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## Parametrization of Möbius groups acting in a disk

MIKA SEPPÄLÄ and TUOMAS SORVALI

### Introduction

We consider groups  $G$  generated by hyperbolic Möbius transformations acting in the upper half-plane  $H$  or in the unit disc  $D$ . It is an interesting problem to find invariants which determine  $G$  up to conjugation. For properly discontinuous groups  $G$  this is the same thing as parametrizing the Teichmüller space of the Riemann surface  $H/G$ . In our considerations the discontinuity of  $G$  plays no role. Consequently the results hold for rather general groups  $G$ .

The proofs are elementary. We use multipliers of transformations of  $G$  to parametrize  $G$ . In the considerations we suppose that  $G$  is finitely generated even though the methods and computations could be easily generalized for infinitely generated groups.

To formulate the main result suppose that the hyperbolic transformations  $g_1, h_1, \dots, g_s, h_s$  generate  $G$  and satisfy certain technical conditions described in Theorem 2.2. Assume that the commutator  $c_s = h_s \circ g_s^{-1} \circ h_s^{-1} \circ g_s$  is hyperbolic and suppose that a relation is given such that  $c_s$  has a known representation in the group generated by  $g_1, h_1, \dots, g_{s-1}, h_{s-1}$ . Then the multipliers of the following  $6s - 4$  transformations

$$\begin{aligned} &g_j, h_j \text{ and } g_j \circ h_1 \quad \text{for } j = 1, \dots, s, \\ &h_j \circ h_1 \quad \text{for } j = 2, \dots, s, \\ &g_j \circ g_1 \text{ and } h_j \circ g_1 \quad \text{for } j = 2, \dots, s - 1 \text{ and} \\ &g_s \circ h_s \end{aligned}$$

determine  $G$  up to conjugation (Corollary to Theorem 2.2). Further, the multipliers of  $g_s \circ h_s$  and  $g_s \circ h_1$  can both have only two different values for fixed values of the other multipliers.

Note that we allow here, in fact, conjugation by Möbius-transformations mapping the upper half-plane (or the unit disk) onto its complement. Then the conjugacy class of  $G$  becomes uniquely determined by the multipliers of the above transformations.

For discontinuous groups  $G$  this result can be used to parametrize the Teichmüller space  $T_S$  of  $S = H/G$  globally by the multipliers of the above  $6s - 4$  transformations. This is the same thing as parametrizing  $T_S$  by the lengths of  $6s - 4$  closed geodesics. Irwin Kra has shown that such a parametrization is not even locally possible by  $6s - 6$  fixed curves only. Corollary to Theorem 2.2 gives a set of  $6s - 4$  closed curves whose lengths parametrize the Teichmüller space globally. Together with Kra's result this implies that the minimal number of curves parametrizing the Teichmüller space is either  $6s - 4$  or  $6s - 5$ .

Deleting the transformations  $g_s \circ h_1$  and  $g_s \circ h_s$  from the above list we obtain a set of  $6s - 6$  Möbius transformations whose multipliers parametrize an open set of the Teichmüller space. A remarkable fact is that such a parametrization can be obtained by elementary computations concerning the group  $G$  and that the discontinuity of  $G$  or the geometry of  $H/G$  plays no role here.

We start with showing how certain basic results about the geometry of a Riemann surface  $H/G$  are actually implied by elementary computations concerning hyperbolic Möbius transformations with intersecting axes. The multipliers of such transformations with a hyperbolic commutator satisfy an inequality (1.6) that, in the case of a Fuchsian group  $G$ , implies an inequality between the lengths of closed intersecting geodesics. The considerations here are of the same nature as those in [A], II.3.3, but the result is somewhat stronger.

The parametrization of  $G$  is achieved by considering a suitably normalized set of generators of  $G$ . That normalization is explained in Section 2 (e.g. Fig. 3). It allows us to avoid huge technical difficulties. For Fuchsian groups a usual standard set of generators satisfies the conditions of the normalization.

Using this set of generators we first parametrize certain free subgroups of  $G$ . This is done in detail in Theorem 2.1. These considerations should be contrasted with those in [S-S] where a similar parametrization was obtained for the Teichmüller space of a Riemann surface  $H/G$ . There we applied rather strong results of the geometry of  $H/G$ . The present paper reflects, in our opinion, the nature of these things more truly. Many of the results which, a priori, look rather deep are actually simple consequences of elementary computations.

