabla u, \nabla u)_{\mathcal{M}} \varphi^2] \, d\mathcal{M}\, dt$$
$$- c \cdot \int_{\mathcal{M}^T} |\partial_t u| \, |\nabla u| \, |\nabla \varphi| \, \varphi \, d\mathcal{M}\, dt,$$

and the latter may be estimated

$$\int_{\mathcal{M}^T} |\partial_t u| \, |\nabla u| \, |\nabla\varphi| \, \varphi \, d\mathcal{M}\, dt \le \int_{\mathcal{M}^T} |\partial_t u|^2 \varphi^2 \, d\mathcal{M}\, dt$$
$$+ cR^{-2} \int_{\mathcal{M}^T \cap \operatorname{supp} \varphi} |\nabla u|^2 \, d\mathcal{M}\, dt.$$

Also

$$c \cdot \int_{\mathcal{M}^T} |\Delta_{\mathcal{M}} u|^2 \varphi^2 \, d\mathcal{M}\, dt \ge \int_{\mathcal{M}^T} |\nabla^2 u|^2 \varphi^2 \, d\mathcal{M}\, dt - cR^{-2} \int_{\mathcal{M}^T \cap \operatorname{supp} \varphi} |\nabla u|^2 \, d\mathcal{M}\, dt$$

while $\int_{\mathcal{M}^T} |\nabla u|^4 \varphi^2 \, d\mathcal{M}\, dt$ may be estimated using Lemma 3.2 instead of Lemma 3.1.

As in the proof of Lemma 3.7 we then obtain

$$\int_{\mathcal{M}^T} |\nabla^2 u|^2 \varphi^2 \, d\mathcal{M}\, dt \le c \int_{\mathcal{M}} |\nabla u_0|^2 \varphi^2 \, d\mathcal{M} + c \int_{\mathcal{M}^T} |\partial_t u|^2 \varphi^2 \, d\mathcal{M}\, dt$$
$$+ cR^{-2} \int_{\mathcal{M}^T \cap \operatorname{supp} \varphi} |\nabla u|^2 \, d\mathcal{M}\, dt$$

provided $\varepsilon(R; x_i) \leq \varepsilon_1$ is sufficiently small. Summing over those indices i with $B^{\mathcal{M}}_{R/2}(x_i) \cap \mathcal{M}' \neq \emptyset$ the claim follows. q.e.d.

LEMMA 3.8'. *For any ε, τ, $E_0 > 0$, $R \in]0, R_0]$ there exists a number $\delta > 0$ such that for any $\mathcal{M}' \subset \mathcal{M}$ and any solution $u \in V((U_R(\mathcal{M}'))^T; \mathcal{N}) \cap_{T'<T} V(\mathcal{M}^{T'}, \mathcal{N})$ of* (1.7) *on any $I \subset [\tau, T]$ with measure $|I| < \delta$ there holds the estimate*

$$\int_I \left(\int_{\mathcal{M}} |\nabla u|^4 \, d\mathcal{M} \right) dt < \varepsilon,$$

provided $\varepsilon(R) \leq \varepsilon_1$, $E(u_0) \leq E_0$.

Proof. To show the contended uniformity again let $\{u_m\}$ be a sequence of solutions of (1.7) satisfying the hypotheses of Lemma 3.8'. By the estimates

$$\int_{\mathcal{M}^T} |\partial_t u_m|^2 \, d\mathcal{M} \, dt \leq c; \qquad \int_{(U_{R/2}(\mathcal{M}'))^T} |\nabla^2 u|^2 \, d\mathcal{M} \, dt \leq c$$

that are implied by Lemmata 3.4, 3.7' we may assume that $u_m \to u$ a.e., $\partial_t u_m \rightharpoonup \partial_t u$, $\nabla^2 u_m \rightharpoonup \nabla^2 u$ weakly and $\nabla u_m \to \nabla u$ strongly in L^2 on $(U_{R/2}(\mathcal{M}'))^T$ as $m \to \infty$. Hence u solves (1.7) on $(U_{R/2}(\mathcal{M}'))^T$. Moreover, by weak lower semi-continuity and Lemma 3.4

