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A non-quasicircle with almost smooth mapping functions

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1. Introduction

A Jordan curve Γ in the ω-plane is a quasicircle (or quasiconformal curve) if, for all $\omega_1, \omega_2 \in \Gamma$ and any ω on $C(\omega_1, \omega_2)$, the arc of smaller diameter between ω_1 and ω_2,

$$\frac{|\omega_1 - \omega| + |\omega - \omega_2|}{|\omega_1 - \omega_2|} < M, \tag{1.1}$$

for a constant $M > 0$ depending on Γ.

Let f be a conformal mapping of the disk $D = \{\zeta : |\zeta| < 1\}$ onto Ω, the interior of Γ, and let f^* be a conformal mapping of $D^* = \{\zeta : |\zeta| > 1\}$ onto Ω^*, the exterior of Γ. Since Γ is a Jordan curve, these functions extend continuously to homeomorphisms of ∂D with Γ. We shall say that a function g is Lip (α), or Hölder continuous with exponent α in its domain if there exist $K > 0$ and $\alpha > 0$ for which

$$|g(x) - g(y)| \leq K |x - y|^{\alpha} \tag{1.2}$$

for all x and y in the domain of g.

It is well known that if Γ is a quasicircle, then f, f^{-1}, f^* and f^{*-1} are Hölder continuous on the closure of their domains, and in fact the Hölder exponents can be expressed in terms of the M in (1.1) ([5, 8]). The question then arose as to whether Hölder continuity of the four functions implies that Γ is a quasicircle. This is true if f and f^{-1} (or f^* and f^{*-1}) are Lip (1). The question was settled by Becker and Pommerenke [1] who constructed a curve Γ which is not a quasicircle, but for which the functions are all Hölder continuous. The exponents, however, are less than $\frac{1}{4}$, and the question remained as to how large the exponents can be with Γ not a quasicircle. Since f^{-1} and f^{*-1} are Lip (α) for $\alpha > \frac{1}{2}$ whenever Γ is a quasicircle [5], one might conjecture that Γ is a quasicircle if the exponents are sufficiently large (but still less than 1). We prove the following

THEOREM. *There exists a non-quasicircle Γ for which the mapping f of $\bar{D}$ onto $\bar{\Omega}$ is* Lip (α) *for all $\alpha < 1$ and the mapping f^* of $\bar{D}^*$ onto $\bar{\Omega}^*$ is* Lip (1). *The inverse mappings f^{-1} and f^{*-1} are* Lip (α) *for all $\alpha < 1$ on $\bar{\Omega}$ and $\bar{\Omega}^*$ respectively. In fact, $|f'(e^{i\theta})|$ is exponentially integrable while $1/|f'(e^{i\theta})|$, $|f^{*\prime}(e^{i\theta})|$ and $|1/f^{*\prime}(e^{i\theta})|$ are uniformly bounded on ∂D.*

Let μ be Lebesgue measure on $[0, 2\pi)$ and define

$$m(\lambda, f') = \mu(\{\theta \in [0, 2\pi) : |f'(e^{i\theta})| > \lambda\}), \tag{1.3}$$

to be the distribution function of $|f'|$. Using the fact that for constant A,

$$\int_0^{2\pi} e^{A\,|f'(e^{i\theta})|}\,d\theta = 2\pi + A\int_0^{\infty} e^{A\lambda} m(\lambda, f')\,d\lambda, \tag{1.4}$$

the exponential integrability of $|f'|$ follows if there exists M such that for $\lambda > M$ and $B > A$,

$$m(\lambda, f') \le e^{-B\lambda}. \tag{1.5}$$

We shall construct Γ such that an inequality like (1.5) holds for any $B > 0$, so that the integral in (1.4) will be finite for any $A > 0$. Γ will be constructed so that $|f'|$ is non-zero and finite on ∂D and $m(\lambda, f')$ will be estimated using the techniques in [4] and [6], where curves were constructed with all mappings Lip (1), but Γ respectively not smooth or "asymptotically conformal." In the last section we shall mention some other phenomena exhibited by the example, in connection with the Muckenhoupt A_∞ condition for $|f'|$.

2. Construction of the curve and estimation of derivatives

Let $S_1 = \{z = x + iy : |y| < \pi/2\}$ and $S_2 = \{z = x + iy : \pi/2 < y < 3\pi/2\}$. We shall construct a strip domain Σ_1, in the $w = u + iv$ plane, which is bounded by $C_2 = \{w : v = -\pi/2\}$ and a Jordan arc C_1 with $-\infty$ and $+\infty$ as endpoints. This C_1 will be very close to the line $v = \pi/2$. The strip Σ_2 will be the "complementary" strip bounded by C_1 and $C_2' = \{w : v = 3\pi/2\}$.

