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On foliations of $\mathbb{R}^{n+1}$ by minimal hypersurfaces

BRUCE SOLOMON

Introduction and preliminaries

This paper sets forth several basic theorems regarding foliations of $\mathbb{R}^{n+1}$ by minimal hypersurfaces; here $n+1 \geq 8$, for reasons discussed below. First we outline our primary results.

In §1, we prove a local result (Theorem 1.1), concerning smoothness. It is shown that any *a priori* merely continuous foliation of an open subset of $\mathbb{R}^{n+1}$ by minimal hypersurfaces is actually Lipschitz, and oriented by a Lipschitz unit normal. This result is sharp, as evidenced by several accompanying examples, and perhaps surprisingly so, since minimal hypersurfaces are themselves always real analytic. We proceed in §2 to study the global structure of such foliations, when they are defined on all of Euclidean space. In this regard, Theorem 2.3 establishes that, in any dimension, the leaf space of the foliation is $\mathbb{R}$, the real line. Geometers, it seems, are often unaware that this is far from being true for general hypersurface foliations of $\mathbb{R}^{n+1}$, even when leaves are assumed proper and real-analytic, even if $n+1=2$ (cf. §2 below). In Theorem 2.4, we specialize to foliations which have an asymptotically regular leaf. This natural hypothesis always obtains in the critical case $n+1=8$, and the theorem states that such a foliation is diffeomorphic to the cartesian product of a *contractible* area-minimizing hypersurface with $\mathbb{R}$, and that, near infinity, each leaf looks like the central cone over a homology $(n-1)$-sphere embedded "symmetrically" in the unit sphere S^n (cf. remarks following Theorem 2.4).

Our interest in foliations of $\mathbb{R}^{n+1}$ by minimal hypersurfaces derives from a very natural problem which has yet to be solved. We precede its statement with a few words of background.

Recall that a solution of the minimal surface equation in $\mathbb{R}^n$ is a function whose graph in $\mathbb{R}^{n+1}$ is a minimal hypersurface. The classical Bernstein problem, which asks whether an *entire* solution is necessarily a linear polynomial (i.e. whether its graph must be an affine hyperplane) is well-known, and has a long, fascinating history. Bombieri, DeGiorgi, and Giusti solved it completely in 1968 ([BDG]), building upon a number of other important works (e.g. [B], [FW], [D], [A], and [SJ]) in doing so. Their result (the "Bernstein Theorem") states that entire minimal graphs are always hyperplanes when $n+1 \leq 8$, whereas in any higher dimension, this assertion fails.

Consider, however, that $\mathbb{R}^{n+1}$ is foliated by the vertical translates of any entire graph, minimal or not. In light of this simple observation, the following deeper, but perhaps more natural (from a geometric standpoint) problem seems inevitable.

Is every foliation of $\mathbb{R}^{n+1}$ by minimal hypersurfaces a foliation by parallel affine hyperplanes?

When $n+1 \neq 8$, the theory developed in connection with Bernstein's problem yields an answer: affirmative if $n+1<8$, negative if $n+1>8$. But in the critical dimension $n+1=8$, it appears quite difficult to settle this question. More generally, one might ask, when $n+1>8$, whether such foliations always arise by translating an entire minimal graph, or, whether the assumption of an asymptotically regular leaf implies that all leaves are parallel hyperplanes. We would guess that both these questions have negative answers in high dimensions. Some discussion of this is given following the proof of Theorem 2.4.

Henceforth our basic setting will be a codimension one, C^k foliation of $\mathbb{R}^{n+1}$ (or some open subset thereof) which will generally be denoted by $\mathscr{F}$. Here $k, n \geq 0$ are unrestricted non-negative integers. We thus have a decomposition ot $\mathbb{R}^{n+1}$ into a union of n-dimensional submanifolds, called the *leaves* of $\mathscr{F}$, and each point of $\mathbb{R}^{n+1}$ has a neighborhood U where there is a distinguished C^k coordinate system

$$(x, y): U \to \mathbb{R}^n \times \mathbb{R}.$$

That is, for each $t \in \mathbb{R}$, $y^{-1}(t)$ is a connected component of $\lambda \cap U$ for some leaf $\lambda \in \mathscr{F}$. Furthermore, we will always assume that leaves of $\mathscr{F}$ are *minimal*, i.e., smoothly immersed submanifolds having zero mean curvature.

Of special interest, particularly in §2, will be the case in which leaves are *hypersurfaces*. By this we mean a codimension one, locally integral current S of the form

$$S = \partial[\![V]\!], \qquad V \subset \mathbb{R}^{n+1} \text{ open},$$

where $[\![V]\!]$ denotes the current corresponding to oriented integration of $(n+1)$-forms over V. (The reader may wish to consult one of the books by H. Federer [FH] or L. Simon [SL], if unfamiliar with the theory of integral currents.) This point of view will be useful to us because there is a geometrically natural topology on the space of hypersurfaces; the *integral flat topology* [FH, 4.3.16] or (SL, §31). Roughly, two hypersurfaces are close in this topology if their difference bounds an open set having "small" volume in any ball.

When the support of a hypersurface S (denoted $\mathrm{spt}(S)$) is a Riemannian submanifold of $\mathbb{R}^{n+1}$, we may sometimes refer to this submanifold as "the hypersurface S". No serious ambiguity will arise through this practice.

