Changes of sign of error terms related to Euler's function and to divisor functions.

Comment. Math. Helvetici 61 (1986) 84–101 0010-2571/86/010084-18$01.50 + 0.20/0

Changes of sign of error terms related to Euler's function and to divisor functions

Y.-F. S. Pétermann

1. Introduction

Let

$$R(x) := \Phi(x) - \frac{3}{\pi^2}x^2 \quad (x \geq 1), \tag{1}$$

where $\Phi(x) := \sum_{n \leq x} \phi(n)$ and $\phi(n)$ is Euler's function. If one computes values of $R(n)$ and of

$$R(n-) := \sum_{m<n} \phi(m) - \frac{3}{\pi^2}n^2 = R(n) - \phi(n),$$

one comes to suspect that $R(x)$ changes sign very frequently between consecutive integers, but that there are very few integers n for which $R(n) < 0$.

Sylvester even conjectured in 1883 ([32] and [33]; the reference to [31] in [7] and [20] is mistaken) that $R(n) > 0$ for all positive integers n. But [33] contains a table of $\phi(n)$, $\Phi(n)$ and $3n^2/\pi^2$ for $1 \leq n \leq 1000$; Sylvester does not seem to have noticed that the entries $\Phi(820) = 204376$ and $3.820^2/\pi^2 = 204385.09\ldots$ disprove his conjecture. Sarma [23] (attributing the conjecture to Pillai and Chowla) rediscovered this counterexample in 1931.

Let $X_R(x)$ denote the number of changes of sign of $R(t)$ in the interval $1 < t < x$, and $N_R(x)$ the number of changes of sign of $R(n)$ on the integers n with $1 < n < x$ (i.e. the number of integers n, $1 < n < x$, such that $R(n)R(n-1) < 0$).

In 1967 Erdös conjectured [5] that

$$N_R(x) = Cx + o(x) \quad (x \to \infty) \tag{2a}$$

for some positive constant C; in 1985 he proposed [6] the weaker

$$N_R(x) = \Omega(x) \quad (x \to \infty). \tag{2b}$$

In 1951 Erdös and Shapiro [7] proved that

$$R(x) = \Omega_{\pm}(x \log\log\log\log x), \tag{3}$$

and hence that

$$N_R(x) \to \infty \quad (x \to \infty).$$

The only other result in the literature is due to Proschan (1971, [20]):

$$N_R(x) \geq IL(x) + O(1) \quad (x \to \infty), \tag{4}$$

where $IL(x)$ is the smallest integer k such that $\log_{4k}(x)$, the $4k$-fold iterated logarithm of x in a sufficiently large basis, is either smaller than 2 or undefined.

We show in Section 3 of this paper that

$$X_R(x) = Cx + o(x) \quad (x \to \infty), \tag{5}$$

where

$$C \geq \frac{8}{3}\left(1 - \frac{\pi^2}{24}\right) = 1.57004\ldots, \tag{6}$$

and in another article [18] that

$$N_R(x) \geq \left(\frac{2}{\log 2} - \varepsilon\right) \log\log x + O_\varepsilon(1), \quad \text{for any} \quad \varepsilon > 0. \tag{7}$$

For the divisor functions $\sigma_k(n) := \sum_{d|n} d^k$, we consider the error term

$$F_k(x) := S_k(x) - T_k(x), \tag{8}$$

where

$$S_k(x) := \sum_{n \leq x} \sigma_k(n) \tag{9}$$

and

$$T_k(x) := \begin{cases} \frac{\pi^2}{6}x - \frac{\log x}{2} - \frac{(\gamma + \log 2\pi)}{2} & (k = -1) \\ \frac{\zeta(1+k)}{1+k}x^{1+k} + \zeta(1-k)x - \frac{\zeta(-k)}{2} & (k \neq 0, -1) \\ x \log x + (2\gamma - 1)x + \frac{1}{4} & (k = 0) \end{cases} \tag{10}$$

(γ is Euler's constant and ζ is Riemann's zeta function). Let $X_{F_k}(x)$ denote the number of changes of sign of $F_k(t)$ in $1 < t < x$, and $N_{F_k}(x)$ the number of changes of sign of $F_k(n)$ on the integers n, $1 < n < x$. It follows from a result of Steinig's [28] that

$$X_{F_k}(x) \geq 4\sqrt{x} + O_k(1), \quad \text{for all} \quad k \in \mathbb{R}; \tag{11}$$

there is no result in the literature concerning $N_{F_k}(x)$.

