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The growth of entire and harmonic functions along asymptotic paths

JOHN ROSSI¹ and ALLEN WEITSMAN

1. Introduction

In a recent paper of Lewis and the two authors [5], the following generalization of a theorem of Huber [4] is proved.

THEOREM A. *Let f be a transcendental entire function. Then there exists a path Γ from 0 to ∞ such that*

$$\lim_{\substack{z \rightarrow \infty \\ z \in \Gamma}} \frac{\log |f(z)|}{\log |z|} = \infty \tag{1.1}$$

$$l(\Gamma(z)) \leq |f(z)|^{\varepsilon(z)} \quad (0 \leq \varepsilon(z) \rightarrow 0, z \rightarrow \infty) \tag{1.2}$$

where $l(\Gamma(z))$ is the length of Γ from 0 to z and

$$\int_{\Gamma} \frac{1}{|f|^{\lambda}} |dz| < \infty \quad (\text{for all } \lambda > 0). \tag{1.3}$$

In [7], one of the authors has proved.

THEOREM B. *Let f be an entire function such that for some $K > 0$ at least one of the level curves $|f| = K$ tends to ∞ . Then there exists a path Γ from 0 to ∞ such that*

$$\log |f(z)| > |z|^{1/2 - \varepsilon(z)} \tag{1.4}$$

and

$$l(\Gamma(z)) \leq (\log |f(z)|)^{c + 2 + \varepsilon(z)} \tag{1.5}$$

where $c > 0$ is an absolute constant and $0 \leq \varepsilon(z) \rightarrow 0$ as $z \rightarrow \infty$.

¹ Research carried out as a NATO Postdoctoral Fellow at Imperial College, London.

In this paper we prove

THEOREM 1. *Let f be as in Theorem B. Then there exists a path Γ from 0 to ∞ such that (1.4) holds and*

$$\int_{\Gamma} (\log |f|)^{-(2+\lambda)} |dz| < \infty \quad (\text{for all } \lambda > 0). \quad (1.6)$$

Whereas (1.1) and (1.2) imply (1.3), we note that because of the presence of c , (1.4) and (1.5) do not imply (1.6). The constant c is a by-product of the proof of Theorem B. We use a totally different approach in proving Theorem 1.

COROLLARY 1. *Let u be a nonconstant harmonic function in \mathbb{C} . Then there exists a path Γ from 0 to ∞ such that (1.4) and (1.6) hold with $\log |f|$ replaced by u .*

The proof of Corollary 1 is immediate from Theorem 1. Indeed, if u is any such harmonic function and v is its harmonic conjugate in \mathbb{C} then $f = e^{u+iv}$ is transcendental and entire with $u = \log |f|$. Clearly by the harmonicity of u every level curve of $|f| = 1$ ($u = 0$) extends to ∞ .

We also prove

THEOREM 2. *Let f be an entire function of order $\rho \leq \infty$ such that for some $K > 0$ the set $\{z : |f| > K\}$ contains at least two components. Then there exists a path Γ from 0 to ∞ such that*

$$\log |f(z)| > |z|^{\lfloor \rho/(2\rho-1) \rfloor - \varepsilon(z)} \quad (0 \leq \varepsilon(z) \rightarrow 0 \text{ as } z \rightarrow \infty) \quad (1.7)$$

and

$$\int_{\Gamma} (\log |f|)^{-\lfloor (2\rho-1)/\rho \rfloor + \lambda} |dz| < \infty \quad (\text{for all } \lambda > 0). \quad (1.8)$$

(We note that by hypothesis and an easy application of the Ahlfors, Denjoy, Carleman method, $\rho \geq 1$ and thus $(2\rho-1)/\rho \geq \frac{1}{2}$.)

Examples in Eremenko [3 p. 681] show that $\varepsilon(z)$ cannot be replaced by 0 in (1.4) and (1.6).

By modifying his examples slightly, we can find an entire function f of order ρ , $1 \leq \rho \leq \infty$ such that

$$\int_{\Gamma} (\log |f(z)|)^{-(2\rho-1)/\rho} |dz| = \infty$$

for every path Γ on which $|f| > 1$. This shows that (1.5) and (1.7) are “sharp” independent of (1.4) and (1.6).

