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Abelian normal subgroups of two-knot groups

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Introduction

Using the algebraic classification of high dimensional knot groups, Hausmann and Kervaire have shown that any finitely generated abelian group is the centre of some n-knot group, for each $n \geq 3$ [21]. On the other hand the only 1-knots whose groups have nontrivial abelian normal subgroups are the torus knots. (For the commutator subgroup can contain no nontrivial abelian normal subgroup [38: Chapter IV.5], so any such subgroup must map 1–1 to the abelianization and be central. Now use [8]). In [25] we considered the remaining case of 2-knots, showing that the centre must have rank at most 2, but a gap was later found in the key lemma of that paper. Here we shall repair that gap and treat the more general question suggested by our title. We shall show that if a 2-knot group contains a torsion free abelian normal subgroup of rank $r > 1$ then the group is an orientable Poincaré duality group of formal dimension 4 and so $r \leq 4$. There are only two such groups with $r = 4$, while the groups with $r = 3$ are just the groups of the fibred 2-knots constructed by Cappell in [9]. Many examples may be constructed with $r = 1$ or 2 by twist spinning classical knots; our results in these cases are less conclusive, but suggest that $r = 2$ arises only by generalized twist spinning [40] torus knots. However there are examples of 2-knot groups with rank 1 abelian normal subgroups which cannot be realized by fibred knots [15, 52]. An example due to Fox [15] has commutator subgroup the dyadic rationals; we show that any virtually solvable 2-knot group must be either virtually poly-$\mathbb{Z}$ or Fox's group or admit no nontrivial torsion free abelian normal subgroup. All the virtually poly-$\mathbb{Z}$ groups allowed by our theorems may be realized by fibred 2-knots.

Our argument is based on the idea used in [24] and [25] of embedding the group ring $\mathbb{Z}[G]$ into a larger ring R in which an annihilator for the augmentation module becomes invertible and for which nontrivial stably free R-modules have well-defined strictly positive rank, with rank $R^n = n$. (Rings with the latter property were called "hopfian" in [25]). Under suitable hypotheses on the group G, Poincaré duality then implies that the equivariant homology of a 4-manifold with fundamental group G is concentrated in degree 2 and is stably free as an R-module. Its rank may be computed by an Euler characteristic counting argument. If the Euler characteristic is 0 the manifold is aspherical and so G is a Poincaré duality group. The remainder of our argument rests upon properties of groups with small cohomological dimension and large centre (principally Theorem 8.8 of [5]) and special features of certain matrix groups.

There are six numbered sections. The first gives some notation and terminology from group theory and the second states some of the results on asphericity from [25] that may be recovered in strengthened form by means of a result of Rosset [43]. The next three sections treat the cases when the maximal rank of an abelian normal subgroup is greater than 2, equal to 2 or 1 respectively. In the last section the preceding results are applied to the consideration of virtually solvable 2-knot groups and it is indicated how most such groups can be realized by (fibred) 2-knots.

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§1. Notation and terminology

We shall say that a torsion free abelian normal subgroup of a group G is *maximal,* if it has maximal rank and is not properly contained in any other such subgroup. Note that if G contains an abelian normal subgroup which is either finitely generated or central, then it contains one of the same rank which is torsion free.

The *centralizer* of a normal subgroup A of G is the kernel of the homomorphism from G to Aut (A) determined by the conjugation action of G on A, and shall be denoted $C_G(A)$. (We shall write $C(G)$ for $C_G(G)$, the centre of G). If A is abelian then it is a central subgroup of $C_G(A)$.

A group is a $PD_n^{(+)}$ group if it is a finitely presentable (orientable) Poincaré duality group of formal dimension n. A group is *polycyclic* if it has a composition series with cyclic factors; the number of infinite cyclic factors is then independent of the composition series chosen, and is called the *Hirsch length* of the group [42:5.4.13]. More generally we may define the Hirsch length of any solvable group as the sum of the ranks of the factors of its derived series. A group is a solvable PD_n group if and only if it is a torsion free polycyclic group of Hirsch length n [5: Theorem 9.23].

An automorphism ϕ of a group K is *meridional* if the normal closure in K of $\{k^{-1}\phi(k) \mid k \text{ in } K\}$ is K, and then the abelianization of the HNN extension K_ϕ^* presented by $\langle K, t \mid tkt^{-1} = \phi(k) \text{ for } k \text{ in } K\rangle$ is infinite cyclic. The automorphism of an n-knot commutator subgroup induced by conjugation by a meridian is meridional. Two meridional automorphisms ϕ and ψ determine isomorphic HNN extensions ($K_\phi^* \approx K_\psi^*$) if and only if ϕ is conjugate to $\psi^{\pm 1}$ in $\mathrm{Out}(K)$, the group of outer automorphisms of K.

A *weight class* for a group is a conjugacy class whose normal closure is the whole group. The group is then said to have *weight* 1. A group G is a *high dimensional knot group* if it is finitely presentable, has weight 1, $H_1(G;\mathbb{Z}) \approx \mathbb{Z}$ and $H_2(G;\mathbb{Z}) = 0$ [32]. Two elements of the same weight class of such a group G determine meridional automorphisms of G' which are conjugate in $\mathrm{Aut}(G')$ by an inner automorphism of G'.

The subquotient G'/G'' may be considered as a (G/G')-module, via the conjugation action of G on G'. If G is a knot group, a choice of meridians for the knot determines an isomorphism $\mathbb{Z}[G/G'] \approx \Lambda = \mathbb{Z}[t, t^{-1}]$. The module G'/G'' is then a finitely generated Λ-torsion module on which $t-1$ acts invertibly. In particular the annihilator ideal $\mathrm{Ann}(G'/G'')$ is nonzero. The *Alexander polynomial* of G is the characteristic polynomial of the meridian acting on $H_1(G';\mathbb{Q})$ and is an element Δ of Λ such that $|\Delta(1)| = 1$. It generates a proper ideal in Λ (i.e. is not of the form $\pm t^n$) if and only if G'/G'' is infinite. The highest common factor of the annihilator ideal divides the Alexander polynomial. (See Chapters III and IV of [26]).

§2. Rosset's lemma and asphericity

The proof of the key lemma of [25] was fallacious, as pointed out by M. N. Dyer (c.f. [25 bis]), and the results became moot for several years. Fortunately, however, Rosset has since provided a correct proof of a closely related result that may be used instead. We shall restate Rosset's result as:

ROSSET'S LEMMA [43]. *Let G be a group which contains a nontrivial torsion free abelian normal subgroup A. Let S be the multiplicative system $\mathbb{Z}[A]\backslash\{0\}$ in $\mathbb{Z}[G]$. Then the (noncentral!) localization $R = S^{-1}\mathbb{Z}[G]$ exists and has the property that nontrivial finitely generated stably free R-modules have well-defined strictly positive rank, with rank $R^n = n$. Furthermore R is flat as a $\mathbb{Z}[G]$-module and $R \otimes_{\mathbb{Z}[G]} \mathbb{Z} = 0$.*

The prototype of such a result was given by Kaplansky who showed that for any group G the group ring $\mathbb{Z}[G]$ has this "hopfian" property [31: page 122]. By means of this lemma we may redeem the results of [25] and restate some of them in the following strengthened forms. (We shall not repeat the proofs as, apart from using Rosset's lemma instead of the key lemma of [25], they are otherwise unchanged).

THEOREM 1. *Let X be a finite 2-dimensional cell complex with fundamental group containing a nontrivial torsion free abelian normal subgroup. Then* $\chi(X) \geq 0$, *and* $\chi(X) = 0$ *if and only if X is aspherical.*

The assumption that X be 2-dimensional is not needed in order to show that X aspherical implies $\chi(X) = 0$; this is in fact Rosset's application of his lemma. Gottlieb obtained the first such result under the further assumption that $C(\pi_1(X)) \neq 1$ [19]. Kaplansky's lemma has been used several times in a related way in connection with the Whitehead conjecture on the asphericity of subcomplexes of 2-dimensional $K(\pi, 1)$ complexes. (See [7], for instance).

COROLLARY. *If a finitely presentable group G contains a nontrivial torsion free abelian normal subgroup then it has deficiency at most* 1. *If* def $G = 1$ *and G is neither $\mathbb{Z}$ nor $\mathbb{Z}^2$ then G has cohomological dimension* 2 *and the centre of G is infinite cyclic or trivial.*

This partially settles (and goes beyond) a conjecture of Murasugi, that the centre of a finitely presentable group of deficiency ≥ 1 other than $\mathbb{Z}^2$ be infinite cyclic or tirival, and be trivial if the group has deficiency ≥ 2 [37]. Some of the arguments of this paper can be seen in microcosm in the following discussion. If c.d. $G = 2$ and G has an abelian normal subgroup $A \neq 1$, either $A \approx \mathbb{Z}$ and so $[G : C_G(A)] \leq 2$ or c.d.$A =$ c.d.$C_G(A) = 2$ and so $C_G(A)$ is abelian, by [5: Theorem 8.8]. If A has rank 1 then Aut (A) is abelian so $G' \subseteq C_G(A)$ and G is solvable. (Such groups have been classified by Gildenhuys [18]). Otherwise $A \approx \mathbb{Z}^2 \approx C_G(A)$. As $C_G(A)$ with an element of infinite order modulo $C_G(A)$ would generate a subgroup of cohomological dimension 3, which is impossible, $G/C_G(A)$ must be a torsion group, and so finite, as it is a subgroup of Aut $(A) \approx GL(2, \mathbb{Z})$, by [31: page 105]. Since G is torsion free it must be $\mathbb{Z}^2$ or the Klein bottle group.

THEOREM 2. *Let M be a closed* 4-*manifold with fundamental group G such that G contains a nontrivial torsion free abelian normal subgroup and* $H^s(G; \mathbb{Z}[G]) = 0$ *for* $s \leq 2$. *Then M is aspherical if and only if* $\chi(M) = 0$.

§3. Rank greater than 2

Let $K: S^2 \to S^4$ be a 2-knot with group $G = \pi_1(S^4 \backslash K(S^2))$, and let M be the closed orientable 4-manifold obtained from S^4 by surgery on K. Then $\pi_1(M) = G$ and $\chi(M) = 0$. We shall show that if G contains a torsion free abelian normal subgroup of sufficiently large rank then the cohomological hypotheses of Theorem 2 also hold, and so M is aspherical.