We show in Section 4 that

$$X_{F_k}(x) \geq \frac{8}{3}\left(1 - \frac{\zeta(2|k|)}{4\zeta(2 + 2|k|)}\right)x + o_k(x), \quad \text{for all} \quad k \in \mathbb{R}, \tag{12}$$

and in [18] that

$$N_{F_{\pm 1}}(x) \geq \left(\frac{2}{\log 2} - \varepsilon\right) \log\log x + O_\varepsilon(1), \quad \text{for all} \quad \varepsilon > 0. \tag{13}$$

Estimate (12) improves (11) when it is non-trivial, that is for

$$|k| > k_0 = 0.6236622\ldots \tag{14}$$

In Section 5 we consider error terms associated with the lattice points in certain four-dimensional ellipsoids, which are closely related to the error terms F_{-1} and F_1. The author wishes to thank Prof. J. Steinig for the time he spent to read the manuscript of this article and for his many useful suggestions.

2. Two general theorems

Let us first define what we mean by the number of changes of sign of a real-valued function f in a non-empty interval I.

DEFINITION.

1) We say that f *is of constant sign* in I if either $f \geq 0$ or $f \leq 0$ throughout I.
2) We say that f *has N changes of sign* in I if I can be partitioned into $N+1$ subintervals I_i, $i = 0, 1, \ldots, N$ (I_i and I_{i+1} being consecutive), with the following properties:
   i) f is not identically zero in any I_i;
   ii) f is of constant sign in each I_i;
   iii) f is of opposite signs in I_i and I_{i+1}.
3) We say that f *has a finite number of sign changes* in I if there is an $N \geq 0$ such that f has N changes of sign in I.

Throughout this article, we consider functions $f : [1, \infty) \to \mathbb{R}$ which have a finite number of sign changes in $(1, x)$ for all $x > 1$, and we denote this number by $X_f(x)$.

We also set $I_n = (n, n+1)$ and $\bar{I}_n = [n, n+1)$ for each integer $n \geq 1$, and $\{x\} := x - [x]$ if x is real. If E is a finite set, $|E|$ denotes its cardinality.

THEOREM 1. *Let $f : [1, \infty) \to \mathbb{R}$ be such that for each $n \geq 1$,*

$$f(x) = f(n) - C\{x\} + \theta(x) \quad \text{if} \quad x \in \bar{I}_n, \tag{15}$$

where C is a constant, $C \neq 0$, and

$$\theta(x) = o(1) \quad (x \to \infty). \tag{16}$$

Suppose further that there is a constant $K > 0$ such that

$$\int_1^x f^2(u)\, du \leq Kx + o(x) \quad (x \to \infty). \tag{17}$$

Then, as $x \to \infty$,

$$X_f(x) \geq \frac{8}{3}\left(1 - \frac{3K}{C^2}\right)x + o(x). \tag{18}$$

If in addition the distribution function for f,

$$D_f(u) := \lim_{x\to\infty} \frac{|\{n \le x, f(n) \ge u\}|}{x} \tag{19}$$

exists and is continuous, and if f itself is monotonic on each interval $\bar{I}_n$ (decreasing if $C>0$, increasing if $C<0$), then as $x\to\infty$

$$X_f(x) = 2\,|D_f(0) - D_f(C)|\,x + o(x). \tag{20}$$

THEOREM 2. *Let $f:[1,\infty)\to\mathbb{R}$ satisfy conditions* (15) *through* (17) *of Theorem* 1. *Let $h:[1,\infty)\to\mathbb{R}$ be positive, and $g:[1,\infty)\to\mathbb{R}$ be such that as $x\to\infty$,*

$$g(x) = h(x)(f(x) + o(1)). \tag{21}$$

Then as $x\to\infty$,

$$X_g(x) \ge \frac{8}{3}\left(1 - \frac{3K}{C^2}\right)x + o(x). \tag{22}$$

If in addition f satisfies condition (19), *and if the function g/h is monotonic on each $\bar{I}_n$ (decreasing if $C>0$, increasing if $C<0$), then as $x\to\infty$,*

$$X_g(x) = 2\,|D_f(0) - D_f(C)|\,x + o(x). \tag{23}$$

Proof of Theorem 1. We may suppose $C>0$ (if $C<0$, consider $-f$ instead of f). We may also restrict ourselves to the case where x is an integer. For $r>0$, set

$$A_r(x) = \{n \le x, |f(n) - C/2| < r\},$$
$$B_r(x) = \{n \le x, |f(n) - C/2| \ge r\}.$$