A further useful aspect of hypersurfaces is the naturality with which one can impose a more global variational hypothesis than that of minimality; namely, that of *area-minimization.* A hypersurface S is said to be area-minimizing in an open set $U \subset \mathbb{R}^{n+1}$ if

$$\|S\|(B_r) \leq \|S+Z\|(B_r)$$

for all sufficiently large $r>0$, whenever Z is a hypersurface having compact support in U. (We use B_r to denote the open ball of radius $r>0$ and center at the origin $O \in \mathbb{R}^{n+1}$). Roughly speaking, this means that no compact piece of S in U can be replaced by a piece having less n-dimensional area, without introducing boundary.

Before we proceed, it is our pleasure to thank the Mathematical Sciences Research Institute in Berkeley, Indiana University, Bloomington, and the Centre for Mathematical Analysis in Canberra, for supporting this research.

§1. Local regularity

In this section, we discuss the smoothness of foliations having codimension one minimal leaves. Our basic result is the following.

(1.1) THEOREM. *Let $\mathcal{F}$ be a codimension one, C^0 foliation of an open set $U \subset \mathbb{R}^{n+1}$. If all leaves of $\mathcal{F}$ are minimal, then $\mathcal{F}$ is oriented by a locally Lipschitz unit normal vector field, and admits a locally Lipschitz structure.*

Though it gives only local $C^{0,1}$ regularity, this theorem is actually sharp. We illustrate by some simple examples.

Let $U \subset \mathbb{R}^2$ be the open disc of radius 1 and center at the point $(2,0)$. Consider the foliation of U by straight line segments, given as level sets of the Lipschitz (but not C^1) function

$$f(x,y)=\begin{cases} y/x & y \geqslant 0 \\ y, & y \leqslant 0 \end{cases}.$$

A moment's meditation on this example makes it clear that more regularity than Theorem 1.1 provides cannot be expected, at least locally. On the other hand, it may be true that a global C^0 foliation of $\mathbb{R}^{n+1}$ by minimal hypersurfaces is necessarily C^1 or smoother, possibly even real analytic. This is true when $n+1<8$ of course (at least, assuming leaves are proper) but it appears to be a considerably more subtle matter when $n+1>8$. Indeed, if true, it must depend on the metric properties of $\mathbb{R}^{n+1}$. For, if we interpret the open disc above as the "Klein model" for hyperbolic 2-space [HCV, §35], we obtain a C^0 foliation of the hyperbolic plane by proper minimal (in fact, totally geodesic) hypersurfaces, which is locally Lipschitz, but not C^1.

It may also be worth mentioning here that the normal to a C^0 foliation of $\mathbb{R}^{n+1}$ by real analytic – but not minimal – leaves, is generally discontinuous. For example, one can foliate the upper half plane in $\mathbb{R}^2$ with leaves which are all homothetic images of the graph $\{(x, f(x)): x \in \mathbb{R}\}$, where $f(x) = (\pi/2) + \tan^{-1}(x)$. The x-axis then completes this foliation continuously to the closed half plane, and reflection across the x-axis even gives an entire foliation of $\mathbb{R}^2$. Each leaf is real analytic, but the normal to this foliation is clearly discontinuous at the origin.

Having brought the content of Theorem 1.1 into focus with the above examples, we now proceed to its proof.

Proof of Theorem 1.1. First, by the purely local nature of the theorem, it will suffice to assume that $\mathcal{F}$ is comprised of the level sets of a single continuous function $y: U \to \mathbb{R}$ having no extrema in U, and which separates leaves. Denote by λ_t the leaf $y^{-1}(t)$.

Next, we claim that (in contrast to the last example above) the unit normal ν to $\mathcal{F}$ is continuous. Since we are free to reparametrize y, it will suffice to derive the continuity of ν on λ_0 (i.e. at $t=0$). If $p \in \lambda_0$ and $B_r(p) \subset\subset U$ for some $r>0$, then one easily sees that as $t \to 0$, $\lambda_t \to \lambda_0$ as hypersurfaces in $B_r(p)$, in the integral flat topology. Under our present hypothesis that all leaves are minimal, it is a basic fact that whenever $K \subset\subset U$ is open and convex, and $\lambda \in \mathcal{F}$ is a leaf, $\lambda \cap K$ is area-minimizing. (We do not argue this fact here, because a stronger version of it is proved later, in Lemma 2.2.) In particular, therefore $\lambda_t \cap B_r(p)$ is minimizing for each $t \in \mathbb{R}$, and hence the convergence $\lambda_t \to \lambda_0$ is governed by the basic regularity theory for minimizing hypersurfaces [FH, 5.3.14], which provides that integral flat convergence of minimizers to a smooth limit is actually smooth convergence. More precisely, for sufficient small $|t| \geq 0$, $\lambda_t \cap B_r(p)$ can be expressed as the graph of a function $f_t: \lambda_0 \to \mathbb{R}$ (relative to the unit normal ν on λ_0) in $B_r(p)$, where $f_t \to 0$ in the C^α norm for every $\alpha>0$, as $t \to 0$. This clearly gives the continuity of ν on $\lambda_0 \cap B_r(p)$, hence throughout U.