Similar questions have been considered by many authors. We would especially like to mention the work of H. Helling [H]. His methods are closely related to those of this paper and his results were a starting point of this investigation.

We thank the referee for valuable comments.

### 1. A collar inequality

Let  $g$  be a hyperbolic Möbius transformation. Denote by  $a(g)$  and  $r(g)$  its *attracting* and *repelling* fixed points, respectively. Then its *multiplier*  $k(g)$  is given by the cross-ratio

$$k(g) = (g(z), z, r(g), a(g))$$

for any  $z \in \hat{\mathbb{C}}$  not fixed by  $g$ .

For the values  $k > 0$  let

$$f(k) = \sqrt{k} + 1/\sqrt{k}$$

and denote

$$f(g) = f(k(g)).$$

Then  $f(g) > 2$ . For a parabolic  $g$  set  $k(g) = 1$  and  $f(g) = 2$ .

If

$$z \rightarrow \frac{az + b}{cz + d}, \quad ad - bc = 1,$$

is any transformation conjugate to  $g$ , then

$$f(g) = |a + d|.$$

Let  $(g, h)$  be a pair of hyperbolic transformations of the unit disc  $D$  onto itself. Suppose that  $g$  and  $h$  have no common fixed points. Denote

$$k_1 = k(g)$$

$$k_2 = k(h)$$

$$t = (r(g), r(h), a(h), a(g)).$$

Then a calculation yields

$$f(g \circ h) = |tf(k_1 k_2) + (1 - t)f(k_1/k_2)|.$$

Divide the pairs  $(g, h)$  in the classes  $\mathcal{H}$ ,  $\mathcal{P}$  and  $\mathcal{E}$  as follows:

$$\begin{aligned} (g, h) \in \mathcal{H} &\Leftrightarrow f(g \circ h) = tf(k_1k_2) + (1 - t)f(k_1/k_2) \geq 2 \\ &\Leftrightarrow t \geq t_1 = (2 - f(k_1/k_2))/(f(k_1k_2) - f(k_1/k_2)) \end{aligned} \tag{1.1}$$

$$\begin{aligned} (g, h) \in \mathcal{P} &\Leftrightarrow f(g \circ h) = -tf(k_1k_2) - (1 - t)f(k_1/k_2) \geq 2 \\ &\Leftrightarrow t \leq t_2 = (-2 - f(k_1/k_2))/(f(k_1k_2) - f(k_1/k_2)) \end{aligned} \tag{1.2}$$

$$(g, h) \in \mathcal{E} \Leftrightarrow t_2 < t < t_1. \tag{1.3}$$

Suppose that  $k_1$  and  $k_2$  are kept fixed but  $t$  is let to vary. If  $t$  increases from  $-\infty$  to  $t_2$ , then  $(g, h) \in \mathcal{P}$  and  $f(g \circ h)$  decreases from  $\infty$  to 2. If  $t_2 < t < t_1$  then  $(g, h) \in \mathcal{E}$  and  $g \circ h$  is elliptic. If  $t$  increases from  $t_1$  to  $\infty$ , then  $(g, h) \in \mathcal{H}$  and  $f(g \circ h)$  increases from 2 to  $\infty$ .

If we conjugate by a Möbius transformation sending  $r(h) \rightarrow 1$ ,  $a(h) \rightarrow 0$ ,  $a(g) \rightarrow \infty$  and  $r(g) \rightarrow t$ , we get Fig. 1. Here the open interval  $]t_2, t_1[$  separates the classes  $\mathcal{P}$  and  $\mathcal{H}$ . Note that the pairs  $(g, h)$  and  $(h, g)$  belong to the same class.

We state two theorems (cf. [S-S], §1).