From (16), (17) and Cauchy's inequality,

$$\int_1^x \theta(u) f(u)\,du = o(x); \tag{24}$$

then from (15) and (16),

$$\int_1^x f^2(u)\,du = \int_1^x ((f(u)-\theta(u))^2 + 2\theta(u)f(u) - \theta^2(u))\,du$$
$$= \sum_{n=1}^{x-1} \int_0^1 (f(n) - Ct^2)\,dt + \int_1^x (2\theta(u)f(u) - \theta^2(u))\,du$$
$$= \sum_{n=1}^{x-1} ((f(n) - C/2)^2 + C^2/12) + o(x),$$

whence

$$\int_1^x f^2(u)\,du \geq r^2\,|B_r(x)| + \frac{C^2}{12}x + o(x),$$

that is

$$\int_1^x f^2(u)\,du \geq (r^2 + C^2/12)x - r^2\,|A_r(x)| + o(x). \tag{25}$$

From (17) and (25) we have

$$|A_r(x)| \geq \left(1 - \frac{K}{r^2} + \frac{C^2}{12r^2}\right)x + o(x). \tag{26}$$

Now take $r = C/2 - \varepsilon$, with $0 < \varepsilon < C/2$. Condition (15) implies that f decreases by $C + o(1)$ on $\bar{I}_n$. Hence by definition of $A_r(x)$ there is an $N = N(\varepsilon)$ such that f changes sign from + to − on I_n whenever $n \geq N$ and $n \in A_r(x)$. This means that the number of sign changes of f from + to − on $(1, x)$, say $X_f^+(x)$, is at least

$$\left(1 - \frac{K}{(C/2-\varepsilon)^2} + \frac{C^2}{12(C/2-\varepsilon)^2}\right)x - N + p(x),$$

where $p(x) = o(x)$ as $x \to \infty$. Hence if x is large enough to ensure that $x \geq N/\varepsilon$ and $|p(x)| < \varepsilon x$, then

$$X_f^+(x) \geq (\tfrac{4}{3} - 4K/C^2)x - \delta(\varepsilon)x - 2\varepsilon x, \tag{27}$$

where $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0+$. Since ε can be arbitrarily small, and as between two changes of sign from + to − there must be one from − to +, we have proved (18).

Suppose now that we also have (19) and that f is monotonically decreasing on each $\bar{I}_n$. For $r > 0$, let

$$D_r := D_f(C/2 - r) - D_f(C/2 + r); \tag{28}$$

then we have

$$|A_r(x)| = D_r x + o(x). \tag{29}$$

With the same argument we used to deduce (18) from (26) we obtain from (29) and the continuity of D_f

$$X_f(x) \geq 2D_{C/2}x + o(x). \tag{30}$$

We will now show that

$$X_f(x) \leq 2D_{C/2}x + o(x); \tag{31}$$

(20) then follows from (28), (30), and (31).

Proof of (31). As we pointed out above, $X_f(x) \geq 2X_f^+(x) - 1$. Since f decreases on each $\bar{I}_n$, f changes sign at most once there (necessarily from + to −). And since $f(n) - f(n+1^-) = C + o(1)$, there is for each $\varepsilon > 0$ an $N = N(\varepsilon)$ such that if f changes sign on I_n and $n \geq N$, then $f(n) \in (0, C + \varepsilon)$. So we have

$$\begin{aligned} X_f^+(x) &\leq (D_f(0) - D_f(C+\varepsilon))x + N + o(x) \\ &\leq (D_f(0) - D_f(C+\varepsilon))x + \varepsilon x, \end{aligned} \tag{32}$$

for x sufficiently large; (31) now follows from (32) and the continuity of D_f. ∎

The proof of Theorem 2 is straightforward, since Theorem 1 can be applied to the function $f^* := g/h$. Indeed, if D_f exists and is continuous, then D_{f^*} also exists and $D_f = D_{f^*}$. ∎

3. Error terms associated with Euler's function

We first define the summatory functions Φ and Φ' and the corresponding error terms R and H: for $x \geq 1$,

$$\Phi'(x) := \sum_{n \leq x} \frac{\phi(n)}{n} =: \frac{6}{\pi^2}x + H(x) \tag{33}$$

and

$$\Phi(x) := \sum_{n \le x} \phi(n) =: \frac{3}{\pi^2} x^2 + R(x). \tag{34}$$

We consider the changes of sign of H and R, and prove

THEOREM 3. *As* $x \to \infty$,

$$X_H(x) \ge \frac{8}{3}\left(1 - \frac{\pi^2}{24}\right)x + o(x) = (1.57004\ldots)x + o(x), \tag{35}$$

$$X_R(x) \ge \frac{8}{3}\left(1 - \frac{\pi^2}{24}\right)x + o(x), \tag{36}$$

$$X_H(x) = 2(D_H(0) - D_H(6/\pi^2))x + o(x), \tag{37}$$

$$X_R(x) = 2(D_H(0) - D_H(6/\pi^2))x + o(x). \tag{38}$$

Proof. The hypotheses of Theorem 1 are satisfied by $f(x) = H(x)$, with $C = 6/\pi^2$ and $K = 1/2\pi^2$. Indeed

$$\int_1^x H^2(u)\,du \sim \frac{1}{2\pi^2} x \quad (x \to \infty) \tag{39}$$

is a theorem of Chowla's [3, (48)] (see Remark 2 in Section 6). And (33) shows that

$$H(x) = H(n) - \frac{6}{\pi^2}\{x\} \quad \text{for} \quad x \in \bar{I}_n. \tag{40}$$