Barth, Brannan and Hayman [2, Theorem 2] show that $\varepsilon(z)$ cannot be replaced by 0 in (1.4) where $\log |f| = u$ is harmonic. Brannan has pointed out in private communication that their example can be modified to show that (1.5) is also “sharp” for harmonic functions. Specifically one can construct a harmonic function u such that

$$\int_{\Gamma} u(z)^{-2} |dz| = \infty$$

for all paths Γ where $u > 0$.

2. Preliminary lemmas

Let D be an unbounded regular plane domain. We let $\theta^*(r) = \infty$ if $\{|z| = r\} \subseteq D$. Otherwise we let $r\theta^*(r)$ equal the length of the longest arc in the intersection of $\{|z| = r\}$ and D . Recall that a set G has log density one if $(\log r)^{-1} \int_{G \cap [1, r]} dt/t \rightarrow 1$ as $r \rightarrow \infty$. We state

LEMMA 1. *Let D be as above and suppose*

$$\inf_G \overline{\lim}_{\substack{r \rightarrow \infty \\ r \in G}} \theta^*(r) = \frac{\pi}{\alpha} \quad (\frac{1}{2} \leq \alpha < \infty) \tag{2.1}$$

where the inf is taken over all sets G of log density one. Then there exists $v > 0$ harmonic in D such that for all $z \in D$

$$v(z) \geq |z|^{\alpha - \varepsilon(|z|)} \quad (0 \leq \varepsilon(|z|) \rightarrow 0 \text{ as } |z| \rightarrow \infty). \tag{2.2}$$

We remark that without the log density statement, (2.2) was proved in [2] with $\alpha = \frac{1}{2}$.

Before we prove Lemma 1 we need the following lemma which asserts that the inf in (2.1) is attained.

LEMMA 2. *There exists a set G of log density one such that*

$$\overline{\lim}_{\substack{r \rightarrow \infty \\ r \in G}} \theta^*(r) = \frac{\pi}{\alpha}. \tag{2.3}$$

Proof. Let $\text{l.m.}(E) = \int_E dt/t$ for any measurable set $E \subseteq [0, \infty)$. By (2.1) we may find G_n , $n=1, 2, \dots$ such that

$$\theta^*(r) \leq \frac{\pi}{\alpha} + \frac{1}{n} \quad (2.4)$$

and

$$\text{l.m.}(G_n \cap [1, r]) \geq \left(1 - \frac{1}{n}\right) \log r \quad (2.5)$$

provided $r \in G_n$, $r \geq r_n$. We may choose r_n so large that

$$\frac{1}{n-1} \log r_n \geq \log r_{n-1}, \quad n = 2, 3, \dots \quad (2.6)$$

Define $G = \bigcup_{n=1}^{\infty} G_n \cap [r_n, r_{n+1}]$. To see that \log dens $G = 1$, choose $\varepsilon > 0$ and let N be such that $3/N < \varepsilon$. Suppose $r \in G$ and $r_n \leq r < r_{n+1}$ for some $n \geq N+1$. We have by (2.5) and (2.6)

$$\begin{aligned} \text{l.m.}(G \cap [1, r]) &\geq \text{l.m.}(G_{n-1} \cap [r_{n-1}, r_n]) + \text{l.m.}(G_n \cap [r_n, r]) \\ &\geq \left(1 - \frac{1}{n-1}\right) \log r_n - \log r_{n-1} + \left(1 - \frac{1}{n}\right) \log r - \log r_n \\ &= -\frac{1}{n-1} \log r_n - \log r_{n-1} + \left(1 - \frac{1}{n}\right) \log r \\ &\geq -\frac{2}{n-1} \log r_n + \left(1 - \frac{1}{n}\right) \log r \\ &\geq \left(1 - \frac{3}{n-1}\right) \log r \\ &\geq \left(1 - \frac{3}{N}\right) \log r \\ &\geq (1 - \varepsilon) \log r. \end{aligned}$$

Since ε was arbitrary G has \log density one.

Furthermore given $\varepsilon > 0$, there exists N such that $1/N < \varepsilon$ and if $r \geq r_N$ we

have by (2.4) and the definition of G that $\theta^*(r) \leq (\pi/\alpha) - \varepsilon$. This implies

$$\overline{\lim}_{\substack{r \rightarrow \infty \\ r \in G}} \theta^*(r) \leq \frac{\pi}{\alpha}. \quad (2.7)$$

Since G has log density one, (2.7) and (2.1) imply (2.4). Lemma 2 is now proved.