LEMMA. *Let A be a torsion free abelian group of rank r and $\mathcal{M}$ a free A-module, $\mathcal{M} = \mathbb{Z}[A]^{(I)}$ say. Then $H^i(A;\mathcal{M}) = 0$ if $i < r$. If A is finitely generated then $H^r(A;\mathcal{M}) \approx \mathbb{Z}^{(I)}$. If $r < \infty$ but A is not finitely generated then $H^r(A;\mathcal{M}) = 0$.*

Proof. Let N be a free abelian subgroup of A of finite rank $s \le r$. If $r < \infty$ we may assume rank $N = r$ and if A is finitely generated we may take $N = A$. Let $Q = A/N$. Since N is an FP group and $\mathcal{M}$ is free as an N-module, $H^i(N;\mathcal{M}) \approx H^i(N;\mathbb{Z}[N]) \otimes_{\mathbb{Z}[N]} \mathcal{M}$ for all i [5: Proposition 2.4]. Therefore $H^i(N;\mathcal{M}) = 0$ if $i < s$ and $H^s(N;\mathcal{M}) \approx \mathbb{Z} \otimes_{\mathbb{Z}[N]} \mathcal{M} \approx \mathbb{Z}[Q]^{(I)}$. If A is not finitely generated then Q is infinite and so $H^0(Q;\mathbb{Z}[Q]^{(I)}) = 0$ [5: Lemma 8.1]. The lemma now follows on applying the LHS spectral sequence $H^p(Q;H^q(N;\mathcal{M})) \Rightarrow H^{p+q}(A;\mathcal{M})$.

THEOREM 3. *Let G be a 2-knot group with a torsion free abelian normal subgroup A of rank $r \ge 2$. Then G is a finitely presentable PD_4^+ group, and so $r \le 4$.*

Proof. Consider the LHS spectral sequence $E_2^{pq} = H^p(G/A;H^q(A;\mathbb{Z}[G])) \Rightarrow H^{p+q}(G;\mathbb{Z}[G])$. By the lemma, if $r \ge 3$ or if $r = 2$ and A is not finitely generated then $E_2^{pq} = 0$ for $q \le 2$. If $A \approx \mathbb{Z}^2$ then $E_2^{pq} = 0$ for $q \le 1$ and $E_2^{p2} = H^p(G/A;\mathbb{Z}[G/A])$. But no group containing $\mathbb{Z}^2$ as a subgroup of finite index can have infinite cyclic abelianization (as a knot group must have) and so G/A must be infinite. Therefore $E_2^{02} = H^0(G/A;\mathbb{Z}[G/A]) = 0$. In all cases we conclude that $H^s(G;\mathbb{Z}[G]) = 0$ for $s \le 2$ and so G is a PD_4^+ group by Theorem 2 and the remarks at the beginning of this section. In particular rank $A \le$ c.d. $A \le 4$.

In our next result we shall determine the 2-knot groups with such subgroups A of rank ≥ 3. (We shall consider the case $r = 2$ in §4).

THEOREM 4. *Let G be a 2-knot group with a maximal torsion free abelian normal subgroup A of rank $r > 2$. Then either*

(i) *$A = C_G(A) = G' \approx \mathbb{Z}^3$ and the meridional map is given by a matrix C in $SL(3,\mathbb{Z})$ such that $|\det(C - I)| = 1$; or*

(ii) *$A \approx \mathbb{Z}^4$ and $G \approx G(+1)$ or $G(-1)$, where $G(\varepsilon)$ is presented by $\langle x, y, t \mid xy^2x^{-1}y^2 = 1,\ txt^{-1} = (xy)^{-\varepsilon},\ tyt^{-1} = x^{\varepsilon} \rangle$, for $\varepsilon = \pm 1$.*

Proof. By Theorem 3, the manifold M obtained by surgery on a knot with group G is aspherical. Therefore the covering space M_A determined by the subgroup A is also aspherical. Hence c.d.$A \leq 4$, with equality if and only if the 4-manifold M_A is also closed, if and only if $[G:A] < \infty$, in which case A is finitely generated. Moreover $r \leq$ c.d. A, with equality if and only if A is finitely generated, and so isomorphic to $\mathbb{Z}^r$, by [5:Theorem 7.14]. Therefore $r \leq 4$, and $r = 4$ implies that $A \approx \mathbb{Z}^4$ and $[G:A] < \infty$.

If $r = 3$ these conditions imply that $A \approx \mathbb{Z}^3$. The quotient G/A must contain an element of infinite order. For otherwise the image of G in $\mathrm{Aut}(A) \approx GL(3, \mathbb{Z})$ under the map determined by conjugation is a finitely generated torsion group, and so finite by [31:page 105]. Since the kernel of this map is $C_G(A)$, we then have $[G:C_G(A)] < \infty$, so c.d. $C_G(A) = 4$. By [5:Theorem 8.8], $C_G(A)'$ must be free. By assumption $r = 3$, so $C_G(A)' \neq 1$. If $C_G(A)' \approx \mathbb{Z}$ then G is solvable-by-finite, therefore polycyclic-by-finite and so must contain a normal subgroup B of finite index which is a poly-$\mathbb{Z}$ group [42; 5.4.15]. Then $B \cap C_G(A)$ is a normal poly-$\mathbb{Z}$ subgroup of finite index in G, and so of Hirsch length 4, which contains $B \cap A$ as a central subgroup. Since $[A:B \cap A] \leq [G:B] < \infty$, we must have $B \cap A \approx \mathbb{Z}^3$, from which it follows that B must be abelian, contrary to the assumption that $r = 3$. If $C_G(A)'$ is a nonabelian free group then $A \cap C_G(A)' = 1$ and so $C_G(A)'$ maps injectively to G/A. Thus there is an element g in G whose image in G/A has infinite order, and so the subgroup of G generated by $A \cup \{g\}$ has cohomological dimension 4, and therefore is of finite index in G.

Thus if $r = 3$ or 4 the group G is a solvable-by-finite PD_4^+-group, hence (poly-$\mathbb{Z}$)-by-finite of Hirsch length 4. Therefore G' is (poly-$\mathbb{Z}$)-by-finite of Hirsch length 3. We claim that G' is virtually abelian. This is clear if $A \cap G' \approx \mathbb{Z}^3$, so suppose that $A \cap G' \approx \mathbb{Z}^2$. Then $A \cap G'$ is normal in G, and $G/C_G(A \cap G')$ is a solvable-by-finite subgroup of $GL(2, \mathbb{Z})$ with cyclic abelianization. Therefore it is either finite (cyclic or S_3) or infinite cyclic. In either case $[G':C_{G'}(A \cap G')] < \infty$. Since $C_{G'}(A \cap G')$ then has Hirsch length 3, and contains $\mathbb{Z}^2$ as a central subgroup, it must be virtually abelian, and so the same is true of G'.

Now a torsion free group which contains $\mathbb{Z}^3$ as a normal subgroup of finite index is the fundamental group of a flat 3-manifold. On examining the lists in [50; pages 117, 120] we see that only the groups $G_1 \approx \mathbb{Z}^3$ and G_6 can occur as knot commutator subgroups. (For the other groups admit no meridional automorphisms, as they have abelianizations of the form $\mathbb{Z} \oplus$ (finite) or (free) $\oplus$ $(\mathbb{Z}/2\mathbb{Z})$).

If $G' \approx \mathbb{Z}^3$ and $r = 4$, then some power of a meridian would commute with G'. But the characteristic polynomial of an automorphism of $\mathbb{Z}^3$ of finite order must be a product of cyclotomic polynomials, of total degree 3, and therefore must have $t-1$ as a factor. Since such an automorphism cannot be meridional, we must have $r = 3$ when $G' \approx \mathbb{Z}^3$. The further details in this case are taken from [24].

Thus to determine the possibilities for G when $r = 4$ we must find the conjugacy classes in $\mathrm{Out}(G_6)$ which contain meridional automorphisms. The group G_6 has a presentation $\langle x, y, z \mid xy^2x^{-1}y^2 = yx^2y^{-1}x^2 = 1,\ z = xy\rangle$. The subgroup A_6 generated by $\{x^2, y^2, z^2\}$ is a maximal abelian normal subgroup, isomorphic to $\mathbb{Z}^3$, with $G_6/A_6 \approx (\mathbb{Z}/2\mathbb{Z})^2$. Define automorphisms i and j of G_6 by $i(x) = y$, $i(y) = x$ (hence $i(z) = x^{-2}y^2z^{-1}$ and $i^2 = id$) and $j(x) = xy$, $j(y) = x$ (hence $j(z) = xyx = z^2y^{-1}$ and $j^6 = 1$). Then the images of i and j generate $\mathrm{Aut}(G_6/A_6) \approx GL(2, \mathbb{F}_2) \approx S_3$. Let H be the subgroup of $\mathrm{Aut}(G_6)$ generated by the automorphisms listed in the following table.

	Effect on					
Automorphism	x	y	z	x^2	y^2	z^2
α	x^{-1}	y	$x^{-2}z$	-1	1	1
β	x	y^{-1}	y^2z	1	-1	1
γ	x	z^2y	z^{-1}	1	1	-1
δ	x	x^2y	x^2z	1	1	1
ε	y^2x	y	y^2z	1	1	1
ϕ	z^2x	z^2y	z	1	1	1