We then define $w_1(z) = u_1(z) + iv_1(z)$ and $w_2(z) = u_2(z) + iv_2(z)$ to be the conformal mappings of S_1 and S_2 respectively onto Σ_1 and Σ_2, with $w_j(-\infty) = -\infty$,

$w_j(+\infty) = +\infty$, $w_j(\pi i/2) = \pi i/2 \in \partial\Sigma_j$ for $j = 1, 2$. We denote by $z_j(w)$ the inverse of $w_j(z)$, for $j = 1, 2$.

Next, define

$$\omega(w) = \frac{e^w - 1}{e^w + 1}, \qquad w \in \overline{\Sigma_1 \cup \Sigma_2}$$

and

$$\zeta(z) = \frac{e^z - 1}{e^z + 1}, \qquad z \in \overline{S_1 \cup S_2}.$$

Then S_1 and S_2 correspond to the interior and exterior of the unit disk in the $\zeta = \xi + i\eta$ plane, while Σ_1 and Σ_2 correspond to the interior Ω and exterior Ω^* of a closed Jordan curve Γ in the $\omega = s + it$ plane. C_1 will be constructed so that the image Γ of $C_1 \cup C_2$ is not a quasicircle. The function $f(\zeta) = \omega(w_1(z(\zeta)))$ is a conformal mapping of D onto Ω and $f^*(\zeta) = \omega(w_2(z(\zeta)))$ is a conformal mapping of D^* onto Ω^*. Both functions may be assumed to be extended continuously to the closures of their domains.

Now, by the chain rule we have, for $\zeta \neq \pm 1$,

$$\begin{aligned} \left|\frac{df}{d\zeta}(\zeta)\right| &= \left|\frac{d\omega}{dw}\right| \left|\frac{dw_1}{dz}\right| \left|\frac{dz}{d\zeta}\right| \\ &= \left|\frac{2e^w}{(e^w + 1)^2}\right| \left|\frac{dw_1}{dz}\right| \left|\frac{(e^z + 1)^2}{2e^z}\right| \\ &= \left|\frac{dw_1}{dz}\right| e^{x - u_1(z)} \left|\frac{1 + e^{-z}}{1 + e^{-w}}\right|^2 \\ &= \left|\frac{dw_1}{dz}\right| e^{x - u_1(z)}(1 + o(1)), \quad \text{as} \quad x \to +\infty. \end{aligned} \tag{2.1}$$

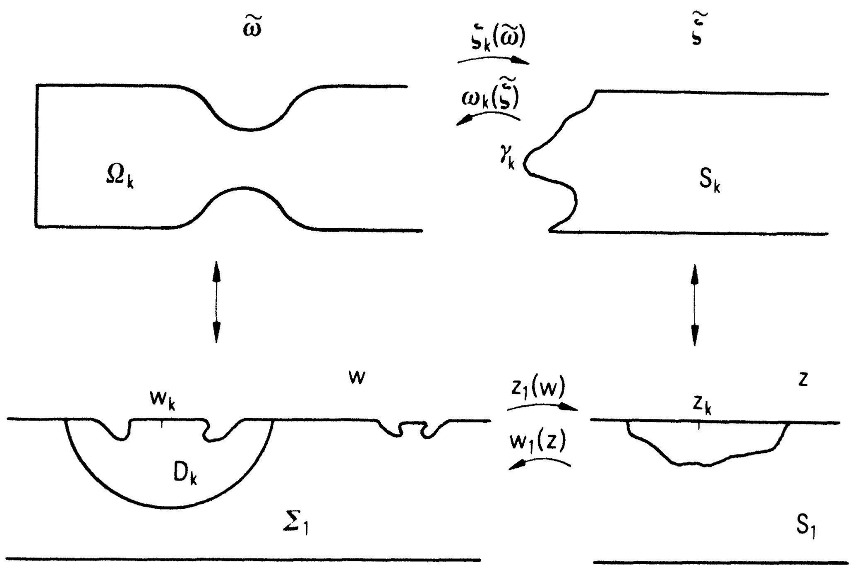
Our goal here is to estimate the distribution function (1.3) and to show that $1/|f'(e^{i\theta})|$ is bounded uniformly from above, so that we must estimate $|dw_1/dz|$ and $x - u_1(z)$ for points on $\partial\Sigma_1$. Similar considerations connect $|f^{*\prime}|$ to $|dw_2/dz|$ and $x - u_2(z)$ on $\partial\Sigma_2$.