We emphasize that this C^α convergence of nearby leaves to λ does *not* imply ν

is $C_{\alpha-1}$ for all $\alpha > 0$. Indeed it cannot, by virtue of the first counterexample preceding this proof. We can show, however, that ν is locally Lipschitz; we now proceed to do so.

For this, observe that since ν is continuous, we may restrict our attention to a neighborhood in which ν barely varies. Indeed, it will suffice to consider the case

$$U = B_3 \times I_2,$$

$$\nu(O) = (0, 0, \ldots, 0, 1) =: \mathbf{n},$$

where, for each $r > 0$,

$$B_r = B_r(O) \subset \mathbb{R}^n, \qquad I_r = (-r, r) \subset \mathbb{R}.$$

Moreover, we may assume without loss of generality that

$$\sup_U |\nu - \mathbf{n}| < \delta, \tag{1.2}$$

where $\delta > 0$ will be chosen shortly. It will then be shown that there is a constant $C = C(n, \delta)$ such that

$$|\nu(\bar{x}) - \nu(\bar{y})| < C\,|\bar{x} - \bar{y}| \tag{1.3}$$

whenever $\bar{x}, \bar{y} \in W := B_2 \times I_1$.

For this purpose we now define $V := B_2 \times I_1$, and note that whenever $u : \mathbb{R}^n \to \mathbb{R}$ is any smooth function, the unit normal ν to the graph of u in $\mathbb{R}^{n+1}$ satisfies the elementary estimates

$$\frac{|\nu - \mathbf{n}|^2}{2 - |\nu - \mathbf{n}|^2} < |Du(x)|^2 < \frac{|\nu - \mathbf{n}|^2}{1 - |\nu - \mathbf{n}|^2}, \tag{1.4}$$

for each $x \in \mathbb{R}^n$, ν being evaluated at $(x, u(x)) \in \mathbb{R}^{n+1}$, provided $|\nu - \mathbf{n}|^2 < 1$. Hence we may choose $\delta > 0$ in (1.2) sufficiently small to ensure that *for each* $t \in I_1$, *the leaf of* $\mathcal{F}$ *containing* $(0, 0, \ldots, 0, t) \in V$ *is the graph of a function*

$$u_t : B_3 \to I_2.$$

Now, the minimality of leaves of $\mathcal{F}$ implies that, for each $t \in I_1$, u_t satisfies the *minimal surface equation*

$$a^{ij}(Du_t) \cdot D_{ij} u_t = 0.$$

Here and henceforth we employ the summation convention, and have, for $1 < i, j < n$,

$$a^{ij}(p) = (1 + |p|^2)\delta_{ij} - p_i p_j, \qquad D_{ij} = D_i D_j = \frac{\partial^2}{\partial x_i \, \partial x_j}.$$

δ_{ij} being the Kronecker symbol, and p any vector in $\mathbb{R}^n$. The point here is that since $|u_t| < 2$ on B_3, we may invoke well-known *a priori* estimates for the minimal surface equation [GT, Corollary 16.7] to obtain, for any given multi-index β, a constant $C = C(n, \beta)$ such that

$$\sup \{|D^\beta u_t(x)| : x \in B_3\} < C. \tag{1.5}$$

In particular, our bound on $|u_t|$, *hence the constant in* (1.5), *is independent of* $t \in I_1$.

Suppose then, that $s, t \in I_1$, $s > t$, and denote by v_t^s the *positive* difference

$$v_t^s = u_s - u_t > 0.$$

It is a standard observation that v_t^s satisfies an elliptic differential equation, but we wish to emphasize here that v_t^s satisfies such an equation having *smooth* coefficients. Precisely, we have

$$A^{ij}(x) \cdot D_{ij} v_t^s + B^i(x) \cdot D_i v_t^s = 0,$$

where

$$A^{ij}(x) = a^{ij}[Du_s(x)],$$

$$B^i(x) = D_{jk} u_t(x) \cdot \int_0^1 D_{p_i} a^{jk}[\tau D u_s(x) + (1 - \tau) D u_t(x)] \, d\tau,$$

(see [BJS, §II.7.1]). From these formulas, it is easy to verify, using (1.5), that for any $\alpha \in (0, 1)$, there are constants $\lambda > 0$, $C < \infty$, such that

$$\lambda |p|^2 < A^{ij} p_i p_j < |p|^2 \quad \text{for all} \quad p \in \mathbb{R}^n \quad \text{(ellipticity)}$$

$$|A^{ij}|_{0,\alpha;B_2}, \; |B^i|_{0,\alpha;B_2} < C \quad \text{(uniform Holder continuity)},$$

The Holder norms above being denoted as in [GT, p. 53]. Moreover, these bounds are independent of $s, t \in I_2$, despite the dependence of A^{ij}, B^i on s, t, because (1.5) holds uniformly. Our positive solutions v_t^s are therefore governed by the classical Schauder estimates, and satisfy a Harnack inequality. We proceed to exploit these facts.

