Zeitschrift: Commentarii Mathematici Helvetici

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 60 (1985)

Artikel: Earthquakes are analytic.

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DOI: https://doi.org/10.5169/seals-46297

Comment. Math. Helvetici 60 (1985) 017–030 0010–2571/85/010017–14$01.50 + 0.20/0

Earthquakes are analytic

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Various approaches to the study of Teichmuller space tend to emphasize different properties which are natural in a given context. For example, its complex analytic structure is natural when considering it as the space (T_g) of all marked Riemann surfaces of genus g (up to equivalence) or as a subset of quasi-Fuchsian groups. It is well-known that the complex analytic structure of T_g is quite inhomogeneous. In particular, the only biholomorphic self-mappings come from the properly discontinuous action of the (Teichmuller) modular group ([7]).

On the other hand, T_g (via the uniformization theorem) is also the space of hyperbolic structures on a surface of genus g. From this point of view, T_g is naturally a *real* analytic manifold, its structure coming from the isomorphism between $PSL(2, \mathbb{R})$ and the group of isometries of two-dimensional hyperbolic space. In contrast to the complex analytic case there are many real analytic maps of T_g to itself. It is reasonable, therefore, to further restrict oneself to maps which arise from geometric deformations of the hyperbolic structure, or to those which preserve some geometric quantity on the surfaces themselves.

The maps discussed in this paper are closely related to geodesic length functions (generalized from the length of closed geodesics to the length of geodesic laminations) in that they preserve the hypersurface level sets of these functions. They are the time 1 maps of a $6g$-6 dimensional family of flows, no two of which agree at any point (see Proposition 2.6 at the end of this paper).

The flows are parametrized by the space $\mathcal{ML}$ of geodesic laminations $\mu \in \mathcal{ML}$ and are denoted by $\mathcal{F}_\mu$. The integral curves of these flows are the earthquake deformations of hyperbolic structures which generalize the classical Fenchel-Nielsen twist deformations. That these older deformations are real analytic is well-known; since they are "dense" in the set of earthquake flows, we can think of the general $\mathcal{F}_\mu$ as a limit of these twist flows. (Indeed, that is how they are usually defined.) The primary purpose of this paper is to show that this limiting process is geometrically and analytically well-controlled.

* During preparation of this work the author was supported in part by NSF Grant MCS 82-03806.

THEOREM 1. *The earthquake flows $\mathscr{F}_\mu$ on T_g are real analytic for every geodesic lamination μ.*

The approach taken here is not quite as direct as the preceding discussion suggests, since it doesn't distinguish between the classical twist case and the general case. It is, however, often useful to keep the limiting process in mind.

The proof of Theorem 1 is fairly straightforward, combining an elementary normal families argument with known facts about both the real analytic structure of T_g and about the behavior of the geodesic length function under earthquake deformations.

As a corollary we find that the length, l_μ, of a geodesic lamination, μ, is equally smooth. In particular (Corollary 2.2), for μ a fixed lamination, it is real analytic as a function of T_g. As μ varies the l_μ vary continuously in the C^∞-topology for functions on compact subsets of T_g.

Since much of the background material is hard to reference, we have given an expository account of it in Section I. Further discussion of geodesic laminations and earthquakes may be found in [8] and [4]. The proof of Theorem 1 is contained in Section II.

I.A.

A hyperbolic surface, M, is a surface of genus g, $g \geq 2$, with a metric of constant curvature-1. It is isometric to a surface of the form $\mathbb{H}^2/\Gamma$ where $\mathbb{H}^2$ is two-dimensional hyperbolic space and Γ is a discrete subgroup of isometries isomorphic to $\pi_1 M$. M determines Γ up to conjugacy in the group of isometries of $\mathbb{H}^2$, which we identify with $PSL(2, \mathbb{R})$. Let Σ be a fixed topological surface of genus g. The Teichmuller space of genus g (T_g) is the space of marked hyperbolic surfaces; i.e., hyperbolic surfaces with a fixed isomorphism of $\pi_1 \Sigma$ to Γ where two surfaces are thought to be equivalent if there is an isometry between them respecting this isomorphism. Equivalently, T_g is the subset of discrete representations of $\pi_1 \Sigma$ into $PSL(2, \mathbb{R})$ up to conjugacy. It is known to be diffeomorphic to an open cell of dimension $6g-6$. The space of Fuchsian groups Γ together with an isomorphism from $\pi_1 \Sigma$ to Γ will be denoted by R_g; it is diffeomorphic to $T_g \times PSL(2, \mathbb{R})$ and will be called the representation space of genus g.

For computational purposes, the upper half-space of $\mathbb{C}$ serves as a convenient model for $\mathbb{H}^2$, but the point at infinity has a less (artificially) distinguished character if we identify the upper half-space with the upper hemisphere of the Riemann sphere $\hat{\mathbb{C}}$ $(=\mathbb{C} \cup \infty)$. The extended real axis $\mathbb{R} \cup \infty$ will be denoted by $\hat{\mathbb{R}}$: it is preserved by isometries of $\mathbb{H}^2$. However, since it will be necessary to consider homeomorphism of $\hat{\mathbb{C}}$ to itself which do not preserve $\hat{\mathbb{R}}$, it is useful to consider $\hat{\mathbb{R}}$ as a circle bounding the upper hemisphere, thus emphasizing that its topological character is unchanged under homeomorphism.