We now construct C_1. We start with "building blocks" as in [6]. For each $k \geq 3$ consider the following circles in the $\tilde{\omega} = s + it$ plane.

$$T_1 : (t - \pi)^2 + (s - k)^2 = (\pi - k^{-1/2})^2$$

$$T_2 : t^2 + (s - u_2)^2 = \frac{\pi^2}{4}, \qquad T_3 : t^2 + (s - u_3)^2 = \frac{\pi^2}{4}$$

where $u_2 = k - ((\pi/2 - k^{-1/2})^2 + 2\pi(\pi/2 - k^{-1/2}))^{1/2}$ and $u_3 = k + ((\pi/2 - k^{-1/2})^2 + 2\pi(\pi/2 - k^{-1/2}))^{1/2}$, so that T_2 and T_3 are tangent to T_1. Let $L = \{s + it : s \geq 0,\ t = \pi/2\}$. We trace a curve Γ_k as follows. Starting at $\pi i/2$ move to the right, first along L to T_2, then on T_2 to T_1, on T_1 to T_3, on T_3 to L and then on L to $+\infty$. Let Γ_k' be the reflection of Γ_k across the s-axis and let Ω_k be the "half strip" bounded by $\Gamma_k \cup \Gamma_k' \cup \{ti : |t| \leq \pi/2\}$. (See the figure, upper left.)



For $\tilde{\omega} \in \Omega_k$, let $w_k(\tilde{\omega}) = -ie^{-\tilde{\omega}} + w_k$ for $w_k = g(k) + \pi i/2$. The function $g(k) = e^{k \ln k}$ will guarantee (1.5). Other choices of $g(k)$ will yield corresponding integrability of $|f'|$, as will be evident. We shall work with $g(k) = e^{k \ln k}$ for our purposes. Let D_k be the image of Ω_k under this $w_k(\tilde{\omega})$. Let $\Sigma = \{w = u + iv : |v| < \pi/2\}$. Delete from Σ the half disks $\{w : |w - w_k| < 1,\ v < \pi/2\}$ and replace them with the D_k. The resulting domain is then Σ_1, and we let $\Sigma_2 = \{w = u + iv : -\pi/2 < v < 3\pi/2\} - \bar{\Sigma}_1$. The upper boundary of Σ_1 is then the curve C_1, with a sequence of shrinking and narrowing double bumps going to $+\infty$. Under the mapping $\omega = (e^w - 1)/(e^w + 1)$, Σ_1 corresponds to a domain Ω which is nearly a unit disk, with a sequence of double bumps converging to $\omega = 1$. It is clear that (1.1) fails for $\partial\Omega$, because the bottlenecks on the Ω_k have width $2/\sqrt{k}$. A rigorous argument can be easily obtained from that on page 229 of [6].

As previewed above we consider the conformal mappings $w_j(z)$ ($j = 1, 2$) from S_j onto Σ_j with $w_j(-\infty) = -\infty$, $w_j(+\infty) = +\infty$ and $w_j(\pi i/2) = \pi i/2$, and define f and f^* on $\bar{D}$ and $\bar{D}^*$ accordingly. We shall work with $|f'|$ and $1/|f'|$; the proofs for the exterior mappings are simpler, as will be noted later. In order to use (2.1) we first observe that there exists K_1, constant, such that for all $z \in S_1$,

$$-K_1 < x - u_1(z) < K_1. \tag{2.2}$$

The upper bound follows as in Lemma 5 of [6] from the Ahlfors upper inequality (see [2]) while the lower bound follows similarly from the Ahlfors lower inequality (the Ahlfors distortion theorem). We now turn to the estimation of $|dw_1/dz|$, which depends on the Ω_k. For each k (≥ 3) let $z_k = z_1(w_k)$ and define $\zeta_k(z) = \mathrm{Log}\,(1/(z - z_k)) - \pi i/2$ for $z \in S_1$, so that $|\mathrm{Im}\, \zeta_k(z)| < \pi/2$. For $w_k(\tilde{\omega}) = -ie^{-\tilde{\omega}} + w_k$, the function $\zeta_k(\tilde{\omega}) = \zeta_k(z_1(w_k(\tilde{\omega})))$ maps Ω_k conformally onto a half strip S_k which is bounded by the horizontal rays from $\zeta_k(\pi i/2)$ and $\zeta_k(-\pi i/2)$ to $+\infty$ and by an arc γ_k in $\{|\mathrm{Im}\, \zeta| < \pi/2\}$. As with (2.1) we see that for $z \in z_1(D_k)$, with $z_1(w)$ the inverse of $w_1(z)$ and $\omega_k(\zeta)$ the inverse of $\zeta_k(\tilde{\omega})$, we have