First of all, the Schauder interior estimates [GT, 6.2] yield a constant C independent of $s, t \in I_2$, such that

$$\sup_{B_1} |Dv_t^s| < C \cdot \sup_{B_2} |v_t^s|. \tag{1.6}$$

At the same time, using the Harnack inequality (e.g. [GT, 9.25]), we obtain, for any $x \in B_2$,

$$\sup_{B_2} |v_t^s| < C \cdot \inf_{B_2} |v_t^s| < Cv_t^s(x). \tag{1.7}$$

Combining (1.6) and (1.7) with the definition of v_t^s, we therefore see that for any $x \in B_1$, $s, t \in I_1$,

$$|Du_s(x) - Du_t(x)| < C(n, \delta)\, |u_s(x) - u_t(x)|,$$

so that, by (1.4), we have

$$|v(x, u_s(x)) - v(x, u_t(x))| \leq C\, |u_s(x) - u_t(x)|. \tag{1.8}$$

That is, v satisfies a Lipschitz condition between pairs of points in V which are "vertically" aligned. We extend this relationship to arbitrary pairs of points in V to obtain (1.3), as follows.

Let $\bar{x}, \bar{y} \in V$. Then there exist $x, y \in B_1$, $s, t \in I_1$, such that

$$\bar{x} = (x, u_t(x)), \qquad \bar{y} = (y, u_s(y)).$$

We then define

$$\bar{\bar{y}} = (y, u_t(y)).$$

Consequently,

$$|v(\bar{x}) - v(\bar{y})| \leq |v(\bar{x}) - v(\bar{\bar{y}})| + |v(\bar{\bar{y}}) - v(\bar{y})|,$$

and from (1.8), we have $C = C(n, \delta)$ such that

$$|v(\bar{\bar{y}}) - v(\bar{y})| \leq C\, |\bar{\bar{y}} - \bar{y}|.$$

Furthermore, from (1.4), (1.5), and the mean value theorem we have $C = C(n, \delta)$ such that

$$
\begin{aligned}
|v(\bar{x}) - v(\bar{y})| &\leq C\,|Du_t(x) - Du_t(y)| \\
&\leq C \sup_{B_1} |D^2 u_t|\,|x - y| \\
&\leq C\,|x - y|.
\end{aligned}
$$

Thus

$$|v(\bar{x}) - v(\bar{y})| < C(|\bar{x} - \bar{\bar{y}}| + |\bar{\bar{y}} - \bar{y}|),$$

implying

$$|v(\bar{x}) - v(\bar{y})|^2 < 2C(|\bar{x} - \bar{\bar{y}}|^2 + |\bar{\bar{y}} - \bar{y}|^2). \tag{1.9}$$

Now, if $(\bar{x} - \bar{\bar{y}})$, $(\bar{y} - \bar{\bar{y}})$ were perpendicular (i.e., $(\bar{x} - \bar{\bar{y}})$ horizontal), we would immediately obtain the desired Lipschitz condition (1.3). But recall that $\delta > 0$ bounds $|v - \mathbf{n}|$, hence $|Du_t|$ by (1.4), so that the vertical component of $(\bar{x} - \bar{\bar{y}})$ is controlled. It is therefore elementary to deduce (1.3) from (1.9) despite non-perpendicularity. We have proven that v is locally Lipschitz.

The remaining assertion of Theorem 1.1 requires us to establish that $\mathcal{F}$ has an atlas of distinguished coordinate charts which are bilipschitz homeomorphisms. Again, however, it will suffice to consider the particular neighborhood V studied above, and, in our notation there, to show that the bijection

$$(x, u_t(x)) \leftrightarrow (x, t)$$

is bilipschitz on V. We leave the details of this argument to the reader; it is quite straightforward in view of (1.4), our choice of $\delta > 0$, and the following consequence of the Harnack inequality satisfied by v_t^s in (1.7). Namely,

$$
\begin{aligned}
u_s(x) - u_t(x) &= v_t^s(x) \\
&< C_1 v_t^s(\underset{\sim}{0}) = C_1(s - t) \\
&< C_2 v_t^s(x) = C_2(u_s(x) - u_t(x)).
\end{aligned}
$$

This concludes our proof of Theorem (1.1).

(1.10) *Remark.* Theorem 1.1 remains true with $\mathbb{R}^{n+1}$ replaced by any smooth ambient manifold. The argument of Theorem 1.1 is further complicated by the need to write leaves as graphs over a given base leaf, instead of over $B_2 \subset \mathbb{R}^n$, but no essentially new ideas are involved.

(1.11) COROLLARY. *Let $\mathcal{F}$ be a codimension one, C^0 foliation of a Riemannian manifold M. If $\mathcal{F}$ is oriented, and all leaves are minimal, then any positive linear combination of closed leaves (with orientations induced by $\mathcal{F}$) is homologically area-minimizing in M.*

Proof. Since v is locally Lipschitz (Theorem 1.1), it is differentiable almost everywhere (Rademacher's Theorem [FH, 3.1.6]). A standard calculation using the minimality of leaves then shows that $\operatorname{div}(v) = 0$ almost everywhere. $\mathcal{F}$ therefore corresponds to a *calibration* of M, in the sense of Harvey & Lawson [HL]. The result follows immediately, then, as in [HL] or [SD], by the divergence theorem.

§2. Global structure

In this section we prove two global theorems concerning "entire" foliations of Euclidean space by minimal hypersurfaces. While the first (Theorem 2.3) is valid quite generally, the second (Theorem 2.4) requires that some leaf be *asymptotically regular* (see below). This requirement is always satisfied in $\mathbb{R}^8$, but is a significant restriction in higher dimensions. A few preliminary facts are in order before we state and prove these results.