(Note that these automorphisms act on $A_6 = \mathbb{Z}\cdot x^2 \oplus \mathbb{Z}\cdot y^2 \oplus \mathbb{Z}\cdot z^2$ via diagonal matrices; the last three columns of the table give the diagonal entries of the matrices). Then $H = \ker(:\mathrm{Aut}(G_6) \to \mathrm{Aut}(G_6/A_6))$. For an automorphism inducing the identity on G_6/A_6 must send x to $x^{2p}y^{2q}z^{2r}x$, y to $x^{2s}y^{2t}z^{2u}y$ and hence z to $x^{2(p+s)}y^{2(q-t)}z^{2(r-u)}z$. The squares of these elements are x^{4p+2}, y^{4t+2} and $z^{4(r-u)+2}$, which generate A_6 if and only if $p = -1$ or 0, $t = -1$ or 0 and $r = u - 1$ or u. Composing such an automorphism appropriately with α, β and γ we may achieve $p = t = 0$ and $r = u$. Then by composing with powers of δ, ε and ϕ we may obtain the identity automorphism. These automorphisms satisfy $\alpha^2 = \beta^2 = \gamma^2 = 1$ and each pair commutes except for $\alpha\delta = \delta^{-1}\alpha$, $\beta\varepsilon = \varepsilon^{-1}\beta$ and $\gamma\phi = \phi^{-1}\gamma$. The inner automorphisms are contained in H, and are generated by $\beta\gamma\delta$ (conjugation by x) and $\alpha\gamma\varepsilon\phi$ (conjugation by y). Therefore $\bar{H} = H/\mathrm{Inn}(G_6)$ is a group of exponent 2 generated by the images of α, β, γ and ε. Since the images of these elements in $\mathrm{Aut}(G_6/G_6') \approx GL(2, \mathbb{Z}/4\mathbb{Z})$ are $\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$, $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$, $\begin{pmatrix}1 & 2\\ 0 & -1\end{pmatrix}$ and $\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}$ respectively (with respect to the basis $\{xG_6', yG_6'\}$ for G_6/G_6') and these matrices generate a group of order 16, we must have $\bar{H} \approx (\mathbb{Z}/2\mathbb{Z})^4$. Since in $\mathrm{Aut}(G_6)$ we have $j^3 = \alpha\beta\gamma\varepsilon$, $jiji = \delta$, $i\alpha = \beta i$, $\phi i\gamma = \gamma i$, $j\alpha = \gamma j$, $j\beta = \delta\alpha j$, $j\gamma = \beta\varepsilon j$

and $j\varepsilon = \delta j$, we find that $\mathrm{Out}(G_6)$ is a group of order 96, with a presentation $\langle i, j, \alpha, \beta, \gamma, \varepsilon \mid \alpha^2 = \beta^2 = \gamma^2 = \varepsilon^2 = i^2 = j^6 = 1,\ \alpha,\ \beta,\ \gamma,\ \varepsilon$ commute, $i\alpha = \beta i$, $i\gamma = \alpha\varepsilon i$, $j\alpha = \gamma j$, $j\beta = \alpha\beta\gamma j$, $j\gamma = \beta\varepsilon j$, $j\varepsilon = \beta\gamma j$, $j^3 = \alpha\beta\gamma\varepsilon$ and $jiji = \beta\gamma\rangle$. There is an exact sequence

$$1 \to (\mathbb{Z}/2\mathbb{Z})^4 \to \mathrm{Out}(G_6) \to S_3 \to 1.$$

If ψ is a meridional automorphism of G_6, then it must induce a meridional automorphism of G_6/A_6, and so we must have $\psi \equiv j$ or j^{-1} modulo H. Conversely any such automorphism is meridional, for it implies that G_6 modulo the normal closure of $\{g^{-1}\psi(g) \mid g \text{ in } G_6\}$ is a perfect group, and therefore trivial, since G_6 is solvable. There are 32 elements in the cosets $j\bar{H} \cup j^{-1}\bar{H}$ of $\mathrm{Out}(G_6)$. The centralizer of j in $\mathrm{Out}(G_6)$ is generated by $\alpha\beta$ and j, and has order 12. The distinct cosets of this centralizer in $\mathrm{Out}(G_6)$ are represented by $\{1, \alpha, \gamma, \alpha\gamma, i, i\alpha, i\gamma, i\alpha\gamma\}$. Conjugating j and j^{-1} by these elements we get 16 distinct elements of $j\bar{H} \cup j^{-1}\bar{H}$, which all give rise to the group with presentation

$$\langle x, y, t \mid xy^2x^{-1}y^2 = 1,\ txt^{-1} = xy,\ tyt^{-1} = x\rangle$$

However this group cannot be a PD_4^+ group, as already the subgroup generated by $\{x^2, y^2, z^2, t\}$ is nonorientable. The elements $j\alpha$ and $j\beta$ also have centralizers of order 12 and their conjugates exhaust the remaining 16 elements of $j\bar{H} \cup j^{-1}\bar{H}$. Each of $j\alpha$ and $j\beta$ is conjugate to its inverse (via i), and so the groups $G(+1)$ and $G(-1)$ that they give rise to are distinct. Moreover these automorphisms are orientation preserving on A_6 and hence on G_6 (in fact $j\alpha = (i\alpha\gamma j\gamma\alpha i)^2$) and so $G(+1)$ and $G(-1)$ are PD_4^+ groups.

Finally in both cases the subgroup generated by $A_6 \cup \{t^6\}$ is an abelian normal subgroup of rank 4.

As the characteristic polynomial of a meridional automorphism of $\mathbb{Z}^3$ must be irreducible and not cyclotomic the only nontrivial abelian normal subgroup of a knot group G with $G' \approx \mathbb{Z}^3$ is G' itself. In each of the groups $G(+1)$ and $G(-1)$ the subgroup A_6 is an abelian normal subgroup of rank 3. Since the characteristic polynomial of t acting on the subgroup A_6 of $G(\varepsilon)$ is $X^3 - 1$, the only candidates for normal subgroups of rank <3 contained in A_6 are (essentially) $(t-1)A_6$, generated by $\{x^2y^2, x^2z^{-2}\}$ and $(t^2 + t + 1)A_6$, generated by $\{x^2y^{-2}z^2\}$. It is easily seen that neither of these groups is even normal in G'. Therefore any abelian normal subgroup B of $G(\varepsilon)$ such that $B \cap A_6$ has rank less than 3 must map injectively to G/G' and so have rank 1. Such a subgroup must be central. However the centre of $G(\varepsilon)$ is trivial. For if t^kg (with g in $G(\varepsilon)' \approx G_6$) is central, then g commutes with t, and so g^2 in A_6 is invariant under $j\beta$. Hence

$g^2 = (x^2y^{-2}z^2)^m$ for some m in $\mathbb{Z}$. Since $t^{6k}g^6 = (t^kg)^6$ is also central, the automorphism $(j\beta)^{6k}$ must be conjugation by g^6, i.e. by a power of $x^2y^{-2}z^2$. But $(j\beta)^6(x) = y^{-2}z^4x$, $(j\beta)^6(y) = x^4z^4y$ and $(j\beta)^6(z) = y^{-2}x^{-4}z$. Therefore $(j\beta)^{6n}x = y^{-2n}z^{4n}x$, etc. Since $(x^2y^{-2}z^2)^p x(x^2y^{-2}z^2)^{-p} = y^{-4p}z^{4p}x$, etc., no nontrivial power of $j\beta$ can be conjugation by a power of $x^2y^{-2}z^2$. Hence $k = 0$. But then g is central in $G(\varepsilon)' \approx G_6$, and so $g = 1$. Thus $G(\varepsilon)$ has trivial centre.

Each of the groups allowed by Theorem 4 is the group of some fibred 2-knot, as we shall now show. (Recall that the groups $\mathbb{Z}^3$ and G_6 are the fundamental groups of flat 3-manifolds, which are in particular Seifert fibred. Note also that the assumption in Theorem 5 that G be a PD_4^+ group is redundant, except perhaps when $H^1(G'; \mathbb{Z}/2\mathbb{Z}) = 0$ [24]).

THEOREM 5. *Let G be a 2-knot group which is a PD_4^+ group such that G' is the fundamental group of a closed aspherical 3-manifold $M(G')$ which is either Seifert fibred or sufficiently large. Then G is the group of a fibred 2-knot.*

Proof. The manifold $M(G')$ must be orientable and the meridional automorphism orientation preserving, since G is a PD_4^+ group. The meridional (outer) automorphism may be realized by a self homotopy equivalence of the aspherical manifold $M(G')$ and therefore by a self homeomorphism, φ say (by [13] or [48]). We may assume that φ fixes a point of $M(G')$. A fibred knot with group G can now be constructed by surgery on a cross section of the mapping torus of φ, using [16] to recognize the resulting homotopy sphere as S^4.

§4. Rank 2

We shall now suppose that the maximal rank of a torsion free abelian normal subgroup is 2. Although we have not been able to eliminate the possibilities that $A \subseteq G'$ or that A be not finitely generated, our results suggest that G must be the group of a generalized twist spun torus knot. We shall use repeatedly in this section the fact that the subgroup $SL(2, \mathbb{Z})'$ of $GL(2, \mathbb{Z})$ is a finitely generated free normal subgroup and the quotient group $GL(2, \mathbb{Z})/SL(2, \mathbb{Z})'$ is D_{12}, the dihedral group of order 24. (See [42: Section 6.2]).

THEOREM 6. *Let G be a 2-knot group with a torsion free abelian normal subgroup A of rank 2, such that $A \nsubseteq G'$. Then G is a PD_4^+ group, $[G:C_G(A)] \leq 2$ and G' is a PD_3^+ group with non-trivial centre. If G'/G'' is infinite then $A \approx \mathbb{Z}^2$ and G is the group of a fibred 2-knot.*

Proof. The group G is a PD_4^+ group by Theorem 3. The intersection $A \cap G'$ is a rank 1 abelian normal subgroup of G. The automorphisms of A preserving such a subgroup form a group isomorphic to a subgroup of lower triangular 2×2 matrices with rational coefficients, which must be metabelian. Therefore $G'' \subseteq C_G(A)$, and if $C_G(A)$ is solvable, G is solvable and so polycyclic. If $A \approx \mathbb{Z}^2$ then $G/C_G(A)$ is a metabelian subgroup of $GL(2, \mathbb{Z})$ with finite cyclic abelianization, and so finite. The group $G/C_G(A)$ is also finite if A is not finitely generated, for otherwise $\text{c.d.}C_G(A) = 3 = \text{c.d.}A$, so $C_G(A)$ is abelian by [5: Theorem 8.8], hence G would be polycyclic and so A finitely generated, contrary to assumption. But a finite lower triangular subgroup of $SL(2, \mathbb{Q})$ with cyclic abelianization must have order at most 2. Thus $[G:C_G(A)] \leq 2$, so $G' \subseteq C_G(A)$ and $A \cap G'$ is central in G'. The subgroup H of G generated by $A \cup G'$ has finite index in G and so is a PD_4^+ group. Since A is central in this group and maps onto $H/G' \approx \mathbb{Z}$, we have $H \approx G' \times \mathbb{Z}$, and so G' is a PD_3^+ group. Now a PD_3 group with nontrivial centre and infinite abelianization is the fundamental group of a Seifert fibred 3-manifold [27]. In particular its central subgroups are finitely generated. Therefore $A \approx \mathbb{Z}^2$, since $A \cap G'$ has rank 1. The final assertion follows from Theorem 5.

We do not know whether A need be central in G, nor whether it need be finitely generated. However Theorem 6 applies whenever $C(G)$ has rank greater than 1.

THEOREM 7. *The centre of a 2-knot group has rank* ≤ 2. *If rank* $C(G) = 2$ *then* $C(G) \nsubseteq G'$, *so the conclusions of Theorem* 6 *hold for* $A = C(G)$, *while* $C(G') = G' \cap C(G)$ *and is contained in* G''.