$$E(u(\cdot, t)) \leq \liminf_{m \to \infty} E(u_m(\cdot, t)) \leq E_0$$

for a.e. $t \in [0, T]$. Now let $\varphi \in C_0^\infty(U_{R/2}(\mathcal{M}'))$ satisfy $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ on M', $|\nabla \varphi| \leq c/R$. Upon "testing" the equation (1.7) for u by the function $g_{ij}(u)\, \partial_t u^j \varphi^2$ as in Lemma 3.6 there results

$$\int_{\mathcal{M}^T} \frac{d}{dt} [e(u(\cdot, t))\varphi^2] \, d\mathcal{M} \, dt \leq cR^{-2} \int_{\mathcal{M}^T} |\nabla u|^2 \, d\mathcal{M} \, dt.$$

Hence $\nabla u(\cdot, t)$ is continuous in $L^2(\mathcal{M}')$ as a function of t. The remainder of the proof now proceeds exactly as in Lemma 3.8, using Lemma 3.2 and a partition $\{B^{\mathcal{M}}_{R/2}(x_i)\}$ covering $\mathcal{M}'$. q.e.d.

Remark 3.9'. If $u_m \in V(U_R(\mathcal{M}')^T; \mathcal{N}) \cap_{T'<T} V(\mathcal{M}^{T'}; \mathcal{N})$ are solutions to (1.7) for initial data $u_{m0} \to u_0$ in $H^{1,2}(\mathcal{M}; \mathcal{N})$, and if $\varepsilon(R; \mathcal{M}'; u_m, T) \leq \varepsilon_1$ uniformly for some $R \in]0, R_0]$, then $u_m \to u$ in $V((\mathcal{M}')^T; \mathcal{N})$ as $m \to \infty$.

LEMMA 3.10′. *Let* $u \in V((U_R(\mathcal{M}'))^T;\ \mathcal{N}) \bigcap_{\tau>0} C^2((U_R(\mathcal{M}'))_\tau^T,\ \mathcal{N}) \bigcap_{T'<T} V(\mathcal{M}^{T'};\mathcal{N})$ *be a (locally) regular solution to* (1.7). *Then for any* $\tau>0$ *the Hölder norms of u and its derivatives may be estimated uniformly on* $(\mathcal{M}')_\tau^T$ *by quantities involving only* $E(u_0)$, τ, T, *and* R, *provided* $\varepsilon(R;\mathcal{M}')\le\varepsilon_1$.

Proof. The proof may be carried out exactly as in the case of Lemma 3.10, localizing the estimates on balls $B_R^{\mathcal{M}}(x_i)$ of a suitable partition and using functions like $\partial_t u\varphi^2$ as testing functions, where $0\le\varphi\le1$, $|\nabla\varphi|\le c/R$, $\varphi\equiv1$ on $B_{R/2}^{\mathcal{M}}(x_i)$ and φ depends only on the distance from x_i, and applying Lemma 3.2. q.e.d.

Remark 3.11′. If u_0 is regular on $U_R(\mathcal{M}')$ we obtain a-priori bounds of Hölder norms of u and its derivatives on $(\mathcal{M}')^T$ in terms of u_0, T, and R, provided $\varepsilon(R;\mathcal{M}')\le\varepsilon_1$.

We conclude this section by showing uniqueness of solutions to (1.7) in the class $V(\mathcal{M}^T;\mathcal{N})$.

LEMMA 3.12. *Suppose* u_1, $u_2\in V(\mathcal{M}^T;\mathcal{N})$ *are solutions to* (1.7) *with* $u_1(\cdot,0)=u_2(\cdot,0)=u_0$. *Then* $u_1=u_2$ *in* $\mathcal{M}^T$.