Proof of Lemma 1. We denote by $\eta_i(r)$, $i = 1, 2, \dots$ any nonnegative sequences such that $\eta_i(r) \rightarrow 0$ as $r \rightarrow \infty$. Then with G as in Lemma 2, we have

$$\theta^*(r) \leq \frac{\pi}{\alpha - \eta_1(r)} \quad (r \in G). \quad (2.8)$$

Also if $a \in E$ where E is compact in \mathbb{C} and $|a| \geq 1$

$$\text{l.m. } (G \cap [|a|, r]) \geq [1 - \eta_2(r)] \log \frac{r}{|a|} \quad (2.9)$$

uniformly in E .

By (2.8) we have

$$\int_{G \cap [|a|, r]} \frac{\eta_1(t)}{t} dt \leq \eta_3(r) \log \frac{r}{|a|}. \quad (2.10)$$

uniformly in E .

Let D_R be any component of $D \cap \{|\zeta| < R\}$. Pick $z \in D_R$ with $|z| < R/4$ and let $\omega_R(z)$ be the harmonic measure of $\{|\zeta| = R\} \cap \partial D_R$ with respect to z and D_R . Then by an inequality found in [8, p. 116] we have

$$\omega_R(z) \leq 9\sqrt{2} \exp \left\{ -\pi \int_{2|z|}^{R/2} \frac{dt}{t\theta^*(t)} \right\}. \quad (2.11)$$

By (2.8)–(2.11) we have for $z \in E$ compact in \mathbb{C}

$$\omega_R(z) \leq K \left(\frac{|z|}{R} \right)^{\alpha - \eta_4(R)} \quad (2.12)$$

where K is a constant depending only on E .

Let $\phi(r)$ be any convex increasing function of $\log r$ such that

$$\frac{\log \phi(r)}{\log r} \rightarrow \alpha \quad (r \rightarrow \infty) \quad (2.13)$$

and

$$\phi(2r) \leq \frac{r^{\alpha - \eta_4(r)}}{(\log r)^2}. \quad (2.14)$$

We now employ a technique similar to the one used in Lemma 1 of [2]. Let D_R be as above. Then there exists a unique function $v_R(z)$ harmonic in D_R , continuous in \bar{D}_R such that for $z \in \partial D_R$

$$v_R(z) = \phi(|z|). \quad (2.15)$$

Let $R_n = 2^n$, $n = 0, 1, 2, \dots$ and define D_{R_n} as before making sure that $D_{R_{n+1}} \supseteq D_{R_n}$. Let $\omega_{n,\nu}$, $n \geq \nu$ be the harmonic measure in D_{R_n} of the portion of ∂D_{R_n} in $\{|\zeta| \geq R_\nu\}$. Then for all $z \in D_{R_n}$, $|z| \leq R_\nu/4$, we have

$$\omega_{n,\nu}(z) \leq \omega_{R_\nu}(z). \quad (2.16)$$

Choose R_k to be the smallest radius greater than $4|z|$. Then for $z \in D_{R_n} \cap \{|z| \leq R_k/4\}$, $n \geq k$, we have by (2.12), (2.16) the definition of ϕ , and the fact that $|z| \geq R_k/8$,

$$\begin{aligned} v_{R_n}(z) &\leq \phi(R_k) + \sum_{\nu=k}^{n-1} \phi(R_{\nu+1}) \omega_{n,\nu}(z) \\ &\leq \phi(8|z|) + k|z|^\alpha \sum_{\nu=k}^{n-1} \phi(R_{\nu+1}) R_\nu^{-\alpha + \eta_4(R_\nu)} \\ &\leq \phi(8|z|) + k|z|^\alpha \left(1 + \sum_{\nu=k}^{\infty} \frac{1}{(\nu+1)^2} \right) \\ &\leq k_1 |z|^\alpha \end{aligned} \quad (2.17)$$

where $k_1 > 0$ is a constant depending only on the compact set $|z| \leq R_k/4$.