Proof. The centre of G contains a torsion free subgroup of the same rank, which is necessarily normal in G. Suppose that rank $C(G) > 1$. Then G is a PD_4^+ group by Theorem 3. Since the groups in Theorem 4 have trivial centre, we must have rank $C(G) = 2$. Now $G' \cap C(G)$ is nontrivial and is contained in G'' since $G/G' \approx \mathbb{Z}$. In particular G' is nonabelian. Since G' is the fundamental group of an open aspherical 4-manifold (the infinite cyclic covering of the manifold M of §3), we have $\text{c.d.}G' \leq 3$. Since $4 = \text{c.d.}G \leq \text{c.d.}G' + \text{c.d.}(G/G') \leq 3 + 1$, we have c.d. $G' = 3$. But if c.d. $G' = 3$ and G' contains a central subgroup isomorphic to $\mathbb{Z}^2$ then either G' is abelian or G'' is free [5: Theorem 8.8]. Thus $C(G)$ cannot be contained in G', and so Theorem 6 applies.

Now $\text{c.d.}C(G') \leq 2$ so either $C(G') \approx \mathbb{Z}^2$ or $C(G')$ has rank 1 [5: Theorem 8.8]. In the former case G'' must be free, since G' is nonabelian [ibid]. Since $G' \cap C(G) \subseteq G''$, we must then have $G'' \approx \mathbb{Z}$ and hence G'' is central in G'. Therefore G' is nilpotent and so a nonabelian poly-$\mathbb{Z}$ group of Hirsch number 3.

But such groups have centre $\mathbb{Z}$. Thus $C(G')$ is a rank 1 torsion free abelian group. Since $C(G')$ contains $G' \cap C(G)$ (which also has rank 1) and is characteristic in G, it follows that $C(G')$ is central in G; in other words $C(G') = G' \cap C(G)$ and so must be contained in G''.

The simplest example of a 2-knot whose group has centre $\mathbb{Z}^2$ is the 6-twist-spun trefoil [51]. The r-twist-spun (p, q)-torus knot (for p and q relatively prime) is fibred with fibre the Brieskorn manifold $M(p, q, r)$, and the r^{th} power of some meridian is central in the group G of this knot. If $p^{-1} + q^{-1} + r^{-1} \leq 1$ then $M(p, q, r)$ is aspherical and $G' = \pi_1(M(p, q, r))$ has centre $\mathbb{Z}$; if also r is odd the conjugation action of G/G' on $\mathbb{Z} = C(G')$ must be trivial, and so $C(G) \approx \mathbb{Z}^2$. (Can the assumption that r be odd be lifted?) Note that if p, q and r are pairwise relatively prime then $M(p, q, r)$ is an homology sphere and $G \approx G' \times \mathbb{Z}$, for there is then a central element mapping to a generator of G/G'.

More general constructions based on torus knots lead to similar examples [40]. Does every 2-knot whose group has centre $\mathbb{Z}^2$ derive thus from a torus knot? If the 2-knot is already known to be a twist-spun 1-knot, then this is usually so. (I am indebted to Scott for explaining how the manifold M in the next theorem admits a Seifert fibration invariant under the group action).

THEOREM 8. *Let G be the group of the q-twist spin of a* 1-*knot K, and suppose that G'/G'' is infinite and $C(G') \approx \mathbb{Z}$. Then K is a torus knot.*

Proof. The subquotient G'/G'' is the first homology of the q-fold branched cyclic cover M of S^3, branched over K, which has order $|\prod_{1 \leq i \leq q} \Delta_1(K)(\zeta^i)|$, where ζ is a primitive q^{th} root of unity [26: Chapter VIII]. Therefore G'/G'' infinite implies that q is not a prime power, and so $q \geq 6$. If the manifold M were a connected sum $M_1 \# M_2$, then one of the summands, M_2 say, would have to be a homotopy sphere, since $C(\pi_1(M)) \neq 1$. The knot K would decompose accordingly, so that M_2 would be a branched cyclic cover of S^3, branched over a knot summand of K [34]. But any such homotopy sphere must be standard, by the Smith conjecture. Therefore M is irreducible, and also sufficiently large, and so must be Seifert-fibred, of type $\mathbb{H}^2 \times \mathbb{R}$, $SL(2, \mathbb{R})$, Nil, $\mathbb{E}^3$ or $S^2 \times \mathbb{R}$. Now Meeks and Scott have shown that any finite group action on a closed 3-manifold admitting a geometric structure of one of the first four of these types (or of type Sol) may be assumed to preserve such a structure [33]; as none of $\mathbb{Z}$, $\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$ or $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ can be the commutator subgroup of a knot group the case $S^2 \times \mathbb{R}$ does not arise here. Preservation of such a structure implies that M admits a Seifert fibration invariant under the action. (When M is flat, i.e. of type $\mathbb{E}^3$, one must observe that the orbifold fundamental group of the orbifold $M/(\mathbb{Z}/q\mathbb{Z})$ sits in an exact sequence $1 \to \mathbb{Z}^3 \to \pi \to Q \to 1$ where Q is a finite subgroup of $SL(3, \mathbb{Z})$ mapping onto $\mathbb{Z}/q\mathbb{Z}$. Since $q > 4$ the group Q must be cyclic or dihedral, and so leaves fixed some nonzero vector in $\mathbb{Z}^3$, corresponding to an invariant Seifert fibration on a 3-torus covering M, which passes to an invariant fibration on M. Note that in fact we would have $q = 6$).

Since $q > 2$ the fixed circle (the branch set in M) must be a fibre of the Seifert fibration which therefore passes to a Seifert fibration on the knot complement $S^3\backslash K$. Thus the knot K must be a torus knot [8].

Remark. The condition $C(G') \approx \mathbb{Z}$ alone implies that $q > 1$. Bedient [4] has shown that when $q = 2$ the knot is not uniquely determined by the associated 2-fold branched cyclic cover being a particular Seifert fibre space, and need not be a torus knot. Can the condition "G'/G'' is infinite" be relaxed to "$q \geq 3$"? (Note that the examples of Bedient do not give counter-examples to the question preceding Theorem 8, as the branch involution acts nontrivially on the centre of the fundamental group of the branched cyclic cover).

There are no known examples of the types allowed by the next theorem, which considers the remaining possibilities in the rank 2 case.

THEOREM 9. *Let G be a 2-knot group with a maximal torsion free abelian normal subgroup A of rank 2 such that $A \subseteq G'$. Then G is a PD_4^+ group. If $A \approx \mathbb{Z}^2$ then either*

(i) $C_G(A) = G'$, *G'' is a nonabelian free group and* Ann (G'/G'') *is divisible by one of $t^2 + t - 1$, $t^2 - t - 1$ or $t^2 - 3t + 1$; or*

(ii) $G/C_G(A) \approx \mathbb{Z}/6\mathbb{Z}$, *$G''$ is a nonabelian free group and* Ann (G'/G'') *is divisible by the cyclotomic polynomial $\phi_6 = t^2 - t + 1$; or*

(iii) $G/C_G(A) = D_3$; *or*

(iv) c.d. $C_{G'}(A) =$ c.d. $C_G(A) = 3$ *but* $C_G(A) \not\subseteq G'$.

If A is not finitely generated then $A = C_G(A)$.

Proof. The group G is a finitely presentable PD_4^+ group by Theorem 3. If $A \approx \mathbb{Z}^2$ then $G/C_G(A)$ is isomorphic to a subgroup of $GL(2, \mathbb{Z})$ and so has virtual cohomological dimension ≤ 1. Therefore c.d. $C_G(A) = 3$ or 4. If A is not finitely generated then c.d. $A = 3$ so again c.d. $C_G(A) = 3$ or 4.

If $A \subseteq G'$ and $A \approx \mathbb{Z}^2$, then $2 = \text{c.d.}A \leq \text{c.d.}C_{G'}(A) \leq \text{c.d.}G' = 3$. If $\text{c.d.}C_{G'}(A) = 2$ then $C_{G'}(A) = A$. Since $C_G(A)' \subseteq C_{G'}(A)$ this implies that $C_G(A)$ is nilpotent. Since A is maximal and $\text{c.d.}C_G(A) \geq 3$, $C_G(A)$ cannot be abelian. But nonabelian nilpotent groups of cohomological dimension 3 have infinite cyclic centre, so $\text{c.d.}C_G(A) = 4$. Therefore $[G' : C_{G'}(A)] \leq [G : C_G(A)] < \infty$ and so $\text{c.d.}C_{G'}(A) = \text{c.d.}G' = 3$, contradiction. Thus $\text{c.d.}C_{G'}(A) = 3$ and so $C_{G'}(A)'$ is a free group. If $C_{G'}(A)'$ is abelian then the subgroup generated by $A \cup C_{G'}(A)'$ is an abelian normal subgroup of G, so $C_{G'}(A)' \subseteq A$ (by maximality of A). This leads to a similar contradiction. Therefore $C_{G'}(A)'$ is a nonabelian free group, and $A \cap C_{G'}(A)' = 1$. (Similarly, if c.d.$C_G(A) = 3$ then $C_G(A)'$ is nonabelian free).

If $A \approx \mathbb{Z}^2$ and $C_G(A) \subseteq G'$ then $G/C_G(A)$ is a subgroup of $GL(2, \mathbb{Z})$ with infinite cyclic abelianization. Such a group must be infinite cyclic. For any subgroup of $GL(2, \mathbb{Z})$ contains a free subgroup with quotient a subgroup of D_{12}. If the abelianization is cyclic then this quotient must be either cyclic (of order dividing 12) or $D_3 = S_3$. But an extension of $\mathbb{Z}/n\mathbb{Z}$ by a free group $F(r)$ which has infinite cyclic abelianization must be torsion free, and so free by [45:5.B.3]. If an extension H of D_3 by a free group $F(s)$ has infinite cyclic abelianization, then we may assume $s > 1$ and that the only finite subgroups of H are isomorphic to $\mathbb{Z}/3\mathbb{Z}$. Since $F(s)$ has infinitely many ends, so does H, and therefore $H \approx B * D$ or $B *_C D$ or $B *_C$, where $C \approx \mathbb{Z}/3\mathbb{Z}$, by [45:5.A.10]. But if $H \approx B * D$ then either B or D would be perfect, which is clearly impossible for nontrivial subgroups of H. If $H = B *_C D$ or $B *_C$ then B (say) would have abelianization $\mathbb{Z}/3\mathbb{Z}$. Furthermore $B \not\approx \mathbb{Z}/3\mathbb{Z}$ and B' would be free. But then $M = B'/B''$ would be a finitely generated $\mathbb{Z}$-torsion free module over the ring $\mathbb{Z}[C] = \mathbb{Z}[x]/(x^3 - 1)$ such that $M = (x - 1)M$. Such a module must be 0, contradicting $B \not\approx \mathbb{Z}/3\mathbb{Z}$. Thus there is no such group H, so $C_G(A) = G'$ and $G'' = C_{G'}(A)'$ is a nonabelian free group. Therefore $A \cap G'' = 1$, so we may regard A as a sub-Λ-module of G'/G''. Since A is $\mathbb{Z}$-torsion free, Ann A is principal, generated by λ say. Since $A \approx \mathbb{Z}^2$, the lowest and highest coefficients of λ are ± 1, and the degree of λ is at most 2. Since $G/G' \approx \mathbb{Z}$, we must have $|\lambda(1)| = 1$. Therefore $\lambda = t^2 - 3t + 1$, $t^2 + t - 1$, $t^2 - t - 1$ or $t^2 - t + 1$. But if $\lambda = t^2 - t + 1$, then $t^6 - 1$ would annihilate A, and hence (since $G' \subseteq C_G(A)$) $G/C_G(A) \approx \mathbb{Z}/6\mathbb{Z}$.