Any isometry of $\mathbb{H}^2$ extends continuously to its boundary, denoted by S^1_∞, and called the circle at infinity. (This is the unit circle in the Poincaré disk model, $\hat{\mathbb{R}}$ in the upper half space model.) Since M is a closed, non-singular surface, all of the elements γ of Γ are hyperbolic; i.e., γ acting on the closure of H^2 has exactly two fixed points, both on S^1_∞, one attracting and one repelling. Pairs of points on S^1_∞ are in 1–1 correspondence to geodesics in H^2; the geodesic corresponding to the fixed points of $\gamma \in \Gamma$ projects to the unique geodesic in M in the free homotopy class of $\gamma \in \pi_1\Sigma$ (under the isomorphism of Γ with $\pi_1\Sigma$).

Since there is a given isomorphism between any two Γ, $\Gamma' \in R_g$, there is a canonical 1–1 correspondence between elements in Γ and those in Γ' which induces a like correspondence between closed geodesics on the quotient surfaces M and M'. In other words, we can talk about *the* geodesic corresponding to the conjugacy class of $\gamma \in \pi_1\Sigma$ on every $M \in T_g$. Similarly, since two geodesics in H^2 intersect at most once, different points of intersection between two closed geodesics in M correspond to intersections between distinct lifts of the geodesics to H^2. Thus the correspondence between endpoints of S^1_∞ via the isomorphism between Γ and Γ' induces an identification between points of intersection of geodesics on M and M'.

Because fixed points of Γ and Γ' are both dense in S^1_∞, there is a unique homeomorphism of the circles at infinity for Γ and Γ' extending the correspondence between fixed points. It follows that the identification between geodesics and their intersections on M, $M' \in T_g$ carries over to infinite, non-closed geodesics as well. Nielsen showed that every lift to H^2 of any homotopy equivalence between M and M' (respecting the isomorphisms to $\pi_1\Sigma$ as usual) extends continuously to a homeomorphism on S^1_∞, depending only on M, M' and the choice of lift. These extensions are precisely the maps given by extending continuously the isomorphism between Γ and Γ'. (Different choices of Γ and Γ' with quotients M and M' amount to different lifts.)

R_g inherits a real analytic structure as a subset of the set of representations of $\pi_1\Sigma$ into the real analytic Lie group $PSL(2, \mathbb{R})$. T_g similarly inherits an analytic structure as a quotient space of R_g. If $\Gamma \in R_g$ and an element is represented by a matrix $A \in \Gamma$ (well-defined up to multiplication by $-I$) then it is an elementary fact that the geodesic representing γ in $\mathbb{H}^2/\Gamma = M$ has length $l_\gamma(M)$ where $\cosh l_\gamma(M) = \frac{1}{2}|\mathrm{tr}\, A|$. In particular, l_γ is a real analytic function on T_g and R_g. In fact, the lengths of finitely many closed curves completely determine the hyperbolic structure on M; locally $6g-6$ lengths serve as co-ordinates. (See e.g., [2], [3].) Whenever analycity on T_g or R_g is discussed in this paper, it is with respect to this analytic structure; lengths of closed curves will generally serve as convenient co-ordinates. It should be noted that any two sets of "length co-ordinates" are analytic functions of each other since they are both determined by the traces of finite products of a fixed generating set for $\pi_1\Sigma$.

Finally, we need to know how R_g sits in the space of all representations of $\pi_1\Sigma$ into $PSL(2,\mathbb{C})$, in particular, in the subset $\mathbb{C}R_g$ of quasi-Fuchsian groups. A quasi-Fuchsian group is a quasi-conformal deformation of a Fuchsian group. By this we mean that $\tilde{\Gamma}\subset PSL(2,\mathbb{C})$ satisfies $\tilde{\Gamma}=f\Gamma f^{-1}$ where Γ is Fuchsian and f and f^{-1} are quasi-conformal maps of the Riemann sphere $\hat{\mathbb{C}}$ to itself. These groups act properly discontinuously on two connected, simply-connected domains Ω_i, $i=0,1$ in $\mathbb{C}$, and have as limit set Λ a topological circle, which separates the Ω_i and which is the image under f of the circle limit set of the Fuchsian group.