This proves (35).

Estimate (37) follows from Theorem 1 by using the existence and continuity of D_H, proved by Erdös and Shapiro [8].

For (36) and (38) we use the estimate

$$R(x) = xH(x) + o(x) \tag{41}$$

due to Pillai and Chowla [19, p. 99] (see Remark 1). As it is easy to see that $R(x)/x$ is decreasing on each $\bar{I}_n$, the hypotheses of Theorem 2 are satisfied if we take $f(x)$, C and K as above, $g(x) = R(x)$ and $h(x) = x$. ∎

Theorems 1 and 2 can also be applied to a class of error terms including H and R, first studied by Proschan [20], and for which Sivaramasarma [27, (7.1.45)] determined the constant K of (17). This is done in [17, II.2].

4. Error terms associated with divisor functions

Let F_k be defined as in (8). We shall prove the following theorem about its changes of sign.

THEOREM 4. *Let k_0 be the solution of the equation*

$$\zeta(2k_0) = 4\zeta(2+2k_0) \tag{42}$$

in the interval $(1/2, \infty)$. *Then if $k_0 < |k|$, we have, as $x \to \infty$*

$$X_{F_k}(x) \geq \frac{8}{3}\left(1 - \frac{\zeta(2|k|)}{4\zeta(2+2|k|)}\right)x + o_k(x) \tag{43}$$

(Using a variant of Simpson's approximation method, B. Gisin computed $k_0 = 0.6236622010\ldots$).

In order to deduce Theorem 4 from Theorems 1 and 2 we need three lemmata

LEMMA 1. *As $x \to \infty$,*

$$F(x) := xF_{-1}(x) - F_1(x) = o(x), \tag{44}$$
$$F_k(x) = O(x^{(1+k)/2}) \quad \textit{for} \quad -1 < k < -\tfrac{1}{2}, \tag{45}$$
$$F_k(x) = x^k F_{-k}(x) + o(x^k) \quad \textit{for} \quad \tfrac{1}{2} < k \leq 1. \tag{46}$$

Proof. Estimate (44) is classical (see Remark 4 in Section 6). For (45) see [3, (112)]. An estimate implying (46) can be found in [13, (6)] (see Remark 5). ∎

LEMMA 2. *With F as in* (44), *we have*

$$F(x) = \int_1^x F_{-1}(t)\,dt + O(1) \quad (x \to \infty). \tag{47}$$

Proof. On the one hand,

$$\sum_{n \leq x} \sigma_{-1}(n)(x-n) = \int_1^x S_{-1}(t)\,dt$$

and on the other,

$$\sum_{n \leq x} \sigma_{-1}(n)(x-n) = xS_{-1}(x) - S_1(x).$$

By using (8), (10) and $\zeta(0) = -\frac{1}{2}$, we get (47). ∎

LEMMA 3. *As* $x \to \infty$,

$$\int_1^x F_{-1}^2(t)\,dt \sim \frac{5\pi^2}{144}x \tag{48}$$

and for $-1 < k < -\frac{1}{2}$,

$$\int_1^x F_k^2(t)\,dt \sim \frac{\zeta(-2k)\zeta^2(1-k)}{12\zeta(2-2k)}x. \tag{49}$$

Proof. (48) is due to Walfisz [36, (I)] and (49) to Chowla [3, (7)]. They considered an error term slightly different from F_k (see Remark 3) and proved, respectively, that for $k = -1$,

$$\int_1^x \left(F_{-1}(t) - \frac{(\gamma + \log 2\pi)}{2}\right)^2 dt = \left(\frac{(\gamma + \log 2\pi)^2}{4} + \frac{5\pi^2}{144}\right)x + O(x^{1/2}) \tag{50}$$

and that for $-1 < k < -\frac{1}{2}$,

$$\int_1^x \left(F_k(t) - \frac{\zeta(-k)}{2}\right)^2 dt = \left(\frac{\zeta^2(-k)}{4} + \frac{\zeta(-2k)\zeta^2(1-k)}{12\zeta(2-2k)}\right)x + O(x^{k+3/2}\log x). \tag{51}$$