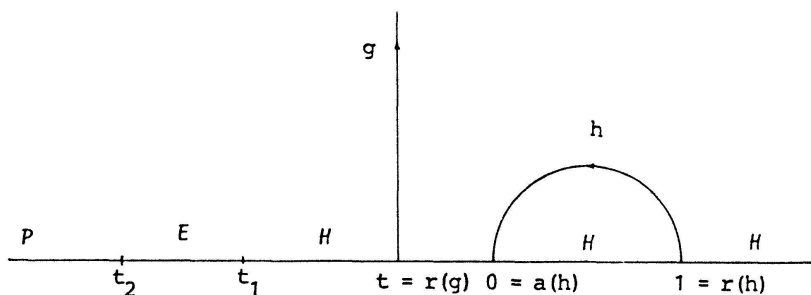


Figure 1.

**THEOREM 1.1.** *For any  $k_1 > 1$ ,  $k_2 > 1$  and  $k_3 > 1$  there exist up to conjugation unique pairs  $(g, h) \in \mathcal{H}$  and  $(g, h) \in \mathcal{P}$  such that  $k_1 = k(g)$ ,  $k_2 = k(h)$  and  $k_3 = k(g \circ h)$ .  $\square$*

**THEOREM 1.2.** *Deform  $a(g)$ ,  $r(g)$ ,  $a(h)$ ,  $r(h)$ ,  $k(g)$  and  $k(h)$  continuously. If  $g$ ,  $h$  and  $g \circ h$  stay hyperbolic during the deformation, then the class of  $(g, h)$  is not changed.  $\square$*

Theorem 1.2 shows that, in later applications, the class of any pair  $(g, h)$  can be kept fixed.

Theorem 1.1 shows that the numbers  $k_1 > 1$ ,  $k_2 > 1$  and  $k_3 > 1$  can be given

freely in the classes  $\mathcal{P}$  and  $\mathcal{H}$ . We consider next the class  $\mathcal{H}$ , impose some natural restrictions on  $(g, h)$  and show that  $k_1$  and  $k_2$  cannot then be simultaneously  $\approx 1$ .

Consider the commutator

$$c = h \circ g^{-1} \circ h^{-1} \circ g.$$

A straightforward calculation (cf. [L], p. 100) yields

$$f(c) = |t(1-t)(f(k_1^2) - 2)(f(k_2^2) - 2) - 2|.$$

Suppose that the axes of  $g$  and  $h$  intersect, i.e.,  $0 < t < 1$  (cf. Fig. 1). If  $c$  is hyperbolic, then either

$$f(c) = t(1-t)(f(k_1^2) - 2)(f(k_2^2) - 2) - 2 > 2 \tag{1.4}$$

or

$$f(c) = 2 - t(1-t)(f(k_1^2) - 2)(f(k_2^2) - 2) > 2$$

Since  $0 < t < 1$ ,  $t(1-t)$  is positive. Hence also  $t(1-t)(f(k_1^2) - 2)(f(k_2^2) - 2) > 0$  which implies that (1.4) holds. Since  $0 < t(1-t) \leq \frac{1}{4}$  for  $0 < t < 1$ , it follows from (1.4) that

$$2 < f(c) \leq \frac{1}{4}(f(k_1^2) - 2)(f(k_2^2) - 2) - 2$$

or

$$(f(k_1^2) - 2)(f(k_2^2) - 2) > 16$$

or

$$(\sqrt{k_1} - 1/\sqrt{k_1})(\sqrt{k_2} - 1/\sqrt{k_2}) > 4. \tag{1.5}$$

If  $k_2 \rightarrow 1$  then  $k_1 \rightarrow \infty$ . We state our result as a theorem.

**THEOREM 1.3.** *Suppose that  $g$  and  $h$  are hyperbolic Möbius transformations of the unit disc  $D$  onto itself. If the axes of  $g$  and  $h$  intersect and the commutator  $h \circ g^{-1} \circ h^{-1} \circ g$  is hyperbolic, then*

$$k(g) > 16/(k(h) - 1)^2. \tag{1.6}$$

*Proof.* Under the hypotheses of the theorem, (1.5) holds with  $k_1 = k(g)$  and  $k_2 = k(h)$ . Since  $0 < 1/\sqrt{k_j} < 1$ ,  $j = 1, 2$ , we get

$$\sqrt{k_1} > \frac{4}{(1/\sqrt{k_2})(k_2 - 1)} > \frac{4}{k_2 - 1}. \quad \square$$