As in the Fuchsian case, $\tilde{\Gamma}\subset\mathbb{C}R_g$ is assumed to possesss an isomorphism to $\pi_1\Sigma$ so that f is uniquely determined on S^1_∞ and the limit sets for different Γ's are canonically identified. (Fixed points of group elements in $\tilde{\Gamma}$ are still dense in Λ.) Moreover, the Riemann surfaces S_i defined by $\Omega_i/\tilde{\Gamma}$ ($\tilde{\Gamma}$ acts conformally on $\hat{\mathbb{C}}$) define points in T_g, and this ordered pair of points determines $\tilde{\Gamma}$ up to conjugacy in $PSL(2,\mathbb{C})$. Thus, $\mathbb{C}R_g\approx T_g\times T_g\times PSL(2,\mathbb{C})$ (although as a complex manifold it is probably best to write it as $T_g\times\bar{T}_g\times PSL(2,\mathbb{C})$ if T_g is given its usual complex structure.) The subset of groups conjugate to a Fuchsian group are characterized by the property that S_0 and S_1 are mirror image surfaces, or equivalently, that Λ is a geometric circle. $R_g\subset\mathbb{C}R_g$ is the subset where Λ is the circle $\hat{\mathbb{R}}\subset\hat{\mathbb{C}}$.

Although T_g has a complex structure, it is not natural in our context; in particular, the functions to be considered in Section 2 are not complex analytic. When extended to $\mathbb{C}R_g$, however, they *are* complex analytic which greatly simplifies convergence questions. The main relationship between $\mathbb{C}R_g$ and R_g which we need in this paper is the following:

PROPOSITION 1.1. *R_g is a real analytic submanifold of $\mathbb{C}R_g$. The induced structure is the analytic structure determined by the geodesic lengths of closed curves.*

This proposition is well-known and there are numerous possible proofs. The proof below is included for completeness and follows Bers' proof in [1] that $\mathbb{C}R_g$ is a $6g-3$ complex dimensional manifold.

Proof. Let Γ be a Fuchsian group and let a_i, b_i be the standard generators for $\pi_1\Sigma$ so that $\prod_{i=1}^{g}[a_i,b_i]=1$ is the single defining relation. If A_i, B_i are matrices representing a_i and b_i respectively (choose $2g-1$ signs arbitrarily), then by conjugation assume that

$$A_g=\begin{pmatrix}\rho & 0\\ 0 & \rho^{-1}\end{pmatrix}\qquad B_g=\begin{pmatrix}\gamma & \sigma\\ \sigma & \delta\end{pmatrix}\qquad \sigma\neq 0,\ \gamma\delta-\sigma^2=1. \tag{1}$$

Then, if $\prod_{i=1}^{g-1}[A_i, B_i] = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$, the equation

$$B_gA_gB_g^{-1}A_g^{-1} = \begin{pmatrix} 1+\sigma^2(1-\rho^{-2}) & \sigma\gamma(1-\rho^2) \\ \sigma\gamma(1-\rho^2) & 1+\sigma^2(1-\rho^2) \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \tag{2}$$

is satisfied. Similarly, the groups $\tilde{\Gamma}$ whose matrices, $\tilde{A}_i$, $\tilde{B}_i$, near those of Γ and satisfying (1) and (2) (with all entries replaced by nearby entries) determine a neighborhood of Γ in the submanifold of $\mathbb{C}R_g$ normalized by (1). It is not hard to see that $\tilde{A}_g$, $\tilde{B}_g$ are uniquely determined (in $PSL(2,\mathbb{C})$) by (2) for arbitrary $\tilde{x}$, $\tilde{y}$, $\tilde{z}$, $\tilde{w}$ so that the matrices $\tilde{A}_i$, $\tilde{B}_i$, $i = 1, 2, \ldots, g-1$, serve as local co-ordinates (i.e., choose three entries from each matrix) for groups in $\mathbb{C}R_g$ normalized by (1). The groups which are Fuchsian have matrices with real entries. Conversely, if x, y, z, w are real, then, by (2), $\tilde{\rho}$, $\tilde{\delta}$, $\tilde{\gamma}$ and $\tilde{\sigma}$ are either real or pure imaginary, and if either $\tilde{\delta}$ or $\tilde{\gamma}$ fails to be real, then so does $\tilde{\sigma}$. Since $\tilde{\sigma}, \tilde{\rho} \neq 0$ by hypothesis, it follows that all solutions near Γ for which $\tilde{A}_i$, $\tilde{B}_i$, $i = 1, 2, \ldots, g-1$, are real have real entries. Thus all $2g$ generators are real iff the first $2g-2$ are; hence in these local co-ordinates, the Fuchsian groups are precisely those with all real co-ordinates. A group is in $R_g \subset \mathbb{C}R_g$ iff it is conjugate by an element in $PSL(2,\mathbb{R})$ (which preserves $\hat{\mathbb{R}}$) to one with real entries satisfying (1). Thus the proposition follows. □

I.B.