Let $\mathcal{F}$ denote a C^k foliation of $\mathbb{R}^{n+1}$ by minimal hypersurfaces. Implicit in our earlier definition of hypersurface, is the assumption that every leaf of $\mathcal{F}$ is *properly embedded* (for short, *proper*). This assumption is rather strong, though it can be deduced from minimality when $n = 2$, or from real-analyticity of $\mathcal{F}$ for arbitrary n. We do not know how to remove it in general.

An important aspect of properness is the fact that each leaf of $\mathcal{F}$ separates $\mathbb{R}^{n+1}$. It follows from this that the *leaf space* of $\mathcal{F}$ is locally homeomorphic to $\mathbb{R}$ (e.g., see [H]). The latter space, denoted here by $\Lambda(\mathcal{F})$, is the set of leaves of $\mathcal{F}$, topologized so that the obvious projection map

$$\pi : \mathbb{R}^{n+1} \to \Lambda(\mathcal{F})$$

is continuous. The fact that $\Lambda(\mathcal{F})$ is locally $\mathbb{R}$ makes it a C^k one-manifold, but consideration of simple examples shows that in general, $\Lambda(\mathcal{F})$ is not a Hausdorff space. Indeed, as is nicely elucidated in a 1957 paper of Haefliger and Reeb

[HR], every simply connected non-Hausdorff one-mainfold is the leaf space for a smooth foliation of $\mathbb{R}^2$ by properly embedded curves.

One property that $\Lambda(\mathscr{F})$ does inherit from $\mathbb{R}^{n+1}$, is simple connectivity. (The proof of this, which makes an amusing exercise, again uses the fact that $\mathbb{R}^{n+1}$, hence $\Lambda(\mathscr{F})$, is separated by each leaf.) It is well-known that a simply connected one-manifold which is Hausdorff, is homeomorphic to $\mathbb{R}$. The following Lemma generalizes this fact to the non-Hausdorff case.

(2.1) LEMMA. (Haefliger & Reeb [HR, §1.2, proposition 1). *Let Λ be a simply connected one-manifold, not necessarily Hausdorff. Then there exists a globally defined local homeomorphism $f: \Lambda \to \mathbb{R}$.*

This result is a key tool in proving the next Lemma, from which Theorem 2.3 will quickly follow. Below, leaves $\lambda_1, \lambda_2 \in \mathscr{F}$ will be termed *inseparable* if, as points in $\Lambda(\mathscr{F})$, they do not have disjoint open neighborhoods.

(2.2). LEMMA. *Let $\mathscr{F}$ be a C^k foliation of $\mathbb{R}^{n+1}$ by proper minimal hypersurfaces, $k \geq 0$. Then any finite sum of arbitrarily oriented, pairwise inseparable leaves, is area-minimizing.*

Proof. Suppose $\lambda_1, \ldots, \lambda_m$ are pairwise inseparable. They by letting

$$f: \Lambda(\mathscr{F}) \to \mathbb{R}$$

be as in Lemma 2.1, and replacing f by $f - f(\lambda_1)$ (without renaming), we clearly obtain a local homeomorphism f such that

$$0 = f(\lambda_1) = f(\lambda_2) = \cdots = f(\lambda_m).$$

Next, give each λ_i an orientation, sum the resulting hypersurfaces to form a locally integral current

$$T = \lambda_1 + \lambda_2 + \cdots + \lambda_m,$$

and let $r > 0$. By using the compactness theorem for integral currents [FH, 4.2.17] or [SL, §27], we may obtain, as the limit of a minimizing sequence, a hypersurface S_r, which is minimizing in $B(O, r)$, and satisfies

$$\operatorname{spt}(T - S_r) \subset B(O, r), \qquad \partial(T - S_r) = 0.$$

The same holds for any indecomposable component [FH, 4.2.25] of $T - S_r$, and we now restrict our attention to one such, call it Q. Note that in view of [FH, 4.5.17], we may reorient Q so that $Q = \partial A$ for some bounded measurable set $A \subset B(O, r)$, i.e. Q is a hypersurface in our terminology.

Now, at some point p in the compact, connected support of Q, the continuous function

$$\mathbb{R}^{n+1} \xrightarrow{\pi} \Lambda(\mathscr{F}) \xrightarrow{f} \mathbb{R}$$

must attain a maximum. Let $\lambda_p \in \mathscr{F}$ be the leaf through p. Since f is locally monotonic, it follows from the constancy theorem [FH, 4.1.7] that A (which is connected because Q is indecomposable) lies in the closure U of one connected component of $\mathbb{R}_{n+1} \sim \lambda_p$. Moreover, Q clearly minimizes area in $\mathbb{R}^{n+1} \sim \operatorname{spt}(T)$, so that $\operatorname{spt}(Q) \cap \operatorname{spt}(T) \neq 0$, and we may apply the generalized maximum principle [SL, 37.10] to deduce that in a neighborhood of p, $\operatorname{spt}(Q)$ coincides with λ_p. But the set of points where Q contacts λ_p is then both open and closed in $\operatorname{spt}(Q) \sim \operatorname{spt}(T)$, whence $f \circ \pi$ must vanish identically on $\operatorname{spt}(Q)$, and we conclude that $\operatorname{spt}(Q) \subset \lambda_p \subset \operatorname{spt}(T)$.

