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Parallelizability of Finite H-Spaces

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This note is a contribution to the study of how closely topological spaces with a multiplication (H-spaces) geometrically resemble Lie groups.

THEOREM. *Let X be a finite H-space and suppose that $X_{(2)}$ has a factor which at the prime 2 is S^7, or a Lie group and that*

1) *X is simply-connected, or*
2) *$\pi_1 X$ is a odd p-group, or*
3) *$\pi_1 X$ is infinite and has no 2-torsion.*

Then X has the homotopy type of a closed parallelizable smooth manifold.

For a history and for our other related results see [2]. For instance, without the assumption on $X_{(2)}$ it is shown there that X is homotopy equivalent to a closed manifold. All known examples of finite H-spaces in fact are products of (just) S^7, RP^7 and Lie groups at the prime 2, so that out theorem for the given fundamental groups includes all the known examples. Here, we just prove the theorem in case $\pi_1 X = 0$; the non-simply connected cases are readily dealt with using similar ideas together with methods of handling fundamental group difficulties developed in [2].

Notice that S^7 is the two-fold cover of RP^7, and every Lie group is the two-fold cover of a closed manifold [2, Lemma 2.1]. Thus $X_{(2)}$ is the two-fold cover of the localization at 2 of a finite complex Y. There is a map $Y \to X_{(0)}$ so that the composite $X_{(2)} \to Y_{(2)} \to X_{(0)}$ is just localization. Let Z be the homotopy pullback:

$$\begin{array}{ccc} Z & \longrightarrow & X[\tfrac{1}{2}] \\ \downarrow & & \downarrow \\ Y_{(2)} & \longrightarrow & X_{(0)} \end{array}$$

Observation 1. Z is a finite Poincaré complex with 2-fold cover homotopy equivalent to X.

Observation 2. There is a map $Z \to BO$ lifting the Spivak normal fibration such that $X \to Z \to BO$ is nullhomotopic.

For the proofs of related statements see [3]. Now, consider a degree-one normal map $f: M \to Z$ associated to the lifting given by (2), and more importantly, the surgery obstruction of the two-fold cover $\tilde{f}: \tilde{M} \to X$. Note that $\tilde{f}$ is covered by trivial bundle data. The only nonzero elements in $L_n(0)$ in the image of the transfer occur for $n \equiv 0 \bmod 4$ and are detected by signature. Now sign $(\tilde{M}) = 0$ by the Hirzebruch signature formula and the stable parallelizability of M and sign $(X) = 0$ by an easy application of the Milnor–Moore theorem on Hopf algebras (e.g. X is rationally a product of odd spheres) so that this obstruction vanishes as well. Thus, we can complete surgery on $\tilde{f}$ producing a stably parallelizable manifold N homotopy equivalent to X. Except in low dimensions where parallelizability is automatic, the obstructions to N being parallelizable are in fact just the Euler characteristic and, if the dimension is $\equiv 1 \bmod 2$, the $Z/2Z$ semicharacteristic, (see e.g. [1]), but these vanish for N since they obviously do for X. Q.E.D.

REFERENCES

[1] G. Bredon and A. Kosinski, *Vector fields on π-manifolds*, Ann. of Math. *84* (1966), 85–90.
[2] S. Cappell and S. Weinberger, *Which H-spaces are manifolds? I*, (preprint).
[3] S. Cappell and S. Weinberger, *Homology propagation of group actions*, (preprint).
[4] S. Weinberger, *Homologically trivial group actions, I: Simply connected manifolds.* Amer. J. of Math., (to appear).

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