Proof. Let $v=u_1-u_2$ and denote $|\nabla U|:=|\nabla u_1|+|\nabla u_2|$. From (1.7) we obtain

$$|\partial_t v-\Delta_{\mathcal{M}}v|\le c(|v|\,|\nabla U|^2+|\nabla v|\,|\nabla U|),$$

whence if we multiply by v and integrate there results

$$\begin{aligned}\frac{1}{2}\int_{\mathcal{M}}|v(\cdot,t)|^2\,d\mathcal{M}&+\int_{\mathcal{M}^t}(\nabla v,\nabla v)_{\mathcal{M}}\,d\mathcal{M}\,dt\\ &\le c\int_{\mathcal{M}^t}(|v|^2\,|\nabla U|^2+|v|\,|\nabla v|\,|\nabla U|)\,d\mathcal{M}\,dt\\ &\le\frac{1}{2}\int_{\mathcal{M}^t}(\nabla v,\nabla v)_{\mathcal{M}}\,d\mathcal{M}\,dt+c_2\int_{\mathcal{M}^t}|v|^2\,|\nabla U|^2\,d\mathcal{M}\,dt,\end{aligned}\tag{3.8}$$

by Young's inequality. Now

$$\int_{\mathcal{M}^t}|v|^2\,|\nabla U|^2\,d\mathcal{M}\,dt\le\left(\int_{\mathcal{M}^t}|v|^4\,d\mathcal{M}\,dt\int_{\mathcal{M}^t}|\nabla U|^4\,d\mathcal{M}\,dt\right)^{1/2}$$

and like Lemma 3.1 we obtain from [7, II. Theorem 2.2, Remark 2.1, p. 63f and

(3.2), p. 74] that for $t \le \min\{1, T\}$

$$\int_{\mathcal{M}^T} |v|^4 \, d\mathcal{M} \, dt \le c \sup_{0\le s\le t} \int_{\mathcal{M}} |v(\cdot, s)|^2 \, d\mathcal{M} \cdot \left(\int_{\mathcal{M}^t} |v|^2 \, d\mathcal{M} \, dt + \int_{\mathcal{M}^t} |\nabla v|^2 \, d\mathcal{M} \, dt \right)$$

$$\le c_3 \left[\sup_{0\le s\le t} \int_{\mathcal{M}} |v(\cdot, s)|^2 d\mathcal{M} + \int_{\mathcal{M}^t} (\nabla v, \nabla v)_{\mathcal{M}} \, d\mathcal{M} \, dt \right]^2.$$

In this last estimate we have again used Young's inequality. By Lemma 3.1 there exists $S \in]0, T]$ such that

$$c_3 \int_{\mathcal{M}^S} |\nabla U|^4 \, d\mathcal{M} \, dt < (2c_2)^{-2}.$$

Now, let $t \in [0, S]$ satisfy

$$\sup_{0\le s\le S} \int_{\mathcal{M}} |v(\cdot, s)|^2 \, d\mathcal{M} = \int_{\mathcal{M}} |v(\cdot, t)|^2 \, d\mathcal{M}.$$

(3.8) then implies that

$$\int_{\mathcal{M}} |v(\cdot, t)|^2 \, d\mathcal{M} + \int_{\mathcal{M}^t} |\nabla v|^2 \, d\mathcal{M} \, dt = 0,$$

and $v \equiv 0$ on $\mathcal{M}^S$. Iterating we obtain the lemma. q.e.d.

4. Results

The estimates of the preceding section imply the following local existence result.

THEOREM 4.1. *For any initial value* $u_0 \in H^{1,2}(\mathcal{M}; \mathcal{N})$ *there exists a number* $T = T(u_0) > 0$ *and a solution* $u \in \bigcap_{T' < T} V(\mathcal{M}^{T'}; \mathcal{N})$ *of* (1.7) *with* $u(\cdot, 0) = u_0$. $T(u_0)$ *is characterized by the condition*

$$\limsup_{T' \to T} \varepsilon(R, T') > \varepsilon_1 \quad \textit{for all} \quad R \in]0; R_0].$$

The solution u *is unique and regular on* $\mathcal{M} \times]0, T]$ *with exception of finitely many* *points* (x^ℓ, T), $1 \le \ell \le L$, *characterized by the condition*

$$\limsup_{T' \to T} E_R(u(\cdot, T'); x^\ell) > \varepsilon_1 \quad \textit{for all} \quad R \in]0, R_0]. \tag{4.1}$$

If u_0 *is regular (on* $\mathcal{M}' \subset \mathcal{M}$*)* u *is regular at* $t = 0$ *(on* $\mathcal{M}' \subset \mathcal{M}$*). Finally,* $E(u(\cdot, t))$ *is finite for* $t \in [0, T]$ *and non-increasing.*