A geodesic lamination $\mathscr{L}$ is a closed subset of a hyperbolic surface which is a union of simple geodesics and which satisfies the following local condition: There are open sets U_i covering $\mathscr{L}$ together with diffeomorphisms ϕ_i from U_i to R^2 such that $\phi_i(U_i \cap \mathscr{L}) = (0, 1) \times B_i$ where B_i is a closed subset of $\mathbb{R}$. On the overlaps $U_i \cap U_j$, $\phi_i \circ \phi_j^{-1}$ is of the form $(x, y) \to (f(x, y), (g(y))$; i.e., it preserves horizontal arcs. In other words, $\mathscr{L}$ is a "partial foliation" of M. We also assume that $\mathscr{L}$ has a transverse Borel measure μ invariant under translation along $\mathscr{L}$ whose support is all of $\mathscr{L}$. We will dop the notational distinction between $\mathscr{L}$ and its measure and denote both by "μ", "ν", etc. The space of all such laminations $\mathscr{ML}$ on M is homeomorphic to $\mathbb{R}^{6g-6}$. (For a discussion of the topology on the space of geodesic laminations see [4] or [8].) If we throw out the "zero" lamination and identify two laminations which are equal under multiplication of the transverse measure by a scalar, we get the space $\mathscr{PL}$ of projective classes of laminations which is homeomorphic to S^{6g-7}.

Given μ we can define $\int_A d\mu$ for any suitable transverse arc A by integrating the transverse measure over A. The *intersection number* $i(\gamma, \mu)$ of μ with any simple closed curve γ on M is $\inf_{\gamma'} \int_{\gamma'} d\mu$ where the γ' run all curves isotopic to γ. Similarly $i(A, \mu)$, A, a transverse arc, is $\inf_{A'} \int_{A'} d\mu$ where A' runs over all arcs isotopic to A with endpoints fixed. In both cases the infinum is realized by the unique geodesic in the corresponding isotopy class.

The simplest example of a geodesic lamination is a simple closed geodesic ϕ with r times the counting measure as its transverse measure. Then $i(\gamma, \mu) = r$ $i(\gamma, \phi)$ where $i(\gamma, \phi)$ is the minimum number of intersections under isotopy of γ between γ and ϕ. The function $i(\gamma, \cdot)$ is continuous on $\mathcal{ML}$; one way to define the topology on $\mathcal{ML}$ is to embed it as a subset of function space $\mathbb{R}^S$, where S is the set of isotopy classes of non-trivial, simple closed curves.

If γ is a closed geodesic on M, we can define the *total cosine* $\cos(\gamma, \mu) = \int_\gamma \cos\theta \, d\mu$ of γ with μ where θ is the angle of intersection of γ with μ (measured counterclockwise from γ to μ). The integral exists because the simplicity of μ uniformly bounds the local variation of θ. (See [4] for a more detailed discussion.) If $\mu = (\phi, r) \in S \times \boldsymbol{R}_+$ then $\cos(\gamma, \mu) = r\Sigma \cos\theta$, where the sum is over the intersections of the geodesics ϕ and γ.

Although weighted simple closed curves are very simple examples of geodesic laminations, Thurston [8] shows that $S \times \mathbb{R}_+$ is dense in $\mathcal{ML}$ and S in dense in $\mathcal{PL}$. This allows one to extend many operations and concepts from simple closed curves to general geodesic laminations. The deformations defined below are one such example.

If a lamination is lifted to $\mathbb{H}^2$ each (infinite) geodesic converges to a point on S^1_∞ in each direction. Conversely, the pairs of points on S^1_∞ determine the geodesic. Therefore, the map between the circles at infinity for two surfaces M and M' discussed in *IA* allows a canonical identification between the laminations on M and those on M'. Simplicity of leaves is invariant under this equivariant map since it is equivalent to nonlinking of the endpoints of a leaf and all of its lifts. We will implicitly make this identification by talking about a lamination μ on all $M \in T_g$ simultaneously.

Given any hyperbolic surface M and simple closed geodesic γ on M we can define a new hyperbolic structure M_t by cutting along γ and glueing it back with a left twist of distance t. To determine a well-defined point in T_g, we must keep track of homotopy classes of curves. This is done by identifying the homotopy class of a closed curve ϕ on M with the homotopy class of the curve ϕ' on M_t determined by following the image of ϕ in M_t until it hits γ (assuming it does), going along γ to the left distance t, continuing along the image of ϕ and so on.

This cutting and glueing operation will be called the *time t twist along* γ (often called a Fenchel-Nielsen twist). As t varies, the surfaces M_t define a path in T_g denoted by $\mathscr{E}_\gamma(t)$ (M will always be implicit) and called the *time t twist deformation along* γ. The time t twist along γ can be generalized to a time t twist deformation determined by $(\gamma, r) \in S \times \mathbb{R}_+ \subset \mathcal{ML}$ by taking it to be the time tr twist along γ. Since $S \times \mathbb{R}_+$ is dense in $\mathcal{ML}$ we make the following:

DEFINITION. For any $M \in T_g$, $\mu \in \mathcal{ML}$, the *time t earthquake deformation*, $\mathscr{E}_\mu(t)$, determined by μ is the limit in T_g (for each t) of the time t twist deformations of M determined by $(\gamma_i, r_i) \in S \times \mathbb{R}_+$ where $(\gamma_i, r_i) \to \mu$ in $\mathcal{ML}$.