Each indecomposable component of $T - S_r$ is therefore seen to be a compactly supported n-cycle in a non-compact, n-dimensional manifold. Such cycles necessarily vanish (as currents; constancy theorem again), hence $T - S_r = 0$. Since $r > 0$ was arbitrary, we have that T is area-minimizing, as desired.

We may now easily state and prove the first main result of this section.

(2.3) THEOREM. *Let $\mathscr{F}$ be a C^k foliation of $\mathbb{R}^{n+1}$ by proper minimal hypersurfaces, $k \geq 0$. Then $\Lambda(\mathscr{F})$ is C^k diffeomorphic to $\mathbb{R}$ (homeomorphic if $k = 0$).*

Proof. We show that $\Lambda(\mathscr{F})$ is a Hausdorff space. Having noted earlier that $\Lambda(\mathscr{F})$ is a simply connected C^k one-manifold, it then follows immediately that $\Lambda(\mathscr{F})$ is homeomorphic to $\mathbb{R}$, and in the C^k sense, when $\mathscr{F}$ is C^k.

Suppose $\Lambda(\mathscr{F})$ were not Hausdorff. Then we could find a pair of distinct but inseparable leaves $\lambda_1, \lambda_2 \in \mathscr{F}$. Each of these leaves divides its complement in $\mathbb{R}^{n+1}$ into exactly two connected open sets. For each $(i, j) = (1, 2), (2, 1)$, let $\mathcal{O}_i$ be the unique component of $\mathbb{R}^{n+1} \sim \lambda_i$ which does not contain λ_j. Thus $\mathcal{O}_1 \cup \mathcal{O}_2$ is a non-empty, *disconnected* open set, call it $\mathcal{O}$, whose topological boundary is $\lambda_1 \cap \lambda_2$. But we may orient λ_1 and λ_2 so that the resulting hypersurface $\lambda_1 + \lambda_2$ forms the *oriented* boundary of $\mathcal{O}$:

$$\partial \mathcal{O} = \lambda_1 + \lambda_2.$$

On the other hand, since λ_1 and λ_2 are inseparable, it follows from Lemma 2.2 that $\partial\mathcal{O}$ is actually area minimizing. This directly contradicts a theorem of Almgren and DeGiorgi (as attributed by Bombieri & Giusti [BG, Theorem 1]). Namely, an open set $\mathcal{O} \subset \mathbb{R}^{n+1}$ having area-minimizing boundary is always connected. Hence no such pair of inseparable leaves can exist, $\Lambda(\mathcal{F})$ is Hausdorff and the proof of Theorem 2.3 is complete.

We now need to define asymptotic regularity, for which purpose we recall the construction, due to W. H. Fleming [FW], of "tangent cones at infinity". (This construction is also central in the result of Almgren/DeGiorgi to which we reduced Theorem 2.3 above.)

Let S be an area-minimizing hypersurface. By well-known arguments, every sequence of radii $\{r_i\} \to 0$ has a subsequence for which the corresponding sequence of homothetic images $\{(r_i)_{\#}S\}$ converges, in the integral flat topology, to a homothetically invariant area-minimizing hypersurface C; i.e. C is a hypercone. (Here (r_i) signifies the homothety $x \mapsto r_i x$ for $x \in \mathbb{R}^{n+1}$.) We will say in this case, that S is *asymptotic* to C. Note that S may be asymptotic, in this sense, to more than one such cone, depending on the defining sequence $\{r_i\}$. This ambiguity will not be a source of difficulty below, however.

If S is asymptotic to a cone C which is *smooth* away from the origin $\mathcal{O} \in \mathbb{R}^{n+1}$, we will say that S is *asymptotically regular*. In this situation $\operatorname{spt}(C) \cap S^n$ is a smooth minimal hypersurface of the unit sphere $S^n \subset \mathbb{R}^{n+1}$, which will be referred to below as the *associated link at infinity*.

(2.4) THEOREM. *Let $\mathcal{F}$ be a C^k foliation of $\mathbb{R}^{n+1}$ by proper minimal hypersurfaces, and suppose some leaf $\lambda \in \mathcal{F}$ is asymptotically regular. Let $\Sigma^{n-1} \subset S^n$ be the associated link at infinity. Then*

(i) *$\mathcal{F}$ is C^k-diffeomorphic to $\lambda \times \mathbb{R}$.*
(ii) *λ is contractible.*
(iii) *There is a diffeomorphism of S^n which exchanges the two components of $S^n \sim \Sigma$ while leaving Σ pointwise fixed.*
(iv) *Σ is a homology $(n-1)$-sphere.*

Proof. Suppose $\varepsilon > 0$ is given. By Theorem 1.1 and the simple-connectivity of $\mathbb{R}^{n+1}$, there is a global, locally Lipschitz unit vector field v, normal to $\mathcal{F}$. Let

$$\tilde{v} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$$

be a C^∞ approximation to v, with

$$\|\tilde{v} - v\|_\infty < \varepsilon$$

(since v is locally Lipschitz, this can be achieved on annuli by mollification, then on $\mathbb{R}^{n+1}$ by partition of unity.) We henceforth assume $0<\varepsilon<1$, so that $\tilde{v}$ is bounded and everywhere transverse to $\mathcal{F}$; in particular $\tilde{v}$ never vanishes. The fundamental existence/uniqueness/smooth-dependence theorem for ordinary differential equations consequently gives a C^∞ embedding

$$G:\lambda \times \mathbb{R} \to \mathbb{R}^{n+1},$$

such that

$$\frac{\partial}{\partial t} G(x, t) = \tilde{v} \circ G(x, t), \qquad G(x, 0) = x.$$