Since ϕ is a convex function of $\log r$, we have that $\phi(|z|) - v_{R_n}(z)$ is subharmonic in D_{R_n} and equal to 0 on ∂D_{R_n} . Thus for $z \in D_{R_n}$ we have

$$v_{R_n}(z) \geq \phi(|z|). \quad (2.18)$$

Also if $m \geq n$ and $z \in D_{R_n}$ we have

$$v_{R_m}(z) \geq v_{R_n}(z). \quad (2.19)$$

By (2.17)–(2.19), v_{R_n} is an increasing sequence of harmonic functions uniformly bounded on compact sets. By Harnack's Theorem $v(z) = \lim_{n \rightarrow \infty} v_{R_n}(z)$ is harmonic in D . Thus (2.2) follows easily from (2.13) and (2.18).

3. Proof of Theorem 1 when f has no zeros

We assume first that f has no zeros. Then every level curve of $\log |f| = 1$ extends to ∞ . Thus if D is any component of $\{z : \log |f| > 1\}$, D is simply connected and contains no full circle $|z| = r$ for $r \geq r_0$. Thus we may find a function v harmonic in D satisfying (2.2) for $\alpha = \frac{1}{2}$. Now let $z_0 \in D$. We can find $\delta > 0$ such that

$$\log |f(z_0)| - \delta v(z_0) > 1. \quad (3.1)$$

Define $w = \delta v$ and let w^* be the harmonic conjugate of w in D . Then $\phi = e^{w+iw^*}$ is analytic in D with no zeros such that

$$\log |\phi| = w \quad (3.2)$$

satisfies (2.2) (for possibly another $\varepsilon(z)$).

Set $F = f/\phi$ in D . By (2.2), (3.1) and (3.2) $\log F$ has boundary values on ∂D not exceeding 1 and is greater than 1 at $z_0 \in D$. Thus every component \mathcal{F}_R , $R \geq 1$ of $\{z : |F| > R\}$ is nonempty and contained in D .

To construct our path Γ we will use extremal length arguments in each \mathcal{F}_R . We define extremal length as in [1, p. 11]. Let \mathcal{G} be a family of curves. The extremal length $\lambda(\mathcal{G})$ of \mathcal{G} is defined as

$$\lambda(\mathcal{G}) = \sup_{\rho} \frac{L^2(\rho)}{A(\rho)}$$

where

$$L(\rho) = \inf_{\gamma \in \mathcal{G}} \int_{\gamma} \rho |dz|, \quad A(\rho) = \iint_{\mathbb{C}} \rho^2 dx dy$$

and $\rho \geq 0$ ranges over all measurable functions for which $A(\rho) \neq 0, \infty$.

To get the construction started let $R_0 > e$ be such that $F' \neq 0$ when $|F| = R_0$ and take a component $\mathcal{F}_{R_0} \subseteq D$ with $\zeta_0 \in \partial\mathcal{F}_{R_0}$ arbitrarily chosen. It follows from the Cauchy–Riemann equations that $\arg F$ is then monotone on $\partial\mathcal{F}_{R_0}$ so that for some $\eta > 0$ a branch of the function $w = \log F$ maps a neighborhood of an arc of $\partial\mathcal{F}_{R_0}$ containing ζ_0 univalently to a neighborhood of a segment

$$T_0 = \{w = \log R_0 + iv : \psi_0 - \eta \leq v \leq \psi_0 + \eta\}$$

with the arc of $\partial\mathcal{F}_{R_0}$ and the segment T_0 corresponding. By replacing F by F^K where K is a sufficiently large positive integer we may assume that η is arbitrarily large. This modification of F will in no way affect our method and so we assume that $\eta = e$ in the definition of T_0 .

Recall the function $\varepsilon(r)$ in (2.2). Fix $\lambda_0 > 0$ such that

$$(2 + 4e) \sum_{j=0}^{\infty} \left(2\pi \int_0^{\infty} \frac{r dr}{[e^j - 1 + \log R_0 + r^{2-\varepsilon(r)}]^{4+2\lambda_0}} \right)^{1/2} \leq 1. \quad (3.4)$$

This is possible since the left side of (3.4) converges for every $\lambda_0 > 0$.