The following result is proved in [4]:

PROPOSITION 1.2. *The limits $\mathscr{E}_{(\gamma_i, r_i)}(t)$, $(\gamma_i, r_i) \to \mu$ are independent of the approximating sequences so that $\mathscr{E}_\mu(t)$ is well-defined. $\mathscr{E}_\mu(t)$ is a C^1 curve in T_g for all $\mu \in \mathcal{ML}$.*

It follows from the work of Wolpert [9] that $\mathscr{E}_\mu(t)$ is C^2. We will show in this paper that the curves are analytic; in fact, they are the integral curves of an analytic flow defined on T_g. The geodesic lengths of closed curves provide analytic co-ordinates for T_g so the first step is to see what the derivatives of the length function, l_ϕ, of a fixed closed geodesic, ϕ, are along $\mathscr{E}_\mu(t)$. This is contained in

PROPOSITION 1.3 ([4]). *$dl_\phi/dt = \int_\phi \cos\theta \, d\mu$ along the earthquake path $\mathscr{E}_\mu(t)$.*

The goal of Section II is to study how $\cos\theta$ and hence dl_ϕ/dt varies as a function of $M \in T_g$.

II.

With the background material established in the previous section, we proceed here to the proof of the main theorem, which is restated below. As previously discussed, we can identify a fixed geodesic lamination $\mu \in \mathcal{ML}$ on every hyperbolic surface $M \in T_g$ simultaneously. This allows us to identify, for each $t \in \mathbb{R}$, the time t earthquake deformation of M determined by μ for every $M \in T_g$. Thus, for any fixed $\mu \in \mathcal{ML}$ a flow $\mathscr{F}_\mu$ is defined on T_g. Although the earthquake maps on the surfaces are complicated and not generally C^1, the flows are very smooth.

THEOREM 1. *The earthquake flows $\mathscr{F}_\mu$ are real analytic for every $\mu \in \mathcal{ML}$.*

COROLLARY 2.1. *The geodesic length function, l_ϕ, ϕ any closed curve, is analytic along every earthquake path $\mathscr{E}_\mu(t)$.*

The length l_ν of a geodesic lamination $\nu \in \mathcal{ML}$ is defined as the total mass on the surface M of the measure which is the product of Lebesque measure along the leaves of ν and the measure ν transverse to the leaves. Equivalently, l_ν is the limit of $r_i l_{\gamma_i}$ where $(\gamma_i, r_i) \in S \times \mathbb{R}_+ \subset \mathcal{ML}$ converges to ν in $\mathcal{ML}$. (This equivalence is proved during the proof of Corollary 2.2.)

COROLLARY 2.2. *The length l_ν of the lamination $\nu \in \mathcal{ML}$ is analytic along $\mathcal{E}_\mu(t)$ for all $\mu \in \mathcal{ML}$. It is analytic on all of T_g and constant along $\mathcal{E}_\nu(t)$. Hence $\mathcal{F}_\nu$ preserves l_ν. As μ varies the l_μ vary continuously in the C^∞-topology for functions on compact subsets of T_g.*

The proof of Corollary 2.2 is at the end of this section. Corollary 2.1 follows immediately from Theorem 1.

We will show that the vector fields on Teichmuller space which generate the earthquake flows are real analytic. Since the lengths of finitely many closed curves provide local (analytic) co-ordinates, it suffices to show that the first derivative of the geodesic length function, l_ϕ, of any closed curve ϕ in the direction of the flow is an analytic function of the point in T_g. By Proposition 1.3, this derivative at $M \in T_g$ in the direction of $\mathcal{F}_\mu$ equals the total cosine, $\int_\phi \cos\theta \, d\mu$, of μ with ϕ, where θ is the angle on M from ϕ to μ at every point of intersection between ϕ and μ. Our first goal, therefore, is to understand how θ varies with M for each such intersection.

First, notice that points of intersection between ϕ and μ on two distinct surfaces M and M' are in a canonical 1–1 correspondence. This correspondence is induced by the maps on the circles at infinity for M and M' respectively as discussed in Section IA. The angle of intersection between the two geodesics can be computed in terms of the cross-ratio of their endpoints.

DEFINITION. The *cross-ratio* $\chi(a, b, c, d)$ of four points in $\hat{\mathbb{C}}$ is equal to

$$(a, b, c, d) = \frac{(a-c)(b-d)}{(a-d)(b-c)}$$

The cross-ratio is invariant under linear fractional transformations and calculation shows that if a_i, b_i, $i = 1, 2$ lie on $\hat{\mathbb{R}}$ then

$$\chi(a_1, b_1, b_2, a_2) = \cos^2\frac{\theta}{2} = \frac{\cos\theta + 1}{2} \tag{1}$$

where (a_1, b_1, a_2, b_2) are the endpoints of two geodesics l_i, $i = 1, 2$ arranged counterclockwise around $\hat{\mathbb{R}}$ and θ is the angle of intersection between the geodesics. The angle θ is clearly an analytic function of the endpoints so we want to show that these vary analytically on R_g.