If c.d.$C_G(A) = 4$ then $G/C_G(A)$ is a finite subgroup of $GL(2, \mathbb{Z})$ with cyclic abelianization, and so $G/C_G(A) \approx D_3$ or $\mathbb{Z}/n\mathbb{Z}$ with n dividing 12. But if $G/C_G(A)$ is cyclic then A is central in G'. Arguing as before we have G'' is free and $A \cap G'' = 1$, and Ann A is generated by one of the four quadratic polynomials given above. Since $G/C_G(A)$ has order dividing 12, λ must divide $t^{12} - 1$. Therefore $\lambda = \phi_6 = t^2 - t + 1$ and hence $G/C_G(A) \approx \mathbb{Z}/6\mathbb{Z}$.

If $A \subseteq G'$ and A' is not finitely generated then c.d.$A = 3 =$ c.d.$C_{G'}(A)$ so $C_{G'}(A) = A$ as before. If c.d.$C_G(A) = 4$ then $[G':A] = [G:C_{G'}(A)] \leq [G:C_G(A)] < \infty$ so G would be virtually metabelian, hence poly ($\mathbb{Z}$ or finite), contradicting the assumption that A be not finitely generated. Thus c.d.$C_G(A) = 3$ and so $A = C_G(A)$.

We remark finally that none of the cases of Theorem 9 can occur when G' is a

PD_3 group, for otherwise $C_{G'}(A)$ would have finite index in G' by [46] and so would also be a PD_3 group, and therefore abelian, contradicting the maximality of A. In particular these cases cannot occur for any fibred 2-knot. (We expect that in fact these cases do not occur at all).

§5. Rank 1

In the rank 1 case G need not be a PD_4^+ group. Nor is A necessarily finitely generated. For instance, Fox found a 2-knot with group Φ presented by $\langle a, t \mid tat^{-1} = a^2 \rangle$. This group is metabelian, with commutator subgroup $\Phi' \approx \mathbb{Z}[\frac{1}{2}]$, the dyadic rationals, and has cohomological dimension 2. However consideration of how the cohomological hypotheses of Theorem 2 might fail suggests a useful separation of subcases. Recall that a finitely generated group K has 0, 1, 2 or infinitely many ends, and that if K is infinite $H^0(K; \mathbb{Z}[K]) = 0$ while if K has 1 end then $H^1(K; \mathbb{Z}[K]) = 0$ also [49]. I am indebted to Geoghegan and Mihalik for their help with case (ii).

THEOREM 10. *Let G be a 2-knot group with a maximal torsion free abelian normal subgroup A of rank 1. Then either*

(i) *G/A is finite, hence G' is finite; or*
(ii) *G/A has one end and G is a PD_4^+ group; or*
(iii) *G/A has two ends and there is a finite normal subgroup N in G with $G/N \approx \Phi$; or*
(iv) *G/A has infinitely many ends.*

Proof. If G/A is finite then A is finitely generated, so $A \approx \mathbb{Z}$ and G has two ends. But then G must be a finite extension of $\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ by [45:4A.6.5]; since $G/G' \approx \mathbb{Z}$ case (i) follows. If G/A has one end then G is simply connected at ∞, by [35:Theorem 1], and then $H^s(G; \mathbb{Z}[G]) = 0$ for $s \leq 2$, by [17], so G is a PD_4^+ group, by Theorem 2. Thus the only case we need consider at length is when G/A has two ends.

Since G is a knot group, the abelianization of G/A is cyclic. Therefore if it has two ends it must have a finite normal subgroup N_0 with infinite cyclic quotient, so there is an exact sequence $1 \to A \to G' \to N_0 \to 1$. Since A is torsion free abelian of rank 1, its group of automorphisms is abelian; since A is normal in G, it must be central in G', and it has finite index there. Therefore the commutator subgroup of G' is finite [42:Theorem 10.1.4]. Hence the set N of elements of G' of finite order is a finite subgroup which is characteristic in G and A maps monomorphically to $H = G/N$. The group H is thus finitely presentable with $H/H' \approx \mathbb{Z}$ and H' is torsion free abelian of rank 1. According to Trotter [47] the

Λ-module $H' = H'/H''$ must have a presentation matrix of the form $tA + (I - A)$ where A is a $d \times d$ integral matrix (with $d = \text{rank } H' = 1$) and where at least one of A, $I - A$ is unimodular. This implies that (up to inversion of the meridian) $H' \approx \Lambda/(t-2)$ and so $H \approx \Phi$. (This result can also be derived from [20]).

The 2-knot groups in case (i) were determined in [23, 41, 51]. Twist spun non-torus knots usually have groups in case (ii). We shall see that Φ is the only example in case (iii). The group of a (0-twist) spun (p, q)-torus knot has centre $\mathbb{Z}$ with quotient $(\mathbb{Z}/p\mathbb{Z}) * (\mathbb{Z}/p\mathbb{Z})$ and so is in case (iv).

We shall now examine case (iii) more closely and show that N must be trivial. For $m \geq 1$ let $\Phi(m)$ be the group presented by $\langle t, a \mid tat^{-1} = a^{2^m} \rangle$. Then every subgroup of finite index in $\Phi(1) = \Phi$ is isomorphic to $\Phi(m)$ for some m. (For if Ψ is such a subgroup then $\Psi \cap \Phi' \approx \lambda \cdot \mathbb{Z}[\frac{1}{2}]$ for some odd $\lambda \geq 1$, so Ψ is generated by a^λ and $t^m a^\mu$ (for some $m \geq 1$ and μ in $\mathbb{Z}[\frac{1}{2}]$) with a single relation $(t^m a^\mu) a^\lambda (t^m a^\mu)^{-1} = a^{\lambda \cdot 2^m}$). Let $\tilde{\Phi}(m)(\approx \mathbb{Z}[\frac{1}{2}])$ be the kernel of the homomorphism from $\Phi(m)$ onto $\mathbb{Z}$ sending t to 1 and a to 0.

LEMMA. *Let Y be a closed orientable 4-manifold with $\chi(Y) = 0$ and such that there is an epimorphism $f: \pi = \pi_1(Y) \to \Phi(m)$ for some m, with finite kernel. Let $\tilde{Y}$ be the infinite cyclic covering space of Y determined by $\tilde{\pi} = f^{-1}(\tilde{\Phi}(m))$. Then the integral homology groups of $\tilde{Y}$ are finitely generated Λ-torsion modules, and $H_2(\tilde{\pi}; \mathbb{Z})$ is finite cyclic of odd order.*

Proof. Since $\tilde{Y}$ is a covering space of a compact manifold and since $\mathbb{Z}[\pi/\tilde{\pi}] \approx \Lambda$ is noetherian, the groups $H_i(\tilde{Y}; \mathbb{Z}) = H_i(Y; \Lambda)$ are finitely generated as Λ-modules. Since Y is orientable, $\chi(Y) = 0$ and $H_1(Y; \mathbb{Q}) \approx \mathbb{Q}$, we have $H_2(Y; \mathbb{Q}) = 0$ and the rest of the first assertion now follows from the Wang sequence for the projection $\tilde{Y} \to Y$ (cf. [36]). By Hopf's Theorem $H_2(\tilde{\pi}; \mathbb{Z})$ is a quotient of $H_2(\tilde{Y}; \mathbb{Z}) = H_2(Y; \Lambda)$, even as a Λ-module. By equivariant Poincaré duality $H_2(Y; \Lambda) \approx \overline{H^2(Y; \Lambda)}$. Since $H_2(Y; \Lambda)$ is a torsion module, the Universal Coefficient spectral sequence gives an isomorphism $H^2(Y; \Lambda) \approx \text{Ext}^1_\Lambda(H_1(Y; \Lambda), \Lambda)$. Now $H_1(Y; \Lambda) \approx \tilde{\pi}/\tilde{\pi}'$ and so there is an exact sequence

$$0 \to T \to H_1(Y; \Lambda) \to \tilde{\Phi}(m) = \Lambda/(t - 2^m) \to 0$$

where T is a finite module. Therefore $\text{Ext}^1_\Lambda(H_1(Y; \Lambda), \Lambda) \approx \Lambda/(t - 2^m)$ and so $H_2(Y; \Lambda) \approx \overline{(\Lambda/(t - 2^m))} \approx \Lambda/(2^m t - 1) \approx \mathbb{Z}[\frac{1}{2}]$ as an abelian group. Now $\tilde{\pi}/\text{ker} f \approx \mathbb{Z}[\frac{1}{2}]$ also, and $H_2(\mathbb{Z}[\frac{1}{2}]; \mathbb{Z}) = \mathbb{Z}[\frac{1}{2}] \wedge \mathbb{Z}[\frac{1}{2}]$ (by [42: page 334]) $= 0$, and so by the

LHS spectral sequence for the extension

$$1 \to \ker f \to \tilde{\pi} \to \mathbb{Z}[\tfrac{1}{2}] \to 1$$

we have that $H_2(\tilde{\pi}; \mathbb{Z})$ is finite. But a finite quotient of $\mathbb{Z}[\frac{1}{2}]$ is cyclic of odd order.

We shall apply this lemma to certain irregular finite covering spaces of the 4-manifold obtained by surgery on a knot with group as in Theorem 10 (iii).

THEOREM 11. *Let G be a 2-knot group with a finite normal subgroup N such that $G/N \approx \Phi$. Then $N = 1$.*

Proof. Let M be the closed orientable 4-manifold obtained by surgery on a 2-knot with group G (as in §3), so $\pi_1(M) = G$ and $\chi(M) = 0$. Choose once and for all a meridional element t in G.