Inverting G on its image $\mathcal{O} := G(\lambda \times \mathbb{R})$, and projecting $\lambda \times \mathbb{R} \to \lambda$, we obtain a C^∞ submersion

$$\psi : \mathcal{O} \to \lambda.$$

By Theorem 2.3, there is also a C^k submersion

$$\pi : \mathcal{O} \to \mathbb{R}$$

whose level sets are the leaves of $\mathcal{F}$. (Here and below, one must argue slightly differently in case $k=0$. We leave this to the reader.) Since the leaves of $\mathcal{F}$ are infinitely smooth, and transverse to $\tilde{v}$, the product map

$$(\psi, \pi) : \mathcal{O} \to \lambda \times \mathbb{R}$$

is then clearly a C^k embedding. Conclusion (i) of the theorem is now easily derived with the aid of two facts: $\mathcal{O} = \mathbb{R}^{n+1}$, and (ψ, π) is surjective. Both these facts are immediate consequences of the following claim.

(2.5) CLAIM. *Each integral curve of the vector field $\tilde{v}$ meets every leaf of $\mathcal{F}$.*

To verify this claim, let $\alpha, \beta \in \mathcal{F}$ be arbitrary, with $\tilde{v}$ pointing *into* the component U_α of $\mathbb{R}^{n+1} \sim \alpha$ which contains β. Fix also an arbitrary point $y \in \alpha$. It will then suffice to show that the integral curve which enters U_α at y eventually crosses β. We will accomplish this in the process of proving conclusion (iii) of the theorem.

Denote by C_Σ, U^+, and U^- respectively, the cone over Σ, and the two (exactly two, by [BG]) components of $\mathbb{R}^{n+1} \sim C_\Sigma$. Let

$$\Omega^+ := S^n \cap U^+, \qquad \Omega^- := S^n \cap U^-$$

be the corresponding components of $S^n \sim \Sigma$. By our hypothesis of asymptotic regularity, there are radii $r_i \to 0$ for which the corresponding sequence $\lambda_i := (r_i)_{\#}\lambda$ converges to C_Σ in the integral flat topology. For each $i = 1, 2, 3, \ldots$, consider the "rescaled" foliation $\mathscr{F}_i := (r_i)\mathscr{F}$, which contains λ_i, and define λ_i^+, $\lambda_i^- \in \mathscr{F}_i$ to be the unique leaves (one on each side of λ_i) having distance $\varepsilon > 0$ from Σ. Since each λ_i^+, λ_i^- is minimizing (Lemma 2.2), it is standard that (for suitable subsequences) $\{\lambda_i^+\}$, $\{\lambda_i^-\}$ converge, as $i \to \infty$ to area maintaining-hypersurfaces having distance $\varepsilon > 0$ from Σ. In addition, however, since λ_i^+ lies on one side of λ_i for each i, $\lim_{i\to\infty} \lambda_i^+$ is supported on one side of $\lim_{i\to\infty} \lambda_i = C_\Sigma$. Similarly for λ_i^-. This situation is a rather special one in light of the following result of Hardt & Simon [HS, Thm. 2.1]:

There is a unique area-minimizing hypersurface T_ε^+ (respectively, T_ε^-) supported in the open set U^+ (respectively, U^-), having distance $\varepsilon > 0$ from Σ. T_ε^+ and T_ε^- are smoothly asymptotic to C_Σ near infinity, and representable (in polar coordinates) as radial graphs over Ω^+, Ω^- respectively. As $\varepsilon \to 0$, T_ε^+, $T_\varepsilon^- \to C_\Sigma$, and uniformly so outside B, $(O, 1)$.

We immediately see from this that

$$\lambda_i^+ \to T_\varepsilon^+, \qquad \lambda_i^- \to T_\varepsilon^-, \quad \text{as} \quad i \to \infty,$$

and that T_ε^+, T_ε^- are diffeomorphic (by graphing) to Ω^+, Ω^-, respectively.

Combining these facts with the basic regularity theory [FH, 5.3.14] for area-minimizing hypersurfaces (which, again, says that weak (i.e. integral flat) convergence of minimizers to a smooth limit is actually smooth convergence), it follows that by making $\varepsilon > 0$ small, and then rescaling (i.e., choosing i large), we may proceed under the following assumptions.

(2.6) $\mathscr{F}$ contains leaves λ^+, λ^- such that *any* leaf $\gamma \in \mathscr{F}$ between λ^+ and λ^- (i.e. $\pi(\lambda^-) \leq \pi(\gamma) \leq \pi(\lambda^+)$) is expressible in $\mathscr{A} = \{x \in \mathbb{R}^{n+1} : \frac{1}{2} < |x| < \frac{3}{2}\}$ as the graph over a domain in C_Σ (relative to the unit normal ν_C on C_Σ) of a function whose gradient is small; that is $O(\varepsilon)$.