When an endpoint, x, is a fixed point for an element $\gamma \in \Gamma_0$ analyticity is clear, for we can write γ as a finite product of fixed generators of Γ_0 which, by definition, vary analytically as functions of R_g. Since γ varies analytically so does x. Fixed points of group elements are dense in S^1_∞ so the general case will follow if the functions $f_\Gamma(x_i): R_g \to S^1$, ($x_i$ fixed points of γ_i, $x_i \to x$), converge nicely. Since this situation is most simply analyzed, via normal families, when the maps are holomorphic, we allow deformations within the complex Lie group $PSL(2, \mathbb{C})$. This is the reason for the discussion of quasi-Fuchsian groups in Section I.A.

PROPOSITION 2.3. *Fix $\Gamma_0 \in R_g$ and denote by $f_\Gamma: S^1_\infty \to \Lambda \subset \hat{\mathbb{C}}$ the map from the circle at infinity of Γ_0 to the limit set Λ of $\Gamma \in \mathbb{C}R_g$. Then, for any $x \in S^1_\infty$, the map $\phi(\Gamma) = f_\Gamma(x)$ from $\mathbb{C}R_g$ to $\hat{\mathbb{C}}$ is complex analytic.*

Proof. In the case x is a fixed point of some element $\gamma \in \Gamma_0$, the proposition follows as before from the fact that γ is finite product of generators which vary analytically. For a general $x \in S^1_\infty$ let $x_i \to x$, x_i fixed points of $\gamma_i \in \Gamma_0$. Let $\phi_i(\Gamma) = f_\Gamma(x_i)$ and $\phi(\Gamma) = f_\Gamma(x)$.

Since f_Γ is continuous for each Γ and $x_i \to x$, then $f_\Gamma(x_i) \to f_\Gamma(x)$ for each Γ so $\phi_i \to \phi$ pointwise. Furthermore, for Γ restricted to a compact set of $\mathbb{C}R_g$, f_Γ is the restriction to $S^1_\infty = \hat{\mathbb{R}}$ of a family of K-quasiconformal mappings of the Riemann sphere to itself. A sequence of K-quasiconformal mappings converging to another K-quasiconformal mapping converges uniformly on compact sets ([6]). Therefore, for every $\Gamma \in \mathbb{C}R_g$, there is an open set containing Γ for which either $1/|\phi_i|$ or $|\phi_i|$ is bounded (depending on whether or not $\phi(\Gamma) = \infty$) for $i > N$, some N. In other words, the ϕ_i are locally bounded. Since they are all complex analytic, they form a normal family, and the limit $\phi(\Gamma) = f_\Gamma(x)$ is analytic. □

The function $\cos\theta$ can be extended to arbitrary collections (a_1, b_1, a_2, b_2) by formula (1). It will be a complex analytic function of the endpoints $\{a_i, b_i\}$. We can then extend the function $\int_\phi \cos\theta \, d\mu$ to a neighborhood of R_g in $\mathbb{C}R_g$. The integral exists and is approximated uniformly on compact sets of $\mathbb{C}R_g$ by its Riemann sums. To see this, note that it is true on R_g because the leaves of μ do not cross. (This was discussed in I.B.) Furthermore, the maps f_Γ from S^1_∞ to Λ are equicontinuous on compact subsets of $\mathbb{C}R_g$ by the proof of Proposition 2.3 so the local variation of $\cos\theta$ is still uniformly bounded. Thus the integral is approximated uniformly by its Riemann sums as claimed.

We can now prove Theorem 1.

Proof of Theorem 1. By Proposition 2.3 $\cos\theta$ is complex analytic on $\mathbb{C}R_g$, and, by the discussion above, the integral $\int_\phi \cos\theta\, d\mu$ is approximated uniformly on compact subsets of $\mathbb{C}R_g$ by its Riemann sums so that it is also complex analytic. The real part of the integral is analytic and since $\cos\theta$ is real on the real analytic submanifold R_g, $\int_\phi \cos\theta\, d\mu$ is real analytic on R_g.

The real analytic structure of T_g is determined by the lengths l_ϕ of finitely many simple closed geodesics ϕ and the derivative of l_ϕ along the flow $\mathcal{F}_\mu$ is $\int_\phi \cos\theta\, d\mu$ by Proposition 1.3. Thus l_ϕ itself is analytic along $\mathcal{F}_\mu$ and $\mathcal{F}_\mu$ is an analytic flow. □

Before proving Corollary 2.2, we digress for a brief discussion of the length l_ν of a geodesic lamination ν and its derivative along $\mathcal{F}_\mu$.

DEFINITION. The *length,* $l_\nu(M)$, $M\in T_g$, *of a geodesic lamination* ν is the integral over M of the product measure $d\nu\times dl$ where dl is the length measure along the leaves of ν.

LEMMA 2.4. *Given any sequence* $c_i\phi_i$ *of weighted simple closed geodesics converging in* $\mathcal{ML}$ *to* ν, *and any* $M\in T_g$, $c_i l_{\phi_i}(M)$ *converges to* $l_\nu(M)$. *The convergence is uniform on compact subsets of* T_g; *hence* l_ν *is continuous on* T_g. *In fact* $l_\nu(M)$ *is continuous with respect to the pair* (M, ν).