(48) follows from (50) with (47) and (44), and (49) from (51) with

$$\int_1^x F_k(t)\,dt = O(x^{1+k/2}) \quad \text{if} \quad -1 < k \leq -\tfrac{1}{2}, \tag{52}$$

which we proceed to prove. For $-1 < k \leq -\frac{1}{2}$, we have

$$F_k(x) = -\sum_{n \leq \sqrt{x}} n^k \Psi(x/n) - x^k \sum_{n \leq \sqrt{x}} n^{-k}\Psi(x/n) + O(x^{k/2}), \tag{53}$$

where $\Psi(y) := \{y\} - \frac{1}{2}$ [3, (65)], whence

$$\int_1^x F_k(t)\,dt = -\sum_{n\le\sqrt{x}} n^k \int_{n^2}^x \Psi(t/n)\,dt - \sum_{n\le\sqrt{x}} n^{-k}\int_{n^2}^x t^k\Psi(t/n)\,dt + O(x^{1+k/2})$$

$$= -\sum_{n\le\sqrt{x}} n^{k+1}\int_n^{x/n}\Psi(u)\,du - \sum_{n\le\sqrt{x}} n\int_n^{x/n} u^k\Psi(u)\,du + O(x^{1+k/2}) = O(x^{1+k/2}). \quad \blacksquare$$

After this preparation, we pass to the proof of Theorem 4. We shall restrict ourselves to the case $|k|\le 1$ (for the case $|k|>1$, see Remark 6). We consider four subcases.

a) $\underline{k=-1}$: if $n\le x<n+1$,

$$F_{-1}(x) = F_{-1}(n) - \frac{\pi^2}{6}\{x\} + O(1/x), \tag{54}$$

whence with (48), conditions (15) through (17) of Theorem 1 are satisfied by $f(x)=F_{-1}(x)$, with $C=\pi^2/6$ and $K=5\pi^2/144$.

b) $\underline{k=+1}$: with (44), we see that $g(x)=F_1(x)$ and $h(x)=x$ satisfy condition (21) of Theorem 2, if $f(x)$ is as in Case (a).

c) $\underline{k\in(-1,-k_0)}$: we have by (10), if $x\in\bar{I}_n$

$$F_k(x) = F_k(n) - \zeta(1-k)\{x\} + O(x^k), \tag{55}$$

whence with (49), conditions (15) through (17) of Theorem 1 are satisfied by $f=F_k$, with $C=\zeta(1-k)$ and $K=\zeta(-2k)\zeta^2(1-k)/12\zeta(2-2k)$.

d) $\underline{k\in(k_0,1)}$: with (46) we see that if $f=F_{-k}$, and C, K are as in Case (c), condition (21) of Theorem 2 is satisfied by $g(x)=F_k(x)$ and $h(x)=x^k$. $\blacksquare$

5. Error terms associated with the lattice points in certain four-dimensional ellipsoids

Arnold Walfisz considered in [36] and [37] the quadratic forms

$$\begin{cases} Q_0 = n_1^2+n_2^2+n_3^2+n_4^2, \\ Q_1 = n_1^2+n_2^2+2n_3^2+2n_4^2, \\ Q_2 = n_1^2+2n_2^2+2n_3^2+4n_4^2, \\ Q_3 = n_1^2+2n_2^2+4n_3^2+8n_4^2, \end{cases} \tag{56}$$

the associated four-dimensional ellipsoids

$$0 \leq Q_k \leq x \quad (k = 0, 1, 2, 3) \tag{57}$$

of respective volumes

$$W_k(x) = \frac{\pi^2}{2^{k+1}} x^2 \quad (k = 0, 1, 2, 3), \tag{58}$$

and the corresponding error terms

$$P_k(x) = \sum_{Q_k \leq x} 1 - W_k(x) \quad (k = 0, 1, 2, 3). \tag{59}$$

He showed that

$$\begin{cases} P_0(x) = 8E_1(x) - 32E_1(x/4), \\ P_1(x) = 4E_1(x) - 4E_1(x/2) + 8E_1(x/4) - 32E_1(x/8), \\ P_2(x) = 2E_1(x) - 2E_1(x/2) + 8E_1(x/8) - 32E_1(x/16), \\ P_3(x) = E_1(x) - E_1(x/2) + 8E_1(x/16) - 32E_1(x/32) + O(x^{5/6}), \end{cases} \tag{60}$$