With ψ_0 as chosen, we let Q_0 be the square in the w -plane defined by $Q_0 = \{(s, t_0) : \log R_0 < s < 2e + \log R_0, \psi_0 - e < t_0 < \psi_0 + e\}$. Set $\gamma = \gamma_{t_0} = \{(s, t_0) : \log R_0 \leq s < s'\}$ where t_0 ranges between $\psi_0 - e$ and $\psi_0 + e$ and $s' \leq \log R_0 + 2e$. The point s' is chosen to be $\log R_0 + 2e$ if the inverse $h(w)$ of $\log F$ can be uniquely continued on γ_{t_0} from $\log R_0$ to $\log R_0 + 2e$. Otherwise s' is chosen so that (s', t_0) is the first point on the horizontal segment γ_{t_0} where h cannot be continued uniquely. Since $s' > \log R_0$ and since h cannot tend to $\partial\mathcal{F}_{R_0} \subseteq D$ this can only happen if either there exists a point $z_1 \in \mathcal{F}_{R_0}$ such that $\log F(z_1) = (s', t_0)$ and $F'(z_1) = 0$ or if $h \rightarrow \infty$ as $w \rightarrow (s', t_0)$.

By taking unions over all such horizontal segments and their preimages in the z -plane, we obtain a measurable set $\mathcal{F} \subseteq \mathcal{F}_{R_0}$ which maps 1–1 under $\log F$ to a subset \tilde{Q}_0 of Q_0 . Let \mathcal{G} be the family of *all* horizontal segments in Q_0 connecting both sides of Q_0 . Since Q_0 is a square this implies [1, p. 12] that $\lambda(\mathcal{G}) = 1$. Furthermore since the curves in $\tilde{\mathcal{G}}$ are no “longer” than those in \mathcal{G} , we have in the notation of [1, p. 12] that $\tilde{\mathcal{G}} < \mathcal{G}$ and so $\lambda(\tilde{\mathcal{G}}) \leq 1$. Let \tilde{C} be the collection of the images under h of those curves in $\tilde{\mathcal{G}}$ which extend all the way across Q_0 . Then $\tilde{C} = h(\tilde{\mathcal{G}}) - C_1 - C_2$ where C_1 are the curves which run into points where $F' = 0$ and C_2 are the unbounded curves. But the number of curves in C_1 is countable and the curves in C_2 extend to ∞ . Thus it is easy to see [6, Theorems 2.13 and 2.14] that $\lambda(\tilde{C}) = \lambda(h(\tilde{\mathcal{G}}))$. Since $(\log F)' \neq 0$ on $h(\tilde{\mathcal{G}})$, it is easy to show that $\lambda(h(\tilde{\mathcal{G}})) = \lambda(\tilde{\mathcal{G}})$. This gives

$$\lambda(\tilde{C}) \leq 1. \quad (3.5)$$

On \tilde{C} we take (in (3.3)) $\rho = \rho_0 = (\log |f|)^{-2-\lambda_0}$ and $\rho = 0$ off \tilde{C} . Clearly $A(\rho) \neq 0$. To show that $A(\rho) \neq \infty$ we have by (2.2), (3.2) and the fact that the union of the \tilde{C} lies in $\mathcal{F} \subseteq \mathcal{F}_{R_0}$

$$\begin{aligned} A(\rho_0) &\leq \iint_{\mathcal{F}} (\log |f|)^{-4-2\lambda_0} r \, dr \, d\theta \\ &\leq \iint_{\mathcal{F}} (\log R_0 + \delta v)^{-4-2\lambda_0} r \, dr \, d\theta \\ &\leq 2\pi \int_0^\infty (\log R_0 + r^{\frac{1}{2}-\varepsilon(r)})^{-4-2\lambda_0} r \, dr \\ &< \infty. \end{aligned} \tag{3.6}$$

Let us define for R and λ positive

$$K(R, \lambda) = \left(2\pi \int_0^\infty (R + r^{\frac{1}{2}-\varepsilon(r)})^{-4-2\lambda} r \, dr \right)^{1/2}. \tag{3.7}$$