Proof. If we denote by $d\phi_i$ the counting measure on ϕ_i then $l_{\phi_i}(M)=\int_M d\phi_i\times dl$. Cover the support of ν with finitely many quadrilaterals with the following properties:

i) Two opposite ("horizontal") sides are disjoint from ν.
ii) The remaining two ("vertical") sides are transverse to ν and each leaf of ν crosses from one side to the other.

It suffices to prove the lemma on a single quadrilateral Q.

By definition (see [4]), convergence of $c_i\phi_i$ to ν implies that, on any finite set of transverse geodesic arcs, A_j, endpoints disjoint from ν, the intersection numbers and total cosines of $c_i\phi_i$ with the A_j converge to those of ν with the A_j. Moreover, if we let the A_j vary continuously with the hyperbolic structure, the intersection number and total cosine are continuous as functions on $T_g\times\mathcal{ML}$ so the convergence is uniform on compact subsets of T_g.

To see that the intersection number is continuous on the product, note that because the leaves of all laminations and the arcs A_j move continuously with the hyperbolic structure, M, we can assume, by restricting to a small neighborhood of a given structure M_0, that only geodesics in a given neighborhood of the endpoints of the A_j move across the endpoints as we vary M. Since the measure with respect to ν of some neighborhood of the endpoints is zero, the measure of this neighborhood with respect to other laminations can be made arbitrarily small on M_0 by restricting to small neighborhoods of ν in $\mathcal{ML}$. Thus the change in the intersection number with the A_j is uniformly small on this neighborhood of ν as we vary over the chosen neighborhood of M_0. Continuity follows. The same argument shows that the angles of intersection and total cosines are also continuous as functions on the product.

In particular the above discussion applies to any finite subdivision of one of the horizontal arcs A of Q if Q, A, and its subdivision vary continuously with the hyperbolic structure. Restricting to a single sub-arc, we see that the variation of the angle of intersection for the leaves of a lamination is bounded, independent of the lamination, depending only on the length of the sub-arc A_j. This is because the leaves of a lamination do not cross. It follows that the shortest and longest pieces of leaves in Q going from a fixed A_j to the opposite side of Q are universally close, depending only on the shape and size of Q, and going to zero as the length of the A_j go to zero. Since the shape and size of Q and the length of A_j vary continuously with the hyperbolic structure, the estimate for the difference between the shortest and longest pieces of leaves can be made uniform on a compact subset of T_g, independent of the lamination.

It follows that the integrals defining the length, restricted to Q, can be uniformly approximated by their Riemann sums, i.e., for any $\varepsilon > 0$ there is a subdivision A_j of A such that

$$\left| \int_Q d\phi_i \times dl - \sum i(A_j, \phi_i) l_j^{(i)} \right| < \varepsilon i(A, \phi_i)$$

Similarly,

$$\left| \int_Q d\nu \times dl - \sum i(A_j, \nu) l_j \right| < \varepsilon i(A, \nu)$$

where l_j ($l_j^{(i)}$) is the length of any arc of $\nu(c_i \phi_i)$ going from A_j to the opposite side of Q. The estimates are uniform on compact subsets of T_g and the only restriction on the weighted curves $c_i \phi_i$ is that they are close enough to ν so that the total length of the arcs crossing A but hitting the top or the bottom of Q is small. (These arcs are counted in the integral but not in the sum.)

Finally, since for any finite collection of A_j's and a given compact subset of T_g, the intersection numbers and angles of intersection (hence the $l_j^{(i)}$ also) converge uniformly, we can always find an integer N such that for the collection of arcs needed for the first estimates we have

$$\left| \sum i(A_j, \nu) l_j - \sum c_i i(A_j, \phi_i) l_j^{(i)} \right| < \varepsilon, \qquad i > N$$

Since ε is arbitrary the lemma follows from the three estimates and the triangle inequality. □

Given any two geodesic laminations, μ, ν, we can define the product measure $d\mu \times d\nu$ on M. First, consider any quadrilateral Q whose "horizontal" sides, A_1, A_2, are parts of leaves of ν and whose "vertical" sides B_1, B_2 are parts of leaves of μ. Furthermore, we require that $i(A_1, \mu) = i(A_2, \mu)$ and $i(B_1, \nu) = i(B_2, \nu)$; i.e., no leaf of either μ or ν hits the same side twice. Then, by definition, $d\mu \times d\nu(Q)$ equals the product $i(B_1, \nu)i(A_1, \mu)$. The measure of an arbitrary Borel set is defined in the usual way. The measure is defined to be zero at any point of tangency of μ and ν; i.e., on any common leaf.