(where $E_1(x) = F_1(x) - x/2 - \zeta(-1)/2$; see Remark 3) and that

$$\int_1^x P_k^2(t)\, dt = \frac{\pi^2}{3 \cdot 2^{2k+1}} x^3 + \begin{cases} O(x^{5/2})(k = 0, 1, 2; [36]) \\ O(x^{5/2} \log^2 x)(k = 3; [37]). \end{cases} \tag{61}$$

Using

$$F(x) = O(x^{5/6}), \tag{62}$$

where F is as in (44) (see Remark 4), we can rewrite (60) as

$$P_k(x) = xR_k(x) + O(x^{5/6}) \quad (k = 0, 1, 2, 3), \tag{63}$$

where

$$\begin{cases} R_0(x) = 8F_{-1}(x) - 8F_{-1}(x/4), \\ R_1(x) = 4F_{-1}(x) - 2F_{-1}(x/2) + 2F_{-1}(x/4) - 4F_{-1}(x/8), \\ R_2(x) = 2F_{-1}(x) - F_{-1}(x/2) + F_{-1}(x/8) - 2F_{-1}(x/16), \\ R_3(x) = F_{-1}(x) - \frac{1}{2}F_{-1}(x/2) + \frac{1}{2}F_{-1}(x/16) - F_{-1}(x/32). \end{cases} \tag{64}$$

Integrating by parts in (61) and using (63) we obtain

$$\int_1^x R_k^2(t)\,dt = \frac{\pi^2}{2^{2k-1}}x + O(x^{5/6}\log x) \quad (k = 0, 1, 2, 3). \tag{65}$$

It is not difficult, using (64) and (54), to show that for $x \in \bar{I}_n$ we have

$$R_k(x) = R_k(n) - \frac{\pi^2}{2^k}\{x\} + O(1/x) \quad (k = 0, 1, 2, 3). \tag{66}$$

We see with (63) through (66) that Theorems 1 and 2 can be applied; we obtain

THEOREM 5. *For* P_k *as in* (59) ($k = 0, 1, 2, 3$) *we have*

$$X_{P_k}(x) \geq \frac{8}{3}\left(1 - \frac{6}{\pi^2}\right)x + o(x) = (1.045527\ldots)x + o(x). \tag{67}$$

For $k = 0$, *this improves*

$$X_{P_0}(x) \geq 2\sqrt{x} + O(1), \tag{68}$$

which is implied by a general result of Steinig's [28, (4.5)].

6. Remarks

Remark 1. If f is strictly monotonic on each I_n, we have the trivial upper bound

$$X_f(x) \leq 2x + 1. \tag{69}$$

This, with the example below, shows that (18) can be sharp: if

$$\Psi(x) := \{x\} - \tfrac{1}{2},$$

(15) holds with $C = -1$, and we have

$$\int_1^x \Psi^2(t)\,dt \sim x/12 \quad (x \to \infty);$$

thus by Theorem 1

$$X_\psi(x) \geq 2x + o(x).$$

(69) should also be compared with (83).

Remark 2. Chowla's estimate of the error term in (39) was $O(x/\log^4 x)$. For better estimates, and also for estimates of $R(x) - xH(x)$, see [26], [30], [27]; [30] also gives estimates subject to the truth of the Riemann hypothesis.

One can obtain a simpler proof of (39) than in [3] by adapting the arguments of Lemmata 3.2. and 3.3 of [8]. One gains the advantage of not having to prove Lemma 7 of [3] (Hilfssatz 6 of [34]).

Remark 3. Some authors (Walfisz [34–37], Chowla [3]) considered another error term E_k defined by

$$S_1(x) =: \frac{\pi^2}{12} x^2 + E_1(x) \tag{70}$$

$$S_k(x) =: \frac{\zeta(1+k)}{1+k} x^{1+k} + \zeta(1-k)x + E_k(x) \quad (-1 < k < 1,\ k \neq 0) \tag{71}$$

$$S_{-1}(x) =: \frac{\pi^2}{6} x - \tfrac{1}{2} \log x + E_{-1}(x) \tag{72}$$

(hence the estimates (50) and (51)). This is a more natural choice than F_k, in the sense that

$$E_k(x) = \begin{cases} o(x^2) & (k=1) \\ o(x) & (0<k<1) \\ o(x^{1+k}) & (-1<k<0) \\ o(\log x) & (k=-1), \end{cases} \tag{73}$$

whereas

$$F_k(x) \neq o(1) \quad (-1 \leq k \leq 1). \tag{74}$$

F_k is the error term one obtains when dealing with S_k by the complex variable methods developed by Chandrasekharan and Narasimhan to exploit the representation

$$\sum_{n\geq 1} \sigma_k(n)n^{-s} = \zeta(s-k)\zeta(s) \quad (\operatorname{Re} s > \max(1, k+1)) \tag{75}$$