$$\left|\frac{dw_1}{dz}\right| = \left|\frac{d\omega_k}{d\zeta}\right| e^{\xi - s_k(\zeta)}. \tag{2.3}$$

Here $\zeta = \zeta_k(z)$ and $\omega_k(\zeta) = s_k(\xi + i\eta) + it_k(\xi + i\eta)$.

In order to estimate $\xi - s_k(\zeta)$ on the horizontal boundary of S_k, we shall again use the Ahlfors inequalities. For a given Ω_k we let $\sigma(s)$ denote the vertical crosscut $\{\mathrm{Re}\, \tilde{\omega} = s\} \cap \Omega_k$. Let $\theta_k(s)$ be the length of $\sigma(s)$. We then define

$$\underline{\xi}_k(s) = \min \xi_k(\tilde{\omega}) \quad \tilde{\omega} \in \sigma(s), \qquad \bar{\xi}_k(s) = \max \xi_k(\tilde{\omega}) \quad \tilde{\omega} \in \sigma(s)$$

where $\xi_k(\tilde{\omega}) = \mathrm{Re}\, \zeta_k(\tilde{\omega})$. We first prove

LEMMA 1. *For $\zeta = \xi + i\eta \in S_k$, we have, for constant K_2,*

$$-K_2 < \xi - s_k(\zeta) < K_2 k \quad \textit{for each } k. \tag{2.4}$$

Proof. We begin by showing that there exists K_3 for which

$$-K_3 \leq \xi_k(it) < K_3 \quad \text{for all } k. \tag{2.5}$$

Because Σ_1 is so nearly a parallel strip $z_1(w)$ has an unrestricted derivative at $\infty: z_1(w) - w \to l$ as $w \to \infty$ for a real l ([10], [11]). Choose M such that for $\mathrm{Re}\, w > M$ and $w \in \bar{\Sigma}_1$ we have

$$|z_1(w) - w - l| < \tfrac{1}{10},$$

and choose N such that for $k > N$, all D_k lie in the half plane $\{\mathrm{Re}\, w > M\}$. Then for $|t| \leq \pi/2$,

$$|z_1(-ie^{-it} + w_k) - (-ie^{-it} + w_k) - l| < \tfrac{1}{10}$$

and from $|z_k - w_k - l| < \frac{1}{10}$, we obtain

$$|(z_1(-ie^{-it} + w_k) - z_k) - (-ie^{-it})| < \tfrac{2}{10}$$

Thus

$$\tfrac{4}{5} < |z_1(-ie^{-it} + w_k) - z_k| < \tfrac{6}{5}$$

and

$$\log \tfrac{5}{6} < \xi_k(it) = -\log |z_1(-ie^{-it} + w_k) - z_k| < \log \tfrac{5}{4}$$

from which (2.5) follows.

From the Ahlfors distortion theorem we have

$$\underline{\xi}_k(s) - \bar{\xi}_k(0) \geq \int_0^s \frac{\pi}{\theta_k(t)}\, dt - 2\pi$$

so that

$$\xi_k(\bar{\omega}) - s \geq \int_0^s \frac{\pi - \theta_k(t)}{\theta_k(t)}\, dt - 2\pi + \bar{\xi}_k(0)$$

and the left side of (2.4) follows from (2.5) and the fact that $\pi - \theta_k(t) \geq 0$. Next we apply the Ahlfors upper inequality as expressed in Theorem 3 of [2], to see that

$$\bar{\xi}_k(s) - \underline{\xi}_k(0) \leq \int_0^s \frac{\pi\, dt}{\theta(t)} + k\frac{\pi}{2} + \pi k^{1/2},$$

so that

$$\xi_k(\bar{\omega}) - s \leq \int_0^s \frac{\pi - \theta(t)}{\theta(t)}\, dt + \frac{\pi}{2} k + \pi k^{1/2} + \underline{\xi}_k(0).$$

Then the right inequality of (2.4) follows from the above, (2.5) and the construction of Ω_k.