When μ and ν are weighted simple closed geodesics, the derivative of l_ν with respect to twisting along μ is easily seen to be the weighted sum of $\cos\theta$ at the finite number of intersections of μ and ν. (See, e.g., [4] Lemma 3.2.) This is just $\int_M \cos\theta \, d\mu \times d\nu$ when $d\mu$ and $d\nu$ are both atomic. Proposition 1.3 covers the case when only $d\nu$ is atomic. For the general case we have the following:

PROPOSITION 2.5. *The function l_ν is C^1 along the earthquake path $\mathscr{E}_\mu(t)$ with derivative $\int_M \cos\theta \, d\mu \times d\nu$, where θ is the angle (measured counterclockwise) from ν to μ at each point of intersection of μ and ν.*

Proof. Take any sequence $c_i\phi_i$ of weighted simple closed curves converging to ν in $\mathscr{ML}$. Then we will show that

$$c_i \int_M \cos\theta \, d\mu \times d\phi_i \to \int_M \cos\theta \, d\mu \times d\nu \tag{2}$$

uniformly on compact sets of T_g. Thus

$$c_i l_{\phi_i} \to l_\nu \quad \text{and} \quad c_i \frac{dl_{\phi_i}}{dt} \to \int_M \cos\theta \, d\mu \times d\nu$$

along $\mathscr{E}_\mu(t)$, uniformly for $t \leq T$. The proposition will then follow.

The proof of (2) is essentially the same as that of Lemma 2.4. Cover the support of $d\mu \times d\nu$ by finitely many geodesic quadrilaterals of the type described above with vertical sides in μ and horizontal sides in ν. Restrict, without loss of generality, to a single such quadrilateral Q which we further subdivide into similar quadrilaterals Q_j with sides of length less than δ. To apply the discussion from the proof of Lemma 2.4 perturb the Q_j slightly so that the vertical sides are disjoint from μ, the horizontal sides from ν and assume that they move continuously with the hyperbolic structure, M. For δ sufficiently small, $\cos\theta$ will vary less than any given ε on every Q_j, so

$$\left|\int_Q \cos\theta\, d\mu \times d\nu - \sum \cos\theta(x_j) \int_{Q_j} d\mu \times d\nu\right| < \varepsilon \int_Q d\mu \times d\nu$$

for any choice of x_j in Q_j. Similarly,

$$\left|c_i \int_Q \cos\theta\, d\mu \times d\phi_i - \sum \cos\theta(y_j) c_j \int_{Q_j} d\mu \times d\phi_i\right| < \varepsilon c_j \int_Q d\mu \times d\phi_i$$

for any choice of y_j in Q_j in the intersection of μ with ϕ_i.

These estimates depend only on δ; hence they are uniform in M. As in the proof of Lemma 2.4, for a fixed subdivision Q_j and compact region of T_g, we can find an N such that for $i > N$ the Riemann sums are approximated uniformly by those of the $c_i\phi_i$. From the estimates above and the triangle inequality the integrals are similarly estimated. But, by choosing δ sufficiently small and N sufficiently large, this holds for any ε and the proposition follows. □

Now the proof of Corollary 2.2 is straightforward.

Proof of Corollary 2.2. From Proposition 2.5, the derivative of l_ν along $\mathscr{E}_\mu(t)$ is:

$$\frac{dl_\nu}{dt} = \int_M \cos\theta\, d\mu \times d\nu$$

where θ is the angle (measured counterclockwise) from ν to μ at each intersection of μ and ν. Proposition 2.3 implies that $\cos\theta$ varies analytically over $\mathbb{C}R_g$ at each intersection of μ and ν. As in the proof of Theorem 1, this implies that l_ν varies analytically along $\mathscr{E}_\mu(t)$. When $\mu = \nu$ the first derivative is identically zero so l_ν is constant. To see that l_ν is analytic on all of T_g note that all of its directional derivatives are analytic either by Proposition 2.6 below or by the fact that tangents to classical twist flows span the tangent space at every point (see [10]).

That the functions l_{μ_i} converge to l_μ in the C^∞-topology (on compacts) as μ_i converges to μ similarly follows from $l_{\mu_i}(M)$ converging to $l_\mu(M)$ and the fact that when complexified, the derivatives of l_{μ_i} along classical twist paths converge uniformly to those of l_μ on compact neighborhoods of R_g in $\mathbb{C}R_g$. □

Although it is not necessary in the proof of Corollary 2.2, it seems worthwhile to point out (Proposition 2.6) that every tangent vector in T_g is tangent to a unique earthquake path. The main point is the following theorem which is proved in [5] where it is of more central importance:

THEOREM. *If two geodesic laminations μ and ν on $M \in T_g$ have the same total cosine with every closed geodesic on M, then $\mu = \nu$.*

PROPOSITION 2.6. *Every tangent vector in T_g is tangent to a unique earthquake path in T_g.*

Proof. Since the space of geodesic laminations and the tangent space at any point M in T_g are homeomorphic to $6g-6$ dimensional balls, it suffices, by invariance of domain, to show that the map associating a lamination μ with the tangent to the integral curve (earthquake path) of $\mathcal{F}_\mu$ through M is continuous, proper, and 1–1. From Proposition 1.3 continuity and properness are immediate. Similarly, from this proposition, it follows that if $\mathcal{F}_\mu$ and $\mathcal{F}_\nu$ are tangent at M, then $\int_\phi \cos\theta \, d\mu = \int_\phi \cos\tilde{\theta} \, d\nu$ for every closed geodesic ϕ. In other words, all the total cosines are equal, so, by the Theorem above, this implies that $\mu = \nu$ and the map is 1–1. □

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Received September 23, 1983