To (1.6) analogous collar inequalities are proved e.g. in [K] and [A]. Retaining the assumptions of Theorem 1.3 we derive an equation satisfied by the multipliers

$$\begin{aligned} k_1 &= k(g), \\ k_2 &= k(h), \\ k_3 &= k(g \circ h), \\ k_4 &= k(h \circ g^{-1} \circ h^{-1} \circ g). \end{aligned}$$

From (1.1) we get

$$t = \frac{f(k_3) - f(k_1/k_2)}{f(k_1 k_2) - f(k_1/k_2)}.$$

Hence by (1.4)

$$\frac{f(k_4) + 2}{(f(k_1^2) - 2)(f(k_2^2) - 2)} = t(1 - t) = \frac{(f(k_3) - f(k_1/k_2))(f(k_1 k_2) - f(k_3))}{(f(k_1 k_2) - f(k_1/k_2))^2}. \quad (1.7)$$

With respect to  $f(k_3)$ , the equation (1.7) is of second degree. This proves the following theorem:

**THEOREM 1.4.** *Suppose that  $g$  and  $h$  are hyperbolic Möbius transformations of the unit disc  $D$  onto itself such that the axes of  $g$  and  $h$  intersect and the commutator  $c = h \circ g^{-1} \circ h^{-1} \circ g$  is hyperbolic. If  $k_1 = k(g)$ ,  $k_2 = k(h)$  and  $k_4 = k(c)$  are known, then  $k_3 = k(g \circ h)$  has two possible values except in the case*

$$f(k_4) + 2 = \frac{1}{3}(f(k_1^2) - 2)(f(k_2^2) - 2) \quad (1.8)$$

when  $k_3$  is uniquely determined.  $\square$

By (1.4), the case (1.8) occurs if and only if  $t = \frac{1}{2}$ .

Denote by  $k_3$  and  $k'_3$  and two possible values in Theorem 1.4. If

$$f(k_3) = tf(k_1 k_2) + (1 - t)f(k_1/k_2), \quad (1.9)$$

then, by (1.7),

$$f(k'_3) = (1 - t)f(k_1 k_2) + tf(k_1/k_2). \quad (1.10)$$

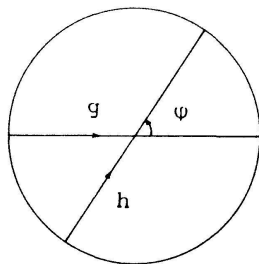


Figure 2. The angle between the axes of  $g$  and  $h$  is  $\varphi$ .

In the case (1.9), conjugate in such a way that  $r(g) \rightarrow -1$ ,  $a(g) \rightarrow 1$ ,  $a(h) \rightarrow e^{i\varphi}$  and  $r(h) \rightarrow -e^{i\varphi}$ ,  $0 < \varphi < \pi$  (Fig. 2). It follows that  $t = \cos^2(\varphi/2)$ . If we denote by  $\varphi'$  the corresponding angle of (1.10), then

$$\cos^2 \frac{\varphi'}{2} = 1 - t = \sin^2 \frac{\varphi}{2}.$$

Hence  $\varphi' = \pi - \varphi$ . In both cases the acute angle between the axes of  $g$  and  $h$  is the same. We have proved the following corollary to Theorem 1.4:

**COROLLARY.** *The axes of  $g$  and  $h$  are orthogonal if and only if (1.8) holds. In other cases  $k(g)$ ,  $k(h)$  and  $k(h \circ g^{-1} \circ h^{-1} \circ g)$  determine uniquely the acute angle between the axes of  $g$  and  $h$ .  $\square$*

Analogous results for Fuchsian groups representing compact Riemann surfaces have been applied e.g. in [S-S].

## 2. Parametrization of a Möbius group with a relation

Let  $\mathcal{H} = \{g_1, h_1, g_2, h_2, \dots, g_s, h_s\}$  be a set of hyperbolic Möbius transformations of  $D$  onto itself. Suppose that the cyclic order of the fixed points is given by Fig. 3 for any  $\gamma \in \{g_2, h_2, \dots, g_{s-1}, h_{s-1}\}$ . Then  $(g_1, h_1) \in \mathcal{H}$  and  $(g_s, h_s) \in \mathcal{H}$ .

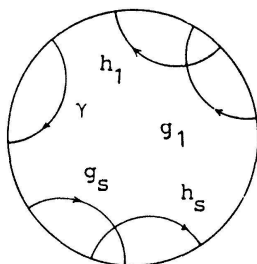


Figure 3.