Let H be the (normal) subgroup of G generated by $N \cup C_G(N)$. Then H has finite index in G and $H/N \approx \Phi(m)$ where t^m generates $HG' \bmod G'$. Let $\tilde{H}$ be the inverse image of $\tilde{\Phi}(m)$ in H. Then $\tilde{H}'$ is finite and $C(\tilde{H})$ has finite index in $\tilde{H}$. Since $C(N) = C(\tilde{H}) \cap N$ is a finite abelian group and $C(\tilde{H})/C(N)$ is isomorphic to $\tilde{\Phi}(m)$, we must have $C(\tilde{H}) \approx C(N) \times \tilde{\Phi}(m)$ by [42:page 106], and so there is a central complement $B \approx \tilde{\Phi}(m)$ for N in $\tilde{H}$. Hence $\tilde{H} \approx N \times \tilde{\Phi}(m)$. The complement B may not be invariant under conjugation by t^m, but $\theta(b) = b^{-2^m} t^m b t^{-m}$ defines a homomorphism from B into $C(N)$ (since B and hence $t^m B t^{-m}$ are central in $\tilde{H}$) which must have image finite of odd order. Therefore if D is any subgroup of N containing U, the odd part of $C(N)$, then BD ($\approx B \times D$) is invariant under conjugation by t^m. Now let p be an odd prime and let $N(p)$ be the p-Sylow subgroup of N. Then $N(p)$ and so $U(p) = U \cdot N(p)$ are invariant under conjugation by t^m and $U(p) \cup B \cup \langle t^m \rangle$ generates a (non-normal) subgroup π ($= U(p) \cdot B \cdot \langle t^m \rangle$) of finite index in H. The quotient $\pi/U(p)$ is isomorphic to $\Phi(m)$, and $\tilde{\pi} \approx U(p) \times B$. Let $Y(p)$ be the (irregular) covering space of M determined by π. Then we may apply the lemma to conclude that $H_2(\tilde{\pi}; \mathbb{Z})$ is finite cyclic of odd order. By the Künneth theorem, $(U(p)/U(p)') \otimes B$ is a direct summand of $H_2(\tilde{\pi}; \mathbb{Z})$. Since $U(p)$ has odd order and $B \approx \mathbb{Z}[\frac{1}{2}]$, it follows that $U(p)/U(p)'$ is cyclic. Hence $[U(p), U(p)'] = U(p)'$. Therefore also $[\tilde{\pi}, \tilde{\pi}'] = \tilde{\pi}'$ and so if $F = \mathbb{F}_p$ the homomorphism from $H^1(\tilde{\pi}; F) \wedge H^1(\tilde{\pi}; F)$ to $H^2(\tilde{\pi}; F)$ determined by cup product is injective [28]. Now let $\tilde{Y}(p)$ be the infinite cyclic covering space of $Y(p)$ determined by $\tilde{\pi}$. Then the classifying map from $\tilde{Y}(p)$ to $K(\tilde{\pi}; 1)$ is 2-connected, so $H^1(\tilde{\pi}; F) \approx H^1(\tilde{Y}(p); F)$ and $H^2(\tilde{\pi}; F) \subseteq H^2(\tilde{Y}(p); F)$. By Milnor duality [36] cup product gives a perfect pairing of $H^1(\tilde{Y}(p); F)$ with $H^2(\tilde{Y}(p); F)$ into $H^3(\tilde{Y}(p); F)$. If $N(p) \neq 1$ then $H^1(U(p); F) \approx F$ (since $U(p)/U$

$\approx N(p)$ and $U(p)/U(p)'$ is cyclic). Then $H^1(\tilde{\pi}; F) \approx F^2$ so the image of $H^1(\tilde{\pi}; F) \wedge H^1(\tilde{\pi}; F)$ in $H^2(\tilde{Y}(p); F)$ is 1-dimensional (by injectivity of cup product) and must be non-trivially paired with some element of $H^1(\tilde{\pi}; F)$. But there can be no nontrivial alternating trilinear form on a 2-dimensional vector space. So $N(p) = 1$, and thus N must be a 2-group.

Now let A be any abelian subgroup of N. Appealing once more to the lemma with $\pi = A \cdot B \cdot \langle t^m \rangle$ and $\tilde{\pi} \approx A \times B$ we find that $H_2(A; \mathbb{Z})$ is cyclic of odd order, as it is a direct summand of $H_2(\tilde{\pi}; \mathbb{Z})$. But $H_2(A; \mathbb{Z}) \approx A \wedge A$ by [42: page 334] and so A must be cyclic. It follows that N must be cyclic ($\mathbb{Z}/2^n\mathbb{Z}$) or $Q(n)$, a generalized quaternion group, presented by $\langle x, y \mid x^2 = y^{2^n}, x^4 = 1 \rangle$ for some $n \geq 1$. (See [42: 5.3.6] or [50: page 161]). Suppose that $N \neq Q = Q(1)$. Then N has no automorphism of odd order and so, returning to the knot group G, the extension $1 \to N \to G' \to \Phi' \to 1$ must split. Hence $G' \approx \Phi' \times N$, and N must admit a meridional automorphism. But this is impossible if N is cyclic of even order, or if $N = Q(n)$ for some $n \geq 2$. Thus $N = 1$ or Q.

Suppose that there is an exact sequence

$$1 \to Q \to G \to \Phi \to 1,$$

where Q is the quaternion group. We shall use equivariant Poincaré duality with coefficients $\Psi = \mathbb{F}_2[\Phi]$ to deduce a contradiction. We shall first describe some of the properties of this noncommutative ring. Since Φ is a torsion free 1-relator group other than $\mathbb{Z}$, it has cohomological dimension 2, and so the ring Ψ has global dimension 2. A presentation for the augmentation module $\mathbb{F}_2$ may be obtained by means of the free differential calculus: there is an exact sequence of left Ψ-modules

$$0 \to \Psi \xrightarrow{\partial_2} \Psi^2 \xrightarrow{\partial_1} \Psi \xrightarrow{\varepsilon} \mathbb{F}_2 \to 0$$

where $\varepsilon(g) = 1$ for all g in Φ, $\partial_1(\theta, \phi) = \theta(a-1) + \phi(t-1)$ and $\partial_2(\psi) = (\psi(t + a + 1), \psi(a^2 + 1))$. As a group ring Ψ has a natural involution, defined by $\bar{g} = g^{-1}$ for all g in Φ. Moreover Ψ is a twisted polynomial ring. Let $E = \mathbb{F}_2[a^{2^{-\infty}}] = \mathbb{F}_2[a_n \mid n \in \mathbb{Z}]/(a_{n+1} - a_n^2 \mid n \in \mathbb{Z})$. This is a commutative (Bezout) domain and its field of fractions $K = \mathbb{F}_2(a^{2^{-\infty}})$ is perfect, i.e. the squaring map σ is onto. Then $\Psi = E[a^{-1}][t, t^{-1}; \sigma]$: each element may be expressed uniquely as a sum $\sum t^p p_m(a)$ over m in a finite subset of $\mathbb{Z}$, where each $p_m(a)$ is in $E[a^{-1}]$, while the multiplication is determined by $a_{n+1}t = ta_n$ for all n, i.e. $t \cdot p(a) = \sigma(p(a)) \cdot t$ for $p(a)$ in $E[a^{-1}]$. The ring Ψ has a skew field of fractions L, which as a right Ψ-module is the direct limit of the system $\{\Psi_\theta \mid 0 \neq \theta \text{ in } \Psi\}$ where each $\Psi_\theta = \Psi$, the index set is ordered by right divisibility ($\theta \leq \phi\theta$) and the map from Ψ_θ to

$\Psi_{\phi\theta}$ sends ψ to $\phi\psi$, and so L is flat. (Note also that L contains the ring $K[t, t^{-1}; \sigma]$ which is a skew Euclidean domain. See [12] for a discussion of fields of fractions of twisted polynomial rings).

Let M^{Φ} be the covering space of M with $\pi_1(M^{\Phi}) \approx Q$ and let H_p denote $H_p(M; \Psi) = H_p(M^{\Phi}; \mathbb{F}_2)$ considered as a Ψ-module via the action of Φ through covering translations. If M has been triangulated, we may lift the triangulation to M^{Φ} and obtain thereby a (cellular) chain complex C_* of finitely generated free Ψ-modules such that $H_p(C_*) = H_p$. Since M^{Φ} is a connected open 4-manifold, $H_0 = \mathbb{F}_2$ (the augmentation module) and $H_4 = 0$. The Ψ-module structure on $H_1(M^{\Phi}; \mathbb{F}_2) = Q/Q' \approx \mathbb{F}_2^2$ is determined by a homomorphism $\Phi \to \mathrm{Aut}\,(Q/Q') \approx GL(2, 2) \approx S_3$. Since the abelianization of the knot group G must be infinite cyclic, the image of Φ cannot be trivial or of order 2, and there are essentially just two possibilities. In the first case the image of Φ is cyclic of order 3, so $a-1$ and t^3-1 act as 0, but $t \neq 1$. It follows that $t-1$ is an automorphism, and that H_1 with this module structure, which we shall denote H_c, is simple, and the annihilator of a generator is the left ideal generated by $a-1$ and t^2+t+1. There is an exact sequence

$$0 \to \Psi \xrightarrow{\partial_2^c} \Psi^2 \xrightarrow{\partial_1^c} \Psi \to H_c \to 0$$

where $\partial_1^c(\theta, \phi) = \theta(a-1) + \phi(t^2+t+1)$ and $\partial_2^c(\psi) = (\psi(t^2+t(a-1)+(a-1)^3), \psi(a^4-1))$. Exactness at three of the modules, and $\partial_1^c\partial_2^c = 0$ are easily verified. We may show that $\ker \partial_1^c = \mathrm{Im}\, \partial_2^c$ as follows. Suppose $\partial_1^c(\theta, \phi) = 0$. Then $\theta(a-1) = \phi(t^2+t+1)$. We may write ϕ as $\phi = (\sum_{m\in F} t^m p_m(a))(a-1)^{4d}$ for some $d \geq 0$ in $\mathbb{Z}[\frac{1}{2}]$ and where $p_m(1) \neq 0$ for some m. Then

$$\phi(t^2+t+1) = \Big(\sum_{m\in F} t^m p_m(a)\Big)(t^2+t(a-1)^d+(a-1)^{3d})(a-1)^d.$$

Since Ψ is a domain we may cancel factors from an equation. Thus if $d<1$ we have

$$\Big(\sum_{m\in F} t^m p_m(a)\Big)(t^2+t(a-1)^d+(a-1)^{3d}) = \theta(a-1)^{1-d},$$

so on substituting $a=1$ we get $\sum_{m\in F} t^m p_m(1) = 0$ in $\mathbb{F}_2[t, t^{-1}]$. This contradicts the assumption on the $p_m(a)$'s. Thus $d \geq 1$, so we may write $\phi = \eta \cdot (a-1)^4$, and it follows that $\theta = \eta(t^2+t(a-1)+(a-1)^3)$.