Proof. Approximate u_0 in $H^{1,2}(\mathcal{M}; \mathcal{N})$ by regular (e.g. C^2-) initial data u_{m0}. (Remark that density of smooth functions in $H^{1,2}(\mathcal{M}, \mathcal{N})$ has been shown by Schoen and Uhlenbeck [9].) By local existence results (cf. e.g. [3, Theorem p. 122]) there exists a local regular solution u_m of (1.7) for which the estimates of Lemmata 3.7, 3.7′, 3.10, 3.10′ are valid. Since $u_{m0} \to u_0$ in $H^{1,2}$ there exists $R > 0$ such that

$$E_{2R}(u_{m0}, x) \le \varepsilon_1/2$$

for all $x \in \mathcal{M}$. By Lemma 3.6 this estimate (with ε_1 instead of $\varepsilon_1/2$ is conserved on balls of radius R for at least a time T_1 of order $\varepsilon_1 R^2$. Hence for large m we obtain uniform estimates of u_m in $V(M^{T_1}; \mathcal{N})$ and uniform pointwise estimates of u_m and its derivatives on any $\mathcal{M}_\tau^{T_1}$, $\tau > 0$. It follows from Remark 3.9 that $u_m \to u$ in $V(\mathcal{M}^{T_1}; \mathcal{N})$, and u solves (1.7) with initial value u_0.

Also $u_m \to u$ uniformly on $\mathcal{M}_\tau^{T_1}$, for any $\tau > 0$, and u is regular. If u_0 is regular (on $\mathcal{M}' \subset \mathcal{M}$) we may take $u_{m0} \equiv u_0$ (on $\mathcal{M}_m''$ exhausting $\mathcal{M}'$) to obtain (local) regularity at $t = 0$ from Remark 3.11, 3.11′. Uniqueness follows from Lemma 3.12.

To obtain the characterization of $T = T(u_0)$, the maximal time of existence of u as a smooth solution to (1.7), we may argue indirectly. If for some $R > 0$ and all $x \in \mathcal{M}$

$$\limsup_{T' \to T} E_R(u(\cdot, T'), x) \le \varepsilon_1,$$

by the regularity estimates of Lemma 3.10 u may be continuously extended to the closed interval $[0, T]$ and $u(\cdot, T)$ is smooth. Hence u may be continued to a larger time interval, contradicting the maximality of $T(u_0)$.

Finiteness of the singular set follows from additivity of the energy, Lemma 3.4 and Lemma 3.6. Moreover, if $\mathcal{M}'$ is compactly contained in $\mathcal{M} \setminus \{x^1, \dots, x^L\}$ there exists $R > 0$ such that $U_R(\mathcal{M}') \subset \mathcal{M} \setminus \{x^1, \dots, x^L\}$ and

$$\sup_{(x, t) \in (\mathcal{M}')^T} E_R(u(\cdot, t), x) < \varepsilon_1.$$

By Lemma 3.10′ u therefore is regular on $(\mathcal{M}')_\tau^T$ for any $\tau>0$. Finally, the energy estimate follows from Lemma 3.4 and since

$$\nabla u(\cdot, T') \rightharpoonup \nabla u(\cdot, T) \text{ weakly in } L^2 \ (T' \to T). \quad \text{q.e.d.}$$

By iteration we obtain global solutions to (1.7) for any initial value $u_0 \in H^{1,2}(\mathcal{M}; \mathcal{N})$.

THEOREM 4.2. *For any initial value* $u_0 \in H^{1,2}(\mathcal{M}; \mathcal{N})$ *there exists a unique solution* u *of* (1.7) *on* $\mathcal{M} \times [0, \infty[$ *which is regular on* $\mathcal{M} \times]0, \infty[$ *with exception of at most finitely many points* (x^ℓ, T^ℓ), $1 \le \ell \le L$, *characterized by the condition that*

$$\limsup_{\substack{T \to T^\ell \\ T < T^\ell}} E_R(u(\cdot, T), x^\ell) > \varepsilon_1 \quad \textit{for all} \quad R \in]0, R_0].$$