LEMMA 2 *There exist positive K_4 and M, independent of k, such that for $\xi \in \partial S_k$ with* $\operatorname{Re} \xi > M$,

$$\frac{1}{K_4} < \left|\frac{d\omega_k}{d\xi}\right| < K_4 k^{1/2}.$$

The proof of Lemma 2 is essentially that of Lemma 3 of [4]. (See also Lemma 7 of [6].) Briefly, the right inequality holds because for each $\tilde{\omega} \in \partial\Omega_k$ with $\operatorname{Re}\tilde{\omega} > \pi/2$, one may inscribe a circle of radius at least $k^{-1/2}$ in Ω_k, tangent to $\partial\Omega_k$ at $\tilde{\omega}$ and with center on the s axis. Furthermore, the image of the s axis is asymptotic to the ξ axis. One then bounds $|d\omega_k/d\xi|$ by a Schwarz lemma argument. The lower bound is simpler in that at each $\tilde{\omega} \in \partial\Omega_k$ there is a circle of radius $\pi - k^{-1/2}$ in the exterior of Ω_k, tangent to $\partial\Omega_k$ at $\tilde{\omega}$.

It is now evident from (2.3) and Lemmas 1 and 2 that for $z \in \partial\Sigma_1 \cap D_k$, we have

$$\frac{1}{K_4} e^{-K_2} \leq \left|\frac{dw_1}{dz}\right| \leq K_4 k^{1/2} e^{K_2 k} < e^{K_5 k}. \tag{2.6}$$

At every other point of $\partial\Sigma_1$, there are tangent circles interior and exterior to $\partial\Sigma_1$ with radius π, and the image of the x axis under $w_1(z)$ is asymptotic to the u axis in S_1. Thus there exists $K_6 > 0$ for which

$$\frac{1}{K_6} < \left|\frac{dw_1}{dz}\right| < K_6$$

on the rest of $\partial\Sigma_1$. It now follows from (2.1), (2.2) and (2.6) that $1/|f'(e^{i\theta})| \in L^\infty(\partial D)$.

Next we must estimate the length of the image of $\partial\Sigma_1 \cap D_k$ under $\omega(w) = (e^w - 1)/(e^w + 1)$, recalling that D_k is centered at $w_k = g(k) + i\pi/2$.

Let $z'_k = z_1(w_k - 1) = x'_k + i\pi/2$, $z''_k = z_1(w_k + 1) = x''_k + i\pi/2$, $\zeta'_k = \zeta(z'_k)$ and $\zeta''_k = \zeta(z''_k)$ so that ζ'_k and ζ''_k are the endpoints of the interval $I_k \subset \partial D$ which corresponds to $\partial D_k \cap \Sigma_1$. Since $z_1(w)$ has an unrestricted derivative l at $+\infty$, $x'_k \to g(k) - 1 + l$ and $x''_k \to g(k) + 1 + l$ as $k \to \infty$. Thus for $|I_k|$ the length of I_k, we have

$$\begin{aligned}
|I_k| &= \operatorname{Arg} \zeta(z'_k) - \operatorname{Arg} \zeta(z''_k) \\
&= \operatorname{Arg}\left(\frac{e^{z'_k} - 1}{e^{z'_k} + 1}\,\frac{e^{z''_k} + 1}{e^{z''_k} - 1}\right) \\
&= \operatorname{Arg}\left(\frac{ie^{x'_k} - 1}{ie^{x'_k} + 1}\,\frac{ie^{x''_k} + 1}{ie^{x''_k} - 1}\right) \\
&= \operatorname{Arg}\left(\frac{e^{x'_k + x''_k} + 1 - i(e^{x'_k} - e^{x''_k})}{e^{x'_k + x''_k} + 1 + i(e^{x'_k} - e^{x''_k})}\right) \\
&= 2\tan^{-1}\frac{e^{x''_k} - e^{x'_k}}{e^{x''_k + x'_k} + 1} \leq 2\tan^{-1}\frac{e^{x''_k - x'_k} - 1}{e^{x''_k}}.
\end{aligned}$$

Since $x_k'' - x_k' \to 2$ and $x_k'' \to g(k) + l + 1$ we see that there exists $K_7 > 0$ for which

$$|I_k| < K_7 e^{-g(k)} \quad \text{for each } k, \text{ noting that } k \geq 3. \tag{2.7}$$

From (2.6), we obtain, for $g(k) = e^{k \ln k}$ and a positive constant K_0

$$\begin{aligned} \mu\{\theta : |f'(e^{i\theta})| > K_0 e^{K_5 k}\} &< \sum_{n=k+1}^{\infty} |I_n| \\ &< \sum_{n=k+1}^{\infty} K_7 \exp(-e^{n \log n}) \\ &< K_8 \exp(-e^{k \log k}) \end{aligned} \tag{2.8}$$

for a positive constant K_8.