Otherwise Φ maps onto $\mathrm{Aut}\,(Q/Q')$, so t maps to an automorphism of order 2 and a maps to a generator of Aut $(Q/Q')'$, of order 3, and the module (now denoted H_f) is again simple. There is an exact sequence

$$0 \to \Psi \xrightarrow{\partial_2^f} \Psi^2 \xrightarrow{\partial_1^f} \Psi \to H_f \to 0$$

where $\partial_1^f(\theta, \phi) = \theta(t+a) + \phi(a^2+a+1)$ and $\partial_2^f(\psi) = (\psi(a^4+a^2+1), \psi(t+a^3+a^2+a))$. (Exactness of this sequence is proved in a similar fashion).

Some information about H_2 and H_3 is given by the Universal Coefficient spectral sequence, which, in conjunction with equivariant Poincaré duality, takes the form $E_2^{pq} = e^q H_p \Rightarrow \bar{H}_{4-p-q}$, with differential of bidegree $(-1, 2)$. Here $e^q H = \mathrm{Ext}^q_\Psi(H, \Psi)$ and $\bar{H}$ is the right Ψ-module with the conjugate action, determined by $h \cdot \psi = \bar{\psi} \cdot h$, for h in H and ψ in Ψ. Now from the three resolutions given above we may compute that $e^1\mathbb{F}_2 = e^0 H_c = e^0 H_f = 0$, and so $H_3 = 0$ in either case. (Note that in fact $e^0\mathbb{F}_2 = e^1\mathbb{F}_2 = 0$, which is equivalent to the fact that the group Φ has one end). The spectral sequence then gives an exact sequence

$$\cdots \to e^0 H_2 \to e^2 H_1 \to \bar{H}_1 \to e^1 H_2 \to 0.$$

Now since the skew field of fractions L is flat as a right module, $H_p(L \otimes_\Psi C_*) \approx L \otimes_\Psi H_p$, and so is nonzero only if $p = 2$. But since M has Euler characteristic 0, which is also the Euler characteristic of $L \otimes_\Psi C_*$ and therefore of $L \otimes_\Psi H_*$, we may concude that $L \otimes_\Psi H_2 = 0$ also. Therefore $e^0 H_2 = 0$ (since $e^0 H_2 = \mathrm{Hom}(H_2, \Psi) \subset \mathrm{Hom}(L \otimes_\Psi H_2, L)$) and we have a short exact sequence

$$0 \to e^2 H_1 \to \bar{H}_1 \to e^1 H_2 \to 0$$

in which the middle term has order 4 (as an abelian group). But this is absurd as $e^2 H_c \approx \Psi/I_c = \Psi/(t^2 + t(a-1) + (a-1)^3,\ a^4 - 1)\Psi$ and $e^2 H_f \approx \Psi/I_f = \Psi/(a^4 + a^2 + 1,\ t + a^3 + a^2 + a)\Psi$ are each infinite right Ψ-modules. (To see this note that for instance $e^2 H_f$ contains $E/E \cap I_f = E/(a^4 + a^2 + 1)$ as a sub E-module). Thus there can be no such 2-knot.

The modules H_c and H_f are realized by the extensions of Φ by Q presented by $\langle a, x, y, t \mid tat^{-1} = a^2,\ txt^{-1} = y,\ tyt^{-1} = xy,\ ax = xa,\ ay = ya,\ x^2 = (xy)^2 = y^2 \rangle$ and $\langle a, x, y, t \mid tat^{-1} = a^2,\ txt^{-1} = y,\ tyt^{-1} = x,\ axa^{-1} = y,\ aya^{-1} = xy,\ x^2 = (xy)^2 = y^2 \rangle$ respectively. These are in fact high dimensional knot groups.

THEOREM 12. *Let G be a 2-knot group with a maximal torsion free abelian normal subgroup A of rank* 1.

(1) *If* $A \cap G' = 1$, *then* $A \approx \mathbb{Z}$, $A \subseteq C(G)$ (*with equality if* $C(G)$ *is torsion free*) *and* G' *is finitely presentable. Moreover* G *is a* PD_4^+ *group if and only if* G' *is a* PD_3^+ *group if and only if* $e(G') = 1$.

(2) If $A \subseteq G'$ *then* $A \subseteq C(G')$ (*with equality if* $C(G')$ *is torsion free*). *If* G' *is finitely generated then* G *is a* PD_4^+ *group. If furthermore* G' *is finitely presentable then* G' *is a* PD_3^+ *group.*

Proof. Since $r = 1$, $\mathrm{Aut}(A)$ is abelian, so $G' \subseteq C_G(A)$. If $A \nsubseteq G'$, then $A \cap G' = 1$ and A maps injectively to $G/G' \approx \mathbb{Z}$, so $A \approx \mathbb{Z}$. Since conjugation by an element of G induces the identity automorphism of G/G', it follows that A is central in G, and therefore $A = C(G)$ if $C(G)$ is torsion free (by maximality of A). The subgroup of G generated by $A \cup G'$ has finite index in G, and is isomorphic to $A \times G'$, so G' is finitely presentable. If G is a PD_4^+-group (which need not be the case when $r = 1$) then $A \times G'$ is also a PD_4^+-group, so G' is a PD_3^+-group and hence $e(G') = 1$. Since $A \cap G' = 1$, $(G/A)' \approx G'$ and has finite index in G/A, so $e(G') = 1$ implies that $e(G/A) = 1$, and so G is a PD_4^+-group by Theorem 10 (ii).

If $A \subseteq G'$, then $A \subseteq C(G')$ (with equality if $C(G')$ is torsion free, by maximality of A). If G' is finitely generated then G'/A must be infinite and so $e(G/A) = 1$. Thus G is a PD_4^+-group, by Theorem 10 (ii). Suppose now that G' is finitely presentable and Z is an infinite cyclic subgroup of $C(G')$. Then $\mathrm{c.d.}G' = 3$ and there is an exact sequence

$$0 \to P_3 \to P_2 \to P_1 \to P_0 \to \mathbb{Z} \to 0$$

where the P_i are projective $\mathbb{Z}[G']$-modules, and are finitely generated for $i \leq 2$, since G' is FP_2. For the same reason the natural map from $H^s(G'; \mathbb{Z}[G']) \otimes \mathbb{Z}[\mathbb{Z}]$ to $H^s(G'; \mathbb{Z}[G])$ is an isomorphism for $s \leq 2$. (Cf [5: Theorem 5.3]). This is also true for $s = 3$. For the spectral sequence for the extension $1 \to Z \to G' \to J \to 1$ gives an isomorphism $H^3(G'; \mathbb{Z}[G']) \approx H^2(J; H^1(Z; \mathbb{Z}[G'])) \approx H^2(J; \mathbb{Z}[J])$ since Z is finitely generated. Likewise, writing $\mathbb{Z}[G] = \mathbb{Z}[G']^{(\mathbb{Z})}$, we have $H^3(G'; \mathbb{Z}[G]) \approx H^2(J; \mathbb{Z}[J]^{(\mathbb{Z})}) \approx H^2(J; \mathbb{Z}[J])^{(\mathbb{Z})}$, since J is FP_2. Therefore $H^3(G'; \mathbb{Z}[G]) \approx H^3(G'; \mathbb{Z}[G'])^{(\mathbb{Z})}$. On keeping track of the direct sum decompositions, we see that in fact $H^3(G'; \mathbb{Z}[G]) \approx H^3(G'; \mathbb{Z}[G']) \otimes \mathbb{Z}[\mathbb{Z}]$ as a $\mathbb{Z}[\mathbb{Z}]$-module.

The LHS spectral sequence for the extension $1 \to G' \to G \to \mathbb{Z} \to 1$, with coefficients $\mathbb{Z}[G]$, reduces to a Wang sequence

$$\cdots \to H^q(G'; \mathbb{Z}[G]) \xrightarrow{t-1} H^q(G'; \mathbb{Z}[G]) \to H^{q-1}(G; \mathbb{Z}[G]) \to \ldots$$

Using the above information on $H^q(G'; \mathbb{Z}[G])$ we find that $H^q(G'; \mathbb{Z}[G']) \approx H^{q+1}(G; \mathbb{Z}[G]) = 0$ for $q \neq 3$ and $H^3(G'; \mathbb{Z}[G']) \approx H^4(G; \mathbb{Z}[G]) \approx \mathbb{Z}$. Thus if we dualize the above $\mathbb{Z}[G']$-resolution of $\mathbb{Z}$ by means of $P^* = \mathrm{Hom}(P, \mathbb{Z}[G'])$ we get an exact sequence

$$0 \to P_0^* \to P_1^* \to P_2^* \to P_3^* \to H^3(G'; \mathbb{Z}[G']) = \mathbb{Z} \to 0$$

The dual P^* of a projective module P is finitely generated if and only if P is. Therefore P_3^* and hence P_3 are finitely generated. Thus G' is FP. As $H^q(G'; \mathbb{Z}[G']) \approx \mathbb{Z}$ if $q = 3$ and 0 otherwise, this implies that G' is a PD_3^+-group.

The group G of a (0-twist) spun torus knot has $C(G) \approx \mathbb{Z}$, $C(G) \cap G' = 1$ and G' free of even rank (so $e(G') \neq 1$). The group of a twist spun prime knot (other than a torus knot or certain rational knots) is usually a PD_4^+-group with $C(G) \approx \mathbb{Z}$ and $C(G) \cap G' = 1$. The groups of Theorem 14(iii) below are PD_4^+-groups with G' finitely presentable and (excepting the group of the 6-twist spun trefoil knot) have centre $\mathbb{Z}$ contained in G'. Yoshikawa has constructed a 2-knot whose group has centre $\mathbb{Z}$ contained in G' and such that $G/C(G)$ has infinitely many ends [46].

Note finally that Theorem 12(2) applies if we assume only that G' has a central element z of infinite order. For the normal closure of $\langle z \rangle$ in G is then a cyclic module over $\mathbb{Z}[G/G'] \approx \Lambda$, and so its $\mathbb{Z}$-torsion subgroup has finite exponent, e say. The normal closure of $\langle z^e \rangle$ in G is then a torsion free abelian normal subgroup of G of positive rank.