Proof. By Theorem 4.1 for any $u_0 \in H^{1,2}(\mathcal{M}, \mathcal{N})$ there exists a unique local solution on some $\mathcal{M}^{T^1}$ which is regular on $\mathcal{M} \times]0, T^1]$ with exception of at most finitely many points (x^ℓ, T^1), $1 \le \ell \le L_1$. Let $u_1 = u(\cdot, T^1) \in H^{1,2}(\mathcal{M}, \mathcal{N})$. Note that

$$\begin{aligned} E(u_1, \mathcal{M}') &\le \liminf_{\substack{T' \to T^1 \\ T' < T^1}} E(u(\cdot, T'), \mathcal{M}') \\ &\le \liminf_{\substack{T' \to T^1 \\ T' < T^1}} E(u(\cdot, T')) - \sum_{\ell=1}^{L_1} E_R(u(\cdot, T'), x^\ell) \\ &\le E(u_0) - L_1 \varepsilon_1 \end{aligned}$$

for any $R \in]0, R_0]$, $\mathcal{M}' \subset \mathcal{M} \setminus \bigcup_{\ell=1}^{L_1} B_R^{\mathcal{M}}(x^\ell)$. Passing to the limit $R \to 0$, $\mathcal{M}' \to \mathcal{M}$:

$$E(u_1) \le E(u_0) - L_1 \varepsilon_1. \tag{4.2}$$

By Theorem 4.1 we may continue u to some larger interval $[0, T^2]$ by solving (1.7) with initial value $u(\cdot, T^1) = u_1$ on $[T^1, T^2]$ and piecing together the solutions at T. Since an isolated point (x^ℓ, T^1) has $(H^{1,2}(\mathcal{M})-)$ capacity zero, u will be a distribution solution to (1.7) on all of $\mathcal{M}^{T^2}$. Iterating, we obtain a global solution u of (1.7) which is regular with exception of points $(x^\ell, T^k)_{1 \le k \le K;\, 1 \le \ell \le L_k}$. Finiteness of the singular set follows from (4.2) and since $E(u_0) < \infty$. q.e.d.

A more detailed description of the behaviour of our solution u near a singular point can be given as follows. Suppose $x_0 \in \mathcal{M}$, $t_0 \in \mathbb{R}$, and let u be a solution to (1.7) on $\mathcal{M}_{t_0}^\infty$. Representing a neighborhood of x_0 by a coordinate chart $\Omega \subset \mathbb{R}^2$ we may introduce the rescaled functions for $R>0$;

$$u_R(x,t)\equiv u_{R,(x_0,t_0)}(x,t)\equiv u(Rx+x_0, R^2t+t_0).$$

Note that as $R\to 0$ the domain of u_R will exhaust all of $\mathbb{R}^2\times[0,\infty[$. Moreover, u_R will solve an equation similar to (1.7) on its domain with coefficients locally tending to the coefficients that correspond to the standard metric on $\mathbb{R}^2$ as $R\to 0$. We may now state:

THEOREM 4.3. *Let u be the solution to* (1.7) *constructed in Theorem* 4.2, *and suppose* (x_0, T), $T\le\infty$, *is a point where*

$$\limsup_{\substack{t\to T\\ t<T}} E_R(u(\cdot,t),x_0)>\varepsilon_1, \qquad \forall R\in\,]0,R_0]$$

Then there exist sequences $x_m\to x_0$, $t_m<T$, $t_m\to T$, $R_m\in\,]0,R_0]$, $R_m\to 0$ and a regular harmonic mapping $\overline{u_0}$: $\mathbb{R}^2\to\mathcal{N}$ such that as $m\to\infty$

$$u_{R_m,(x_m,t_m)}(\cdot,0)\to\overline{u_0} \text{ locally in } H^{2,2}(\mathbb{R}^2;\mathcal{N}).$$

$\overline{u_0}$ has finite energy and extends to a smooth harmonic map $S^2\to\mathcal{N}$.