Thus it follows by (3.3), (3.6) and (3.7) that there exists in \tilde{C} a curve $\tilde{\beta}_0 \subseteq \mathcal{F}_{R_0}$ that joins a point $z \in \partial\mathcal{F}_{R_0}$ to $\partial\mathcal{F}_{e^{2e}R_0}$ for some component $\mathcal{F}_{e^{2e}R_0} \subseteq \mathcal{F}_{R_0}$ of the set $\{z : |F| > e^{2e}R_0\}$. Furthermore

$$\int_{\tilde{\beta}_0} (\log |f|)^{-2-\lambda_0} |dz| \leq 2K(\log R_0, \lambda_0). \tag{3.8}$$

We let $\tilde{\beta}_0$ correspond to γ_{t_0} in Q_0 . Then a similar procedure is applied to the rectangle $S_0 = \{(s, t) : e + \log R_0 < s < 2e + \log R_0, t_0 < t < t_0 + 2e^2\}$ in the w plane where the bottom of S_0 corresponds to half of $\tilde{\beta}_0$ under a branch h of $(\log F)^{-1}$. Here we consider the family of vertical segments $\gamma = \gamma_{s_0} = \{(s_0, t) : t_0 - e^2 < t < t_0 + e^2\}$ in S_0 . As before we obtain a family $\tilde{\mathcal{G}}$ whose union is mapped 1-1 onto a set $\xi \subseteq \mathcal{F}_{e^e R_0}$. Since S_0 is a rectangle of length $2e^2$ and width e we obtain with \tilde{C} as before

$$\lambda(\tilde{C}) = \lambda(\tilde{\mathcal{G}}) \leq \lambda(\mathcal{G}) = \frac{2e^2}{e} = 2e.$$

So in \mathcal{E} we again get a curve $\tilde{\alpha}_0$ whose image γ_{s_0} under $\log F$ is a vertical segment joining the two sides of S_0 and

$$\int_{\tilde{\alpha}_0} (\log |f|)^{-2-\lambda_0} |dz| \leq 4eK(\log R_0, \lambda_0). \tag{3.9}$$

We now cut $\tilde{\beta}_0$ off where it joins $\tilde{\alpha}_0$ at $\log R_1 (\geq \log R_0 + e)$ and obtain the first piece $\beta_0 \subseteq \tilde{\beta}_0$ of our curve Γ . With λ_0 still fixed we continue with the square

$$Q_1 = \{(s, t) : \log R_1 < s < 2e^2 + \log R_1, t_0 < t < t_0 + 2e^2\}$$

and obtain a curve $\tilde{\beta}_1$ on which $F' \neq 0$ joining $\tilde{\alpha}_0$ to the boundary of a component $\mathcal{F}_{e^{2e^2}R_1} \subseteq \mathcal{F}_{R_1}$ of the set $\{z : |F| > e^{2e^2}R_1\}$. Then (3.6) becomes

$$\int_{\tilde{\beta}_1} (\log |f|)^{-2-\lambda_0} |dz| \leq 2K(\log R_1, \lambda_0).$$

We now cut $\tilde{\alpha}_0$ off where it joins $\tilde{\beta}_1$ and obtain the second piece α_0 of Γ . Let $\tilde{\beta}_1$ correspond to γ_{t_1} in Q_1 and define the rectangle

$$S_1 = \{(s, t) : e^2 + \log R_1 < s < 2e^2 + \log R_1, t_1 < t < t_1 + 2e^3\}.$$

Again we find that the extremal length of the vertical lines joining the two sides of S_1 is $2e$. So we again obtain a curve $\tilde{\alpha}_1$ such that

$$\int_{\tilde{\alpha}_1} (\log |f|)^{-2-\lambda_0} |dz| \leq 4eK(\log R_1, \lambda_0).$$

This process is continued yielding a curve $\beta_0 \cup \alpha_0 \cup \beta_1 \cup \alpha_1 \cup \cdots \cup \beta_n \cup \tilde{\alpha}_n$ extending from $\partial\tilde{\mathcal{F}}_{R_0}$ to the boundary of a component \mathcal{F}_{R_n} where

$$\log R_n \geq e^n - 1 + \log R_0 \quad n = 0, 1, 2, \dots \quad (3.10)$$

Our construction yields

$$\int_{\tilde{\beta}_i} (\log |f|)^{-2-\lambda_0} |dz| \leq 2K(\log R_i, \lambda_0)$$

and

$$\int_{\tilde{\alpha}_i} (\log |f|)^{-2-\lambda_0} |dz| \leq 4eK(\log R_i, \lambda_0).$$