and the functional equation satisfied by $\zeta(s-k)\zeta(s)$ (see [1], [2], [9], [10]). It seems to be the "right" error term to consider if one is interested in the change of sign problems. To be concrete, let us say that a good point in favor of F_k for these problems is that for $k<0$, we have

$$\int_1^x F_k(t)\,dt = o(x), \tag{76}$$

which shows that the mean value of $F_k(t)$ is 0. As for Ω or 0 estimates, since

$$E_k(x) - F_k(x) = O(1) \quad \text{for} \quad k<0, \tag{77}$$

the results one obtains for any one of these error terms are also true for the other.

Remark 4. 0-estimates of the error term in (44) were successively improved in [38], [12], [13], [35], [16]. The current record-holder is Recknagel [22] with

$$F(x) = O(x^{109/382}). \tag{78}$$

A special case of a result of Segal's [24] reads

$$\sum_{n\leq x} F_{-1}(n) = \frac{\pi^2}{12}x + O(x^{1/4}), \tag{79}$$

which is equivalent to

$$F(x) = O(x^{1/4}) \tag{80}$$

(use (47) and (54)). Segal pointed out in [25] that his proof of (79) is incorrect. In fact, (79) itself is incorrect: see [17, Appendix]. (However, [25] was sometimes overlooked, as in [14] and [29]).

Remark 5. To our knowledge, the best 0-estimate to date of $F_k(x) - x^kF_{-k}(x)$ for $\frac{1}{2}<k\leq 1$ comes from using [22] instead of the weaker [15] in [11, Corollary 1

p. 403]. One obtains

$$F_k(x) = x^k F_{-k}(x) + O(x^{\theta(1-k)}) \quad (\tfrac{1}{2} < k \le 1), \tag{81}$$

where

$$\theta(t) = \begin{cases} \frac{109}{382} + (\frac{75}{191})t & (0 \le t < \frac{5}{93}) \\ \frac{49}{172} + (\frac{69}{172})t & (\frac{5}{93} \le t < \frac{1}{110}) \\ \frac{211}{744} + (\frac{77}{186})t & (\frac{1}{10} \le t < \frac{41}{224}) \\ \frac{209}{742} + (\frac{45}{106})t & (\frac{41}{224} \le t < \frac{11}{42}) \\ \frac{11}{40} + (\frac{9}{20})t & (\frac{11}{42} \le t < \frac{1}{2}). \end{cases} \tag{82}$$

Remark 6. Most authors who studied the S_k restricted themselves to the case $|k| \le 1$ ("to avoid unnecessary complications" according to Cramér [4]). Estimates of

$$F_k(x), \int_1^x F_k(t)\,dt \quad \text{and} \quad \int_1^x F_k^2(t)\,dt$$

for the case $|k| > 1$ are apparently unavailable in the literature. With the help of the existing proofs [3] of such estimates for $|k| \le 1$, together with Ramanujan's estimate [21] of $F_k(x) - x^k F_{-1}(x)$ for $0 < k < \infty$, extending the domain of validity of (43) to $|k| > 1$ is only a matter of tedious and unoriginal calculation. We now observe that

$$\lim_{k \to \infty} \frac{8}{3}\left(1 - \frac{\zeta(2k)}{4\zeta(2+2k)}\right) = 2; \tag{83}$$

with (69), this shows that the constant in (43) is in some sense best possible.

REFERENCES

[1] K. Chandrasekharan and R. Narasimhan, *Hecke's functional equation and arithmetical identities*, Ann. of Math. 74 (1961), 1–23.

[2] ——, *Functional equations with multiple gamma factors and the average order of arithmetical functions*, Ann. of Math. 76 (1962), 93–136.

[3] S. Chowla, *Contributions to the analytic theory of numbers*, Math. Z. 35 (1932), 279–299.

[4] H. Cramér, *Contributions to the analytic theory of numbers*, Fifth Congress of Scand. Math., Helsingfors (1922), 266–272.