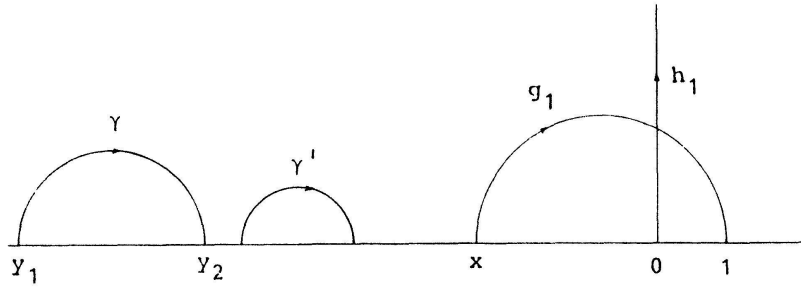


Figure 4.

Suppose the classes of  $(\gamma, g_1)$  and  $(\gamma, h_1)$  are fixed for all  $\gamma \in \{g_2, h_2, \dots, g_s, h_s\}$ .

Conjugate to the upper half-plane  $H$  in such a way that  $r(h_1) \rightarrow 0$ ,  $a(h_1) \rightarrow \infty$ ,  $a(g_1) \rightarrow 1$  and  $r(g_1) \rightarrow x = 1 - 1/t < 0$  (Fig. 4).

Fix the multipliers  $k_1 = k(g_1)$ ,  $k_2 = k(h_1)$  and  $k_3 = k(g_1 \circ h_1)$ . Then, by Theorem 1.1,  $x$  is uniquely determined.

**LEMMA 2.1.** *Under the above assumptions, the numbers  $k(\gamma)$ ,  $k(\gamma \circ g_1)$  and  $k(\gamma \circ h_1)$  determine  $\gamma \in \{g_2, h_2, \dots, g_s, h_s\}$  uniquely.*

*Proof.* Let  $\gamma'$  be another candidate for  $\gamma$ . By Theorem 1.1, the pairs  $(\gamma, g_1)$  and  $(\gamma', g_1)$  as well as the pairs  $(\gamma, h_1)$  and  $(\gamma', h_1)$  are conjugate. Hence we have Möbius transformations  $\psi$  and  $\sigma$  such that

$$\gamma' = \sigma \circ \gamma \circ \sigma^{-1}, \quad \sigma(0) = 0, \quad \sigma(\infty) = \infty$$

$$\gamma' = \psi \circ \gamma \circ \psi^{-1}, \quad \psi(1) = 1, \quad \psi(x) = x.$$

Moreover,  $\psi$  and  $\sigma$  are hyperbolic. Let  $y_1$  and  $y_2$  denote the fixed points of  $\gamma$ . Then

$$\psi(y_1) = \sigma(y_1)$$

$$\psi(y_2) = \sigma(y_2)$$

or

$$\frac{\psi(y_1)}{\psi(y_2)} = \frac{y_1}{y_2}$$

since  $\sigma(z) = kz$ , for  $k = k(\sigma)$  or  $k^{-1} = k(\sigma)$ .

The next Lemma shows that  $\psi = id$  and  $\gamma' = \gamma$ .

LEMMA 2.2. Let  $\psi : H \rightarrow H$ ,

$$\psi(z) = \frac{(ak - r)z - ar(k - 1)}{(k - 1)z - kr + a},$$

$k > 0$ ,  $k \neq 1$ , be a hyperbolic Möbius transformation with the fixed points  $a$  and  $r$  and with the multiplier  $\max(k, 1/k)$ . If there exist real numbers  $y_1, y_2$  such that  $y_2\psi(y_1) = y_1\psi(y_2)$ , then

$$k = \frac{r(y_1 - a)(y_2 - a)}{a(y_1 - r)(y_2 - r)}. \quad \square \tag{2.1}$$

The proof of Lemma 2.2 is a direct calculation. To prove Lemma 2.1 assume that  $\psi$  is not the identity. Then we can apply, to this  $\psi$ , Lemma 2.2 with  $a = 1$ ,  $r = x < 0$ ,  $y_1 < x$  and  $y_2 < x$ . Hence  $k < 0$  which is impossible. Lemma 2.1 is hereby proved.  $\square$