Adding these contributions and taking into account (3.4), (3.7) and (3.10) we

obtain

$$\begin{aligned} \int_{\beta_0 \cup \alpha_0 \cup \dots \cup \beta_n \cup \tilde{\alpha}_n} (\log |f|)^{-2-\lambda_0} |dz| &\leq (2+4e) \sum_{j=0}^{\infty} K(\log R_j, \lambda_0) \\ &\leq (2+4e) \sum_{j=0}^{\infty} K(e^j - 1 + \log R_0, \lambda_0) \\ &\leq 1 \end{aligned}$$

independent of n . We keep λ_0 fixed until N is so large that

$$(2+4e) \sum_{j=0}^{\infty} \left(2\pi \int_0^{\infty} \frac{r dr}{(e^j - 1 + \log R_N + r^{1/2-\varepsilon(r)})^{4+\lambda_0}} \right)^{1/2} \leq \frac{1}{2}. \quad (3.11)$$

At this point we change λ_0 to $\lambda_0/2$ with (3.11) playing the role of (3.4). We then continue from the arc $\tilde{\alpha}_n$ where $|F| = R_N$ in place of the original arc γ_0 on $|F| = R_0$. In the general case we obtain a sequence

$$0 = N_0 < N_1 < \dots < N_j$$

such that

$$\log R_{N_j} \geq e^{N_j - N_{j-1}} + \log R_{N_{j-1}} \quad j = 1, 2, \dots \quad (3.12)$$

The N_j are chosen such that

$$(4+2e) \sum_{n=0}^{\infty} K(e^n - 1 + \log R_{N_j}, \lambda_0/(j+1)) \leq 2^{-j} \quad (3.13)$$

with $\beta_{N_j} \cup \alpha_{N_j} \cup \dots \cup \beta_{N_{j+1}} \cup \tilde{\alpha}_{N_{j+1}}$ extending from $\partial \mathcal{F}_{R_{N_j}}$ to $\partial \mathcal{F}_{R_{N_{j+1}}}$ and satisfying

$$\begin{aligned} \int_{\beta_{N_j} \cup \alpha_{N_j} \cup \dots \cup \beta_{N_{j+1}} \cup \alpha_{N_{j+1}}} (\log |f|)^{-2-\lambda_0/(j+1)} |dz| \\ \leq (4+2e) \sum_{n=0}^{\infty} K(e^n - 1 + \log R_{N_j}, \lambda_0/(j+1)) \end{aligned}$$

Let $\Gamma = \beta_0 \cup \alpha_0 \cup \dots \cup \beta_k \cup \alpha_k \cup \dots$. Then since $\log |f| > 1$ in D and hence on Γ we have

$$\int_{\Gamma} (\log |f|)^{-2-\lambda} |dz| < \int_{\Gamma} (\log |f|)^{-2-\lambda'} |dz|$$

if $\lambda > \lambda'$. Thus it follows from (3.13) and (3.14) that Γ satisfies (1.6) for all $\lambda > 0$.

4. Proof of Theorem 1—general case

When f has zeros the proof in §3 must be modified slightly. First of all by hypothesis there exists a component D of $\{z : \log |f(z)| > \log K\}$ such that $\theta^*(r) \leq 2\pi$ for $r \geq r_0$, where we can assume $K > e$. Thus we can still find v satisfying (2.2) and (3.1). Since D is not necessarily simply connected, we can only define a local conjugate of $w = \delta v$ and so our function F is now multivalued. However $|F|$ and $\log |F|$ are single valued and subharmonic in \mathbb{C} . Thus we see that \mathcal{F}_R is again nonempty for all $R \geq K$.

We then proceed as before taking γ_0 to be a level curve of $|F| = R_0$ extending to infinity, where $F' \neq 0$ and find a curve $\tilde{\beta}_0$. We remark that $\tilde{\beta}_0$ never intersects a level curve $|F| = R$, $R_0 \leq R \leq R + 2e$ which forms a loop. In fact inside such a loop $|F| < R$ so if β_0^* is the portion of $\tilde{\beta}_0$ joining R_0 to R , β_0^* must pass through some point z_0 where $|F(z_0)| > R$. This is impossible since β_0^* is the image under h of the horizontal segment beginning at $\log R_0$ and ending at $\log R$. Hence we can find an $\tilde{\alpha}_0$ as before. We now continue as in §3.