[5] P. ERDÖS, Letter to J. Steinig (December 3, 1967).
[6] ——, Letter to J. Steinig (April 15, 1985).
[7] P. ERDÖS and H. N. SHAPIRO, *On the changes of sign of a certain error function*, Canad. J. Math. *3* and *4* (1951), 375–385.
[8] ——, *The existence of a distribution function for an error term related to the Euler function*, Canad. J. Math. *7* (1955), 63–75.
[9] J. L. HAFNER, *On the average order of a class of arithmetical functions*, J. Number Theory *15* (1982), 36–76.
[10] S. KANEMITSU, *Some results in the divisor problems*, RIMS Kôkyûroku *496* (1983) 44–65.
[11] ——, *Omega theorems for divisor functions*, Tokyo J. of Math. *7* (1984) 399–419.
[12] E. LANDAU, *S. Wigert (Stockholm): Sur quelques fonctions arithmétiques*, Gött. gel. Anz. *177* (1915), 377–414.
[13] ——, *Ueber einige zahlentheoretische Funktionen*, Gött. Nachr. (1924) 116–134.
[14] R. A. MCLEOD, *Fractional part sums and divisor functions*, J. Number Theory *14* (1982) 185–227.
[15] W. G. NOWAK, *On a problem of S. Chowla and H. Walum*, Bull. Number Theory *7* (1983) 1–10.
[16] H. Y. PENG, *A result in divisor problem*, Acad. Sinica Sci. Rec. *1* (1942) 69–72.
[17] Y.-F. S. PÉTERMANN, *Oscillations and changes of sign of error terms related to Euler's function and to the sum-of-divisors function*, Thèse de doctorat; Université de Genève, 1985.
[18] ——, *Changes of sign of error terms related to Euler's function and to divisor functions II*, to appear.
[19] S. S. PILLAI and S. D. CHOWLA, *On the error terms in some asymptotic formulae in the theory of numbers I*, J. London Math. Soc. *5* (1930), 95–101.
[20] J. H. PROSCHAN, *On the changes of sign of a certain class of error functions*, Acta Arith. *17* (1971), 407–430.
[21] S. Ramanujan, *On certain trigonometrical sums and their applications in the theory of numbers*, Trans. Cambridge Phil. Soc. 22, 13 (1918), 259–276, or Collected papers, Cambridge Univ. press (1927), 179–199.
[22] W. RECKNAGEL, *Ueber eine Vermutung von S. Chowla und H. Walum*, Arch. Math. *44* (1985), 348–354.
[23] M. N. L. Sarma, *On the error term in a certain sum*, Proc. Indian Acad. Sci. A, *3* (1931) 338.
[24] S. L. Segal, *A note on the average order of number theoretic error terms*, Duke Math. J. *32* (1965) 279–284; erratum 765.
[25] ——, Erratum, same J. *33* (1966) 821.
[26] R. SITARAMACHANDRARAO and D. SURYANARAYANA, *On the average order of the function* $E(x) = \sum_{n \le x} \phi(n) - 3x^2/\pi^2$, Ark. Mat. *10* (1972), 99–106.
[27] A. SIVARAMASARMA, Some problems in the theory of Farey series and the Euler totient function, Doctoral thesis; Andhra univ., Waltair, India, 1979.
[28] J. STEINIG, *The changes of sign of certain arithmetical error-terms*, Comment. Math. Helv. *44* (1969) 385–400.
[29] S. SURYANARAYANA, *On a conjecture of S. Chowla*, Proc. Amer. Math. Soc. *56* (1976) 27–33.
[30] ——, *On the average order of the function* $E(x) = \sum_{n \le x} \phi(n) - 3x^2/\pi^2$ *(II)*, J. Indian Math. Soc. *42* (1978), 179–195.
[31] J. J. SYLVESTER, *Sur le nombre de fractions ordinaires inégales qu'on peut exprimer en se servant de chiffres qui n'excèdent pas un nombre donné*, C. R. Acad. Sci. Paris *46* (1883) 409–413; or Coll. papers, Cambridge 1904–1912, IV 84–87.
[32] ——, *Note sur le théorème de Legendre cité dans une note insérée dans les "Comptes rendus"*, C. R. Acad. Sci. Paris *46* (1883) 463–465; or Coll. papers IV 88–90.
[33] ——, *On the number of fractions contained in any Farey series of which the limiting number is given*, Phil. Mag. *15* (1883) 230–233; or Coll. papers IV 101–109.
[34] A. WALFISZ, *Teilerprobleme I*, Math. Z. *26* (1927), 66–88.
[35] ——, *Teilerprobleme IV*, Ann. Scuola Norm. Sup. Pisa 2*a* s. *5* (1936) 289–298.

[36] ——, *Teilerprobleme V*, Acta Arith. *2* (1937) 80–133.
[37] ——, *Ueber Gitterpunkte in mehrdimensionalen Ellipsoiden V*, Acta Arith. *I*. (1936) 222–283.
[38] S. Wigert, *Sur quelques fonctions arithmétiques*, Acta Math. *37* (1914), 113–140.

Université de Genève,
Section de Mathématique,
2–4, *rue du Lièvre, C. P. 240,*
CH-1211 Genève 24.
Received August 27, 1985.