Now let $\lambda_k = e^{K_5 k}$. Then for any $A > 0$,

$$\begin{aligned} A \int_{\lambda_3}^{\infty} e^{A\lambda} m(\lambda, f')\, d\lambda &= \sum_{k=3}^{\infty} A \int_{\lambda_k}^{\lambda_{k+1}} e^{A\lambda} m(\lambda, f')\, d\lambda \\ &\leq \sum_{k=3}^{\infty} m(\lambda_k, f') A \int_{\lambda_k}^{\lambda_{k+1}} e^{A\lambda}\, d\lambda \\ &\leq \sum_{k=3}^{\infty} m(\lambda_k, f') e^{A\lambda_{k+1}} \\ &\leq \sum_{k=3}^{\infty} K_8 \exp(-e^{k \log k} + Ae^{K_5(k+1)}), \end{aligned}$$

where the last inequality follows from (2.8). Since this series converges for any $A > 0$, it follows from (1.4) that $|f'(e^{i\theta})|$ is exponentially integrable to any power.

The boundedness of $|f^{*\prime}|$ and $1/|f^{*\prime}|$ follow in the same way as that of $1/|f'|$. The argument is applied to Ω_k^* which is bounded by Γ_k, its reflection across $s = \pi$ and $\{ti : \pi/2 < t < 3\pi/2\}$. Rather than a narrowing, Ω_k^* has a widening, so that a disk inside Ω_k^* with radius $\pi/2$ is tangent to Γ_k at any $\tilde{\omega} \in \Gamma_k$ (Re $\tilde{\omega} > \pi/2$).

3. Hölder continuity and further remarks

Since $|f^{*\prime}|$ is bounded on ∂D, f^* is in Lip (1). An application of Hölder's inequality shows that $f \in \text{Lip}(\alpha)$ on $\bar{D}$, for any $\alpha < 1$, because $f' \in L^p$ for any $p < \infty$. The Hölder continuity of the inverse functions is less obvious, as Γ is not a quasicircle. However, by a theorem of Pommerenke [9, Theorem 1], the fact that

$|f^{*\prime}|$ is bounded above and below on ∂D implies that Γ is an "exterior quasicircle," which is defined as follows. For $\omega_1, \omega_2 \in \Gamma$, let

$$d_{\Omega^*}(\omega_1, \omega_2) = \inf_C \operatorname{diam} C$$

where C ranges over all arcs which lie in Ω^* except for their endpoints, ω_1 and ω_2. We say that Γ is an "exterior quasicircle" if there exists $M > 0$ such that for every $\omega_1, \omega_2 \in \Gamma$,

$$\operatorname{diam} C(\omega_1, \omega_2) \leq M d_{\Omega^*}(\omega_1, \omega_2).$$

With the corresponding definition of "interior quasicircle," it is easy to see that if Γ is both an interior and exterior quasicircle then it is a quasicircle.

In [4, Corollary to Theorem 1] it is shown that if $f \in \operatorname{Lip}(\alpha)$ on ∂D and if Γ is a quasicircle then $f^{*-1} \in \operatorname{Lip}(1/(2-\alpha))$ on Γ. This proof in fact only requires that Γ be an exterior quasicircle, for then the result of Lemma 4 in [7] holds and the fact that $f^{*-1} \in \operatorname{Lip}(1/(2-\alpha))$ on Γ follows exactly as in the proof of Theorem 2 in [7]. Thus f^{*-1} is Hölder continuous for any exponent less than 1.

We now turn to the proof that f^{-1} is $\operatorname{Lip}(\alpha)$ for all $\alpha < 1$. Let $s(\omega)$ denote arclength on Γ, starting at some $\omega_0 \in \Gamma$, proceeding in the positive direction to ω. Choose $\omega_1, \omega_2 \in \Gamma$, and let $f^*(\zeta_i) = \omega_i$, $i = 1, 2$. Since f^* is $\operatorname{Lip}(1)$ on ∂D, we have for some