(2.7) There are diffeomorphisms

$$\phi_\pm : \bar{\Omega}^\pm \to D^\pm := \lambda^\pm \cap B, (O, 1)$$

such that for any $\omega \in \Sigma$, $\rho \circ \phi_{\pm}(\omega) = \omega$. Here ρ is defined, on a tubular neighborhood of Σ containing ∂D^+, ∂D^-, to be the "nearest point retraction" onto Σ.

(2.8) Letting $\alpha, \beta \in \mathcal{F}$ and $y \in \alpha$ be as mentioned earlier in reference to claim 2.5, we have

$$\pi(\lambda^-) < \pi(\alpha) < \pi(\beta) < \pi(\lambda^+), \quad \text{and} \quad |y| < 1.$$

From (2.6) and (2.8), we can now deduce claim 2.5. For, (2.6) shows that in $\mathcal{A}$, any leaf of $\mathcal{F}$ between λ^- and λ^+ is "nearly parallel" to $C_\Sigma \cap \mathcal{A}$. By pre-assigning $\varepsilon > 0$ small enough, therefore, we can clearly arrange that any integral curve of the approximately normal vector field $\tilde{v}$, which meets S^n between λ^- and λ^+, passes through both these leaves, hence *a posteriori* through both α and β. Since $\tilde{v}$ points into U_α on α, and never vanishes, the integral curve of $\tilde{v}$ which enters U_α at $y \in \alpha$ must eventually meet $S^n \cap U_\alpha$. If it does so between λ^- and λ^+, then we have just seen that it must eventually reach β. If not, it clearly meets S^n after having already passed through β. This establishes claim 2.5, hence proves conclusion (i) of our theorem.

Conclusion (ii) of the theorem follows directly from conclusion (i), because for each $i = 1, 2, 3, \ldots$, the homotopy functor π_i commutes with cartesian product, and $\pi_i(\mathbb{R}^{n+1}) = \pi_i(\mathbb{R}) = 0$. It is well-known that contractibility of a manifold is equivalent to triviality of all its homotopy groups [GM, III.B].

To get conclusion (iii), observe that by the argument for conclusion (i), $\mathcal{F}$ itself gives an isotopy from D^+ to D^- through diffeomorphisms. But then by (2.6) and (2.7), the corresponding map

$$\bar{\Omega}^+ \xrightarrow{\phi_+} D^+ \longrightarrow D^- \xrightarrow{\phi_-^{-1}} \bar{\Omega}^-$$

is a diffeomorphism, whose restriction to $\partial\bar{\Omega}^+ = \Sigma = \partial\bar{\Omega}^-$ is isotopic to the identity on Σ. This diffeomorphism can then be smoothly modified in a collar neighborhood of $\partial\bar{\Omega}^+$, so as to leave Σ pointwise fixed. Conclusion (iii) is now evident.

Finally, we deduce conclusion (iv) from (iii) by noting that for each $0 < k < n - 1$, Mayer-Vietoris gives an isomorphism

$$O \longrightarrow H_k(\Sigma) \xrightarrow{(i_*, -j_*)} H_k(\bar{\Omega}^+) \oplus H_k(\bar{\Omega}^-) \longrightarrow O$$

where i and j denote the obvious inclusions. But since i and j are essentially the same map, by conclusion (iii), $(i_*, -j_*)$ cannot be an isomorphism unless $H_k(\Sigma)=0$. Recalling the well-known fact that compact embedded minimal hypersurfaces in S^n are always connected, we see that $H_k(\Sigma)=H_k(S^{n-1})$ for all $k \geq 0$. This completes the proof.

We conclude with a few remarks.

Theorem 2.4 suggests an interesting extension of the spherical Bernstein problem [Y]:

Which $(n-1)$-dimensional homology spheres can be minimally embedded in the unit sphere S^n? More restrictively, which can bound area-minimizing hypercones?

Several authors have found non-equatorial minimal hyperspheres in S^n ([FK], [HW], [TP]). Though none of these examples are known to bound minimizing cones, experience indicates that in sufficiently high dimensions, some of them will (cf. [HsS]).

Regarding more general homology spheres, M. Kervaire [KM] has shown that when $n>5$, there are infinitely many homology $(n-1)$-spheres which bound contractible smooth manifolds. But if $n>5$ and Ω^n is contractible, its double $D(\Omega^n)$ is simply-connected and bounds $\Omega \times [-1,1]$, which is then diffeomorphically an $(n+1)$-ball by the h-cobordism theorem [MJ, §9 Prop. A]. Hence $D(\Omega)$ is a smooth n-sphere. In particular, the situation described in Theorem (2.4), where a homology sphere Σ decomposes S^n as the double of a contractible manifold, is *topologically* very common. This again suggests that in sufficiently high dimensions, there will be minimizing cones on homology spheres.

In such a case, the existence of non-hyperplanar foliations of $\mathbb{R}^{n+1}$ by asymptotically regular minimal hypersurfaces is, for large n, made rather plausible. Neither could such a foliation arise by translating an entire minimal graph because such graphs are *never* asymptotically regular [D]. Thus, although the italicized question posed in our introduction may have an affirmative answer in $\mathbb{R}^8$, the obvious generalizations mentioned there for higher dimensions are probably false.

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Department of Mathematics
Indiana University
Bloomington, IN 47405

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