At this stage we have proved the following result:

THEOREM 2.1. Let  $\mathcal{K} = \{g_1, h_1, \dots, g_s, h_s\}$  be a set of hyperbolic Möbius transformations of  $D$  onto itself. Suppose that the cyclic order of the fixed points is given by Fig. 3 and the classes of the pairs  $(\gamma, g_1)$  and  $(\gamma, h_1)$  are fixed for all  $\gamma \in \{g_2, h_2, \dots, g_s, h_s\}$ . Then the multipliers of the following  $6s - 3$  Möbius transformations determine  $\mathcal{K}$  uniquely up to conjugation:

$$\begin{aligned} g_j, h_j, g_j \circ h_1, & \quad j = 1, \dots, s \\ h_j \circ h_1, g_j \circ g_1, h_j \circ g_1, & \quad j = 2, \dots, s. \quad \square \end{aligned}$$

Suppose next that the commutator

$$c_s = h_s \circ g_s^{-1} \circ h_s^{-1} \circ g_s$$

is hyperbolic and that  $c_s$  has a given representation in the group generated by the set  $\{g_1, h_1, \dots, g_{s-1}, h_{s-1}\}$ . Suppose also that  $c_s$  and  $h_1$  have no common fixed points. Let the multipliers of the following  $6s - 9$  Möbius transformations be given:

$$\begin{aligned} g_j, h_j, g_j \circ h_1, & \quad j = 1, \dots, s - 1. \\ h_j \circ h_1, g_j \circ g_1, h_j \circ g_1, & \quad j = 2, \dots, s - 1. \end{aligned}$$

Then, by Theorem 2.1, the set  $\{g_1, h_1, \dots, g_{s-1}, h_{s-1}\}$  is determined up to conjugation. Hence  $k(c_s)$  is uniquely determined.

Assume that also  $k(g_s)$  and  $k(h_s)$  are given. Then, by Theorem 1.4,  $k(g_s \circ h_s)$  has at most two possible values. Choose one of these. By Theorem 1.1,

$$t_s = (r(g_s), r(h_s), a(h_s), a(g_s))$$

is uniquely determined.

Finally, give  $k(h_s \circ h_1)$ . Let  $(g'_s, h'_s)$  be another candidate for the pair  $(g_s, h_s)$ . Since both have the same cross-ratio  $t_s$ , there exists a Möbius transformation  $\psi$  such that

$$h'_s = \psi \circ h_s \circ \psi^{-1}$$

$$g'_s = \psi \circ g_s \circ \psi^{-1}.$$

Since  $(g_s, h_s)$  and  $(g'_s, h'_s)$  have the same commutator  $c_s$ ,  $\psi$  is hyperbolic and has the same axis as  $c_s$ . Denote by  $a$  and  $r$  the common fixed points of  $c_s$  and  $\psi$ .

Denote  $y_1 = r(h_s)$ ,  $y_2 = a(h_s)$ . Since  $k(h_s \circ h_1) = k(h'_s \circ h_1)$ , it follows as in Lemma 2.1 that  $y_2\psi(y_1) = y_1\psi(y_2)$ . By Lemma 2.2, we have two alternatives. Either  $\psi = id$  or

$$\psi(z) = \frac{(ak - r)z - ar(k - 1)}{(k - 1)z - kr + a},$$

$$k = \frac{r(y_1 - a)(y_2 - a)}{a(y_1 - r)(y_2 - r)}. \tag{2.2}$$

Hence we have in general two alternatives for the pair  $(g_s, h_s)$ .