§6. Virtually solvable 2-knot groups

We shall apply the above results to the determination of the virtually solvable 2-knot groups which contain nontrivial torsion free abelian normal subgroups. All except for Fox's group Φ are virtually poly-$\mathbb{Z}$. If we assume further that G' be nilpotent, we need make no assumption about torsion free abelian normal subgroups. Let $I^* = SL(2, \mathbb{F}_5)$ and for each $k \geq 1$ let $T(k)$ be the extension of $\mathbb{Z}/3^k\mathbb{Z}$ by Q with presentation

$$\langle x, y, z \mid x^2 = (xy)^2 = y^2,\ zxz^{-1} = y,\ zyz^{-1} = xy,\ z^{3^k} = 1 \rangle.$$

THEOREM 13. *Let G be a virtually solvable 2-knot group which has a nontrivial torsion free abelian normal subgroup A. Then either*

(i) *G is a torsion free virtually poly-$\mathbb{Z}$ group of Hirsch length 4 and of orientable type, and G' is virtually nilpotent; or*

(ii) *$G' = (\mathbb{Z}/n\mathbb{Z}) \times P$ where $P = 1$, Q, I^* or $T(k)$ and $(n, 2|P|) = 1$, and we may assume that the meridional map multiplies the cyclic factor by -1, is* *the identity on I^*, and sends x, y in Q to y, xy and x, y, z in $T(k)$ to y^{-1}, x^{-1}, z^{-1} respectively; or*

(iii) $G = \Phi$.

Proof. We may assume A maximal and apply Theorems 3, 10 and 11. (Note that since G/A is a finitely generated virtually solvable group it cannot have infinitely many ends).

If G is a virtually solvable PD_4^+ group then it is torsion free and virtually polycyclic, by [5: Theorem 9.23], and hence virtually poly-$\mathbb{Z}$, by [42:5.4.15]. The commutator subgroup then has a nontrivial maximal (free) abelian normal subgroup, B say, of rank at most 3, the Hirsch length of G'. If B has rank 2, then G'/B has Hirsch length 1 and so is two-ended. It follows that G' has a characteristic subgroup with quotient either $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ or $\mathbb{Z}$. But no such group can be the commutator subgroup of a knot group. Thus B has rank 1 or 3 and G' is then easily seen to be virtually nilpotent.

Otherwise $G = \Phi$ or G' is finite; the further detail in the latter case follows from [14, 23].

There are finitely generated infinite solvable groups that contain no nontrivial torsion free abelian normal subgroups, but we do not know whether such can arise as high dimensional knot groups. (Very likely there are such). However a solvable group which is constructable in the sense of Baumslag and Bieri [3] must contain such a subgroup, for every subgroup of a constructable group is (torsion free)-by-finite. In particular polycyclic groups and finitely presentable groups which are nilpotent-by-(infinite cyclic) (for instance Φ) are constructable [3, 6]. We can be much more specific about 2-knot groups of the latter type.

THEOREM 14. *Let G be a 2-knot group with G' nilpotent. Then either*

(i) *$G' \approx \mathbb{Z}^3$ and the meridional map is given by a matrix C in $SL(3, \mathbb{Z})$ such that* $\det(C - I) = 1$; *or*

(ii) *$G' \approx F(2)/F(2)_3$, the free nilpotent group of class 2 on two generators, presented by $\langle x, y \mid [x, [x, y]] = [y, [x, y]] = 1\rangle$ and the meridional map is given by $x \to x^a y^b$ and $y \to x^c y^d$ where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is one of $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$; or*

(iii) *G' has a presentation $\langle x, y, z \mid [x, y] = z^q,\ [x, z] = [y, z] = 1\rangle$ for some odd $q > 1$ and the meridional map is given by $x \to x^a y^b$, $y \to x^c y^d$, $z \to z^{-1}$ where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$; or*

(iv) *G has a presentation* $\langle a, t \mid a^n = 1,\ tat^{-1} = a^{-1}\rangle$ *for some odd* $n \geq 1$; *or*
(v) *G has a presentation* $\langle a, x, y, t \mid a^n = 1,\ x^2 = (xy)^2 = y^2,\ ax = xa,\ ay = ya,\ tat^{-1} = a^{-1},\ txt^{-1} = y,\ tyt^{-1} = xy\rangle$ *for some odd* $n \geq 1$; *or*
(vi) $G = \Phi$.

Proof. By the remark on page 265 of [6], G is constructable. Since every subgroup of a constructable solvable group is (torsion free)-by-finite [3], G must have a nontrivial torsion free abelian normal subgroup. Moreover G is a virtual duality group and $H^k(G; \mathbb{Z}[G]) = 0$ for $k \neq h(G)$, the Hirsch length of G, by [3:Theorem B]. Thus if $h(G) > 2$ Theorem 2 applies so G is a PD_4^+ group, hence polycyclic. Therefore G' is a nilpotent poly-$\mathbb{Z}$ group of Hirsch length 3, by [42:5.2.20], and thus is an extension of $\mathbb{Z}^2$ by $\mathbb{Z}$; the further details in cases (i), (ii) and (iii) are taken from [24]. If $h(G) = 1$ then G' is a torsion group, so finite by [3:Proposition 1], and cases (iv) and (v) follow from Theorem 13. Finally if $h(G) = 2$ then the elements of finite order of G' form a (characteristic) subgroup T [42:5.2.7] which is again finite by [3] and the torsion free nilpotent group G'/T must be abelian of rank 1, since it has Hirsch length 1. Therefore $G/T \approx \Phi$ by Theorem 10 (iii), and case (vi) follows from Theorem 11.

COROLLARY. *If G is a metabelian 2-knot group then either* $G' \approx \mathbb{Z}^3$ *or* $\mathbb{Z}/n\mathbb{Z}$ *for some odd n, or* $G \approx \Phi$.

The determination of the abelian 2-knot commutator subgroups is essentially due to Yoshikawa [53], who however did not exclude the possibility that $G' \approx \Phi' \oplus (\mathbb{Z}/5\mathbb{Z})$.

Each of the groups allowed by Theorem 13 is the group of some 2-knot. Examples of type (i) were first given by Cappell [9] for the case $G' \approx \mathbb{Z}^3$. Cappell and Shaneson used such knots to give examples of distinct 2-knots with homeomorphic exteriors [10]. These knots have been examined in further detail by Aitchison and Rubinstein [1]. If G' is poly-$\mathbb{Z}$, then it is either $\mathbb{Z}^3$ or nilpotent of class 2. In the latter case G' is a discrete uniform subgroup of the 3-dimensional Lie group of upper triangular 3×3 matrices over $\mathbb{R}$ [24]. (The 6-twist spun trefoil is such a knot). In general the commutator subgroup of a group of type (i) is the fundamental group of an aspherical closed Seifert fibred 3-manifold, by [2] and [44], and so G is the group of a fibred 2-knot, by Theorem 5.

In another paper we shall show that the exterior of a 2-knot whose group is torsion free virtually poly-$\mathbb{Z}$ is determined up to homeomorphism by the group together with a weight class [29]. If $G' = \mathbb{Z}^3$ the weight class for G is unique up to inversion, and the knot is determined up to a finite ambiguity by its Alexander polynomial [24]. (The corresponding assertion in [24] for the other poly-$\mathbb{Z}$ groups is not justified, as the role of the weight class was overlooked there). No 2-knot with $G' = \mathbb{Z}^3$ can be a twist spun 1-knot, for a meridional automorphism of $\mathbb{Z}^3$ cannot have finite order (cf. the proof of Theorem 3).

Fox constructed 2-knots with commutator subgroup $\mathbb{Z}/(2n+1)\mathbb{Z}$, for any $n \geq 1$ [15: Examples 12 and 15]. These have been shown to be the 2-twist-spins of certain twist knots by Litherland (for the case $n = 1$, which is the 2-twist-spun trefoil) and Kanenobu [30] (for all $n \geq 1$). Thus these knots are all fibred, with fibre a punctured lens space [54]. (See also Section 6 of [39]). All the other possibilities for G' finite allowed by [23] have been realised, most as twist-spun classical knots, by Yoshikawa [51]. Plotnick [39] and Gonzales-Acuna have shown that no outer automorphism of $I^* = SL(2, 5)$ can be realized as conjugation by a meridian in a 2-knot group, thus resolving the one uncertainty about 2-knot groups with G' finite remaining in [23]. The method is essentially to show that an outer automorphism induces the identity on $H_3(S(3); \mathbb{Z}) \approx \mathbb{Z}/3\mathbb{Z}$ while it induces -1 on $H_3(S(5); \mathbb{Z}) \approx \mathbb{Z}/5\mathbb{Z}$, where $S(p)$ is the p-Sylow subgroup. If it were geometrically realizable, these calculations would lead to an inconsistency when considering the effect on the universal cover of the manifold obtained by surgery on the knot. Plotnick and Suciu [41] have determined all the fibred 2-knots with fibre a punctured spherical space form, and have found representatives for the weight classes when G' is finite.

Fox's Example 10, with group Φ, is not fibred, as its commutator subgroup is not finitely generated, and so cannot be twist spun. However it is a ribbon knot, as can be seen by "thickening" a suitable immersed ribbon D^2 in S^3 for the stevedore knot 6_2 (the equatorial cross-section of Example 10) to get an immersed ribbon D^3 in S^4. Alternatively, we may construct a ribbon 2-knot with group Φ by using the equivalent (Wirtinger) presentation $\langle u, v, w \mid vuv^{-1} = w, wuw^{-1} = v \rangle$ and the method of [26; Chapter II]. (The presentations are related by $u \to ta$, $v \to t^2at^{-1}$ and $w \to t$). Are all 2-knots with group Φ topologically equivalent?

It is well-known that all 2-knots are slice knots [32]. However knots with groups of type (i) or (ii) cannot be homotopy ribbon, let alone ribbon. For the manifold obtained by surgery on a homotopy ribbon knot bounds a 5-manifold (the complement of some slicing 3-disc in the 5-disc) with Euler characteristic 0 and built out of 0, 1 and 2-handles [11]. Considering the dual handle decomposition relative to the boundary, we see that the fundamental group of this 5-manifold, which has deficiency 1, is the knot group. Thus if Theorem 1 applies this group must have cohomological dimension 2 (and in fact the 5-manifold is aspherical) and so (i) cannot hold, while $H_1(G'; \mathbb{Z})$ must be $\mathbb{Z}$-torsion free by [26; Theorem III.10] and so (ii) cannot hold. (In fact Cochran has shown that the result of surgery on a homotopy ribbon 2-knot is never aspherical. He also raises the question as to whether the group of every ribbon 2-knot has a 2-dimensional Eilenberg–Mac Lane space [11]).

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