Proof. Let $x^1,\dots,x^L$ enumerate all singular points of u at time T, characterized by the condition (4.1), and let $\rho\in\,]0,R_0/2]$ be chosen such that $B_\rho^{\mathcal{M}}(x^k)\cap B_\rho^{\mathcal{M}}(x^\ell)=\emptyset$ for all $\ell\ne k$. By Theorem 4.2 there exists $T_1<T$ such that $u\in V(\mathcal{M}_{T_1}^{T_2};\mathcal{N})$ for all $T_2<T$. Therefore, for any ℓ there exist sequences $x_m\to x^\ell$, $t_m\to T$, $t_m<T$, $R_m\to 0$, $R_m\in\,]0,R_0]$, such that

$$\varepsilon_1=E_{R_m}(u(\cdot,t_m),x_m)=\sup_{\substack{T_1\le t\le t_m\\ x\in B_\rho^{\mathcal{M}}(x^\ell)}} E_{R_m}(u(\cdot,t),x).$$

Note that by Lemma 3.6 this implies that for some constant $c_4=\dfrac{\varepsilon_1}{2c_1E(u_0)}$ and any $t\in[t_m-c_4R_m^2, t_m]$

$$E_{2R_m}(u)\cdot,t),x_m)\ge\varepsilon_1/2.$$

Moreover, by Lemma 3.7

$$\int_{\mathcal{M}_{t_m-c_4R_m^2}^{t_m}} |\nabla^2u|^2\,d\mathcal{M}\,dt\le cE(u_0).$$

Hence after scaling the family $u_m = u_{R_m,(x_m,t_m)}$ satisfies the estimates on $\mathscr{D}(u_m) = \{(x, t) \mid R_m x + x_m \in B_\rho^{\mathcal{M}}(x^\ell);\ R_m^2 t + t_m \geq 0\}$:

$$E_2(u_m(\cdot, t), 0) \geq \varepsilon_1/2, \qquad \forall t \in [-c_4, 0]$$

$$\sup_{\substack{(x,t)\in\mathscr{D}(u_m)\\ -c_4\leq t\leq 0}} E_1(u_m(\cdot, t), x) \leq \varepsilon_1,$$

$$\int_{\substack{\mathscr{D}(u_m)\\ -c_4\leq t\leq 0}} |\nabla^2 u_m|^2\, dx\, dt \leq c,$$

$$\int_{\substack{\mathscr{D}(u_m)\\ -c_4\leq t\leq 0}} |\partial_t u_m|^2\, dx\, dt = \int_{\mathcal{M}^{t_m}_{t_m - c_4 R_m^2}} |\partial_t u|^2\, d\mathcal{M}\, dt \to 0 \qquad (m \to \infty),$$

by absolute continuity of $\int |\partial_t u|^2\, d\mathcal{M}\, dt$. In particular, for some number $\tau_m \in [-c_4, 0]$ we can achieve that as $m \to \infty$

$$\int_{\substack{\mathscr{D}(u_m)\\ t=\tau_m}} |\nabla^2 u_m|^2\, dx \leq c; \qquad \int_{\substack{\mathscr{D}(u_m)\\ t=\tau_m}} |\partial_t u_m|^2\, dx \to 0,$$

while

$$\int_{\substack{B_2(0)\\ t=\tau_m}} |\nabla u_m|^2\, dx \geq c > 0$$

uniformly in m. (Rescaling $t_m \to t_m - \tau_m R_m^2$ we may assume $\tau_m = 0$.) Hence there is a sequence $m \to \infty$ such that $u_m(\cdot, 0)$ converges weakly[1] to some function $\overline{u_0} \in H^{2,2}_{\mathrm{loc}}(\mathbb{R}^1; \mathcal{N})$ and strongly in $H^{1,2}(\Omega; \mathcal{N})$ for any $\Omega \subset\subset \mathbb{R}^2$. Passing to the limit $m \to \infty$ in the equation (1.7) it follows that $\overline{u_0}$ is harmonic map from $\mathbb{R}^2$ onto $\mathcal{N}$. Moreover, since

$$E^{\mathbb{R}^2}(\overline{u_0}) \leq \limsup_{m\to\infty} E^{\mathcal{M}}(u(\cdot, t_m)) \leq E^{\mathcal{M}}(u_0)$$

$\overline{u_0}$ has finite energy. Thus by conformal equivalence $\mathbb{R}^2 \cong S^2 \setminus \{p\}$ and [8, Theorem 3.6] $\overline{u_0}$ extends to a smooth harmonic map of S^2 into $\mathcal{N}$. q.e.d.

Remark 4.4. Let $\varepsilon(\mathcal{N}) = \inf\{E(u) \mid u : S^2 \to \mathcal{N}$ is a non-constant, regular harmonic map$\} > 0$. By Theorem 4.3 for any initial value $u_0 \in H^{1,2}(\mathcal{M}, \mathcal{N})$ with

[1] Even strongly locally in $H^{2,2}(\mathbb{R}^2, \mathcal{N})$

$E(u_0) < \varepsilon(\mathcal{N})$ our solution u of (1.7) will be globally regular on $\mathcal{M} \times]0, \infty[$, and at $t = \infty$ in the sense that

$$\limsup_{t\to\infty} E_R(u(\cdot, t), x) \le \varepsilon_1,$$

for some $R \in]0, R_0]$ and all $x \in \mathcal{M}$. Using the estimate $\partial_t u \in L^2(\mathcal{M}^\infty)$, the proof of Theorem 4.3 shows that for some sequence $t_m \to \infty$ $u(\cdot, t_m)$ converges weakly in $H^{2,2}$ (and hence strongly in $H^{1,2}$) to a harmonic mapping u_∞ from $\mathcal{M}$ into $\mathcal{N}$. Likewise, for arbitrary initial data we may conclude that $u(\cdot, t_m) \rightharpoonup u_\infty$ weakly in $H^{1,2}(\mathcal{M}; \mathcal{N})$ for some sequence $t_m \to \infty$, where u_∞ is a harmonic map from $\mathcal{M}$ into $\mathcal{N}$ with $E(u_\infty) \le E(u_0)$ and regular on $\mathcal{M}$ with exception of at most finitely many points. By [8, Theorem 3.6] therefore u_∞ extends to a smooth harmonic map $\mathcal{M} \to \mathcal{N}$. The proof proceeds as that of Theorem 4.3 in conjunction with our local estimates Lemma 3.7′ and can be omitted.

5

For reference we note the following corollary of our results in the stationary case. The proof is analogous to that of Theorem 4.3 and hence can be left to the reader.

PROPOSITION 5.1. *Suppose $\{u_m\}$ is a sequence in $H^{2,2}(\mathcal{M}, \mathcal{N})$ satisfying the conditions*

$$E(u_m) \le c, \qquad dE(u_m) \to 0 \text{ in } L^2(\mathcal{M})(m \to \infty).$$

Then either the sequence $\{u_m\}$ is relatively weakly compact(2) *in $H^{2,2}(\mathcal{M}; \mathcal{N})$ and a subsequence converges weakly to a harmonic map $u : \mathcal{M} \to \mathcal{N}$ in $H^{2,2}$, or there exist (at most finitely many) points $x^1, \ldots, x^L$ such that $\{u_m\}$ is relatively weakly compact*(2) *in $H^{2,2}_{\text{loc}}(\mathcal{M} \setminus \{x^1, \ldots, x^L\}; \mathcal{N})$ and accumulates at a smooth harmonic map $\overline{u_0} : \mathcal{M} \to \mathcal{N}$, while for $\ell = 1, \ldots, L$ there exist sequences $x^\ell_m \to x^\ell$, $R^\ell_m \to 0$ and smooth harmonic mappings $\overline{u_\ell} : \mathbb{R}^2 \to \mathcal{N}$ such that*

$$u_m(R^\ell_m \cdot + x^\ell_m) \rightharpoonup \overline{u_\ell} \text{ weakly}^{(2)} \text{ in } H^{2,2}_{\text{loc}}(\mathbb{R}^2, \mathcal{N})$$

as $m \to \infty$. Moreover,

$$E^{\mathcal{M}}(\overline{u_0}) + \sum_{\ell=1}^{L} E^{\mathbb{R}^2}(\overline{u_\ell}) \le \liminf_{m\to\infty} E^{\mathcal{M}}(u_m).$$

I.e. a local Palais-Smale condition is satisfied for E and its L^2-gradient.

[2] Again, one can easily show strong $H^{2,2}$-compactness (locally).

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