**THEOREM 2.2.** *Let  $\mathcal{H} = \{g_1, h_1, \dots, g_s, h_s\}$  be a set of hyperbolic Möbius transformations of  $D$  onto itself. Suppose that the cyclic order of the fixed points is*

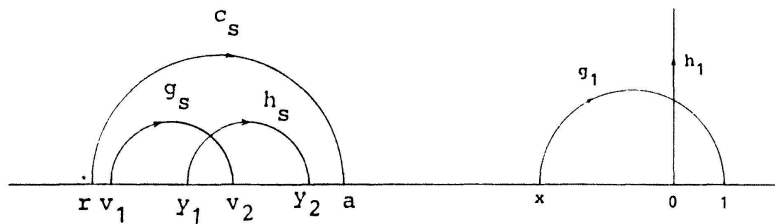


Figure 5.

given by Fig. 3 and the classes of the pairs  $(\gamma, g_1)$  and  $(\gamma, h_1)$  are fixed for all  $\gamma \in \{g_2, h_2, \dots, g_s, h_s\}$ . Suppose that the commutator  $c_s = h_s \circ g_s^{-1} \circ h_s^{-1} \circ g_s$  is hyperbolic and has a given representation in the subgroup generated by  $\{g_1, h_1, \dots, g_{s-1}, h_{s-1}\}$ . Suppose that  $c_s$  and  $h_1$  have no common fixed points. Let the multipliers of the following  $6s - 6$  Möbius transformations be given:

$$\begin{aligned} g_j \circ h_j, & \quad j = 1, \dots, s, \\ g_j \circ h_1, & \quad j = 1, \dots, s - 1, \\ h_j \circ h_1, & \quad j = 2, \dots, s, \\ g_j \circ g_1 \quad \text{and} \quad h_j \circ g_1, & \quad j = 2, \dots, s - 1. \end{aligned} \tag{2.3}$$

Then the set  $\{g_1, h_1, \dots, g_{s-1}, h_{s-1}\}$  is uniquely determined up to conjugation. If  $\{g_1, h_1, \dots, g_{s-1}, h_{s-1}\}$  is fixed then  $(g_s, h_s)$  has at most four possible alternatives.  $\square$

The  $6s - 6$  multipliers in Theorem 2.2 give in fact local parameters for the Teichmüller space of the group generated by  $\mathcal{H}$ . To obtain a global parametrization for the Teichmüller space, additional multipliers are needed.

We retain the assumptions of Theorem 2.2. We add to the  $6s - 6$  multipliers (2.3) also  $k(g_s \circ h_1)$  and suppose that the cyclic order of the fixed points of  $c_s$  is given by Fig. 5. Denote  $v_1 = r(g_s)$  and  $v_2 = a(g_s)$ . Then similarly as for  $h_s$ ,  $v_2 \psi(v_1) = v_1 \psi(v_2)$ . Hence we get a second expression for the number  $k$  in (2.2):

$$k = \frac{r(v_1 - a)(v_2 - a)}{a(v_1 - r)(v_2 - r)}.$$

Hence

$$\frac{v_1 - a}{v_1 - r} \cdot \frac{v_2 - a}{v_2 - r} = \frac{y_1 - a}{y_1 - r} \cdot \frac{y_2 - a}{y_2 - r}$$

or

$$(v_1, y_1, a, r) = (y_2, v_2, a, r).$$

But this is impossible, since  $(v_1, y_1, a, r) > 1$  and  $(y_2, v_2, a, r) < 1$ . Hence  $\psi = id$ .

The set  $\mathcal{H}$  is not uniquely determined up to conjugation by the  $6s - 5$  multipliers (i.e. the  $6s - 6$  multipliers of the transformations (2.3) plus  $k(g_s \circ h_1)$  since  $k(g_s \circ h_s)$  still has in general two possible values. Theorem 1.4 and its Corollary give a clear picture of these two alternatives.

If we finally give  $k(g_s \circ h_s)$ , then  $\mathcal{K}$  is uniquely determined up to conjugation.

**COROLLARY.** *Suppose that  $\mathcal{K}$  satisfies the hypotheses of Theorem 2.2. If the cyclic order of the fixed points of  $c_s$  is given by Fig. 5, then the multipliers of the following  $6s - 4$  Möbius transformations determine  $\mathcal{K}$  uniquely up to conjugation:*

$$\begin{aligned} g_j, h_j, & \quad j = 1, \dots, s \\ g_j \circ h_1, & \quad j = 1, \dots, s \\ h_j \circ h_1, & \quad j = 2, \dots, s \\ g_j \circ g_1, h_j \circ g_1, & \quad j = 2, \dots, s - 1 \\ g_s \circ h_s. & \end{aligned}$$

Note that the  $6s - 3$  multipliers in Theorem 2.1 give global parameters also for a set  $\mathcal{K}$  without any relation.

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