5. Proof of Theorem 2

To prove Theorem 2 we need the following.

LEMMA 3. *Let f be entire of order $\frac{1}{2} < \rho < \infty$. If D is any component of $\{z : |f(z)| > K\}$, $K > e$ then*

$$\sup_G \lim_{\substack{r \rightarrow \infty \\ r \in G}} \theta^*(r) \geq \frac{\pi}{\rho} \quad (5.1)$$

where the sup is taken over all sets G of log density one.

Proof. Suppose on the contrary that the left side of (5.1) equals π/ρ_1 , $\rho_1 > \rho$. As in Lemma 2 we may find a set G of log density one where the sup on the left side of (5.1) is attained. Thus for $r \geq r_0$, $r \in G$

$$\theta^*(r) \leq \frac{\pi}{\rho_2} \quad (\rho_1 \geq \rho_2 > \rho). \quad (5.2)$$

Let $z \in D$ and choose R such that $|z| < R/4$. With the notation of (2.11) we

have

$$\begin{aligned}
\omega_R(z) &\leq 9\sqrt{2} \exp \left\{ -\rho_2 \int_{G \cap [2|z|, R/2]} \frac{dt}{t} \right\} \\
&\leq 9\sqrt{2} \exp \left\{ -\rho_2(1 - \varepsilon_m) \log \left(\frac{R}{|z|} \right) \right\} \\
&= 9\sqrt{2} \left(\frac{|z|}{R} \right)^{\rho_2(1 - \varepsilon_m)}
\end{aligned} \tag{5.3}$$

where (since $\log \text{dens } G = 1$) $\varepsilon_m \rightarrow 0$ as $m \rightarrow \infty$. Thus by (2.12) we have for fixed $z \in D_R$

$$\begin{aligned}
1 \leq \log |f(z)| &\leq \log K + \log M(R, f) \omega_R(z) \\
&\leq K_1 \log M(R, f) \left(\frac{|z|}{R} \right)^{\rho_2(1 - \varepsilon_m)}
\end{aligned}$$

where $K_1 > 0$ is constant. Then

$$\log M(R, f) \geq \frac{1}{K_1} \left(\frac{R}{|z|} \right)^{\rho_2(1 - \varepsilon_m)}.$$

Since z is fixed this implies that f has order at least $\rho_2 > \rho$, a contradiction. Thus (5.1) holds and Lemma 3 is true.

Proof of Theorem 2. Let D_1 be a component of $\{|f| > K\}$ and suppose

$$\inf_G \overline{\lim}_{\substack{r \rightarrow \infty \\ r \in G}} \theta_1^*(r) = \frac{\pi}{\alpha} \quad \left(\frac{1}{2} \leq \alpha < \infty \right) \tag{5.4}$$

where the inf is taken over all sets G , $\log \text{dens } G = 1$ and θ_1^* corresponds to θ^* for D_1 . Since there exists another component D_2 of $\{|f| > K\}$, (5.4) implies

$$\sup_G \overline{\lim}_{\substack{r \rightarrow \infty \\ r \in G}} \theta_2^*(r) \leq 2\pi - \frac{\pi}{\alpha} \tag{5.5}$$

where θ_2^* corresponds to θ^* for D_2 .

By Lemma 3 we must have

$$\frac{\pi}{\rho} \leq 2\pi - \frac{\pi}{\alpha}$$

or

$$\alpha \geq \frac{\rho}{2\rho - 1}. \quad (5.6)$$

By Lemma 1, (5.4) and (5.6) we may find a function v harmonic in D_1 such that for all $z \in D_1$

$$v(z) \geq |z|^{[\rho/(2\rho-1)] - \varepsilon(|z|)} \quad (0 \leq \varepsilon(|z|) \rightarrow 0 \text{ as } |z| \rightarrow \infty). \quad (5.7)$$

We now define ϕ and F as in the proof of Theorem 1. The proof of Theorem 2 now follows in the same way as that of Theorem 1 using $\rho/(2\rho-1)$ instead of $\frac{1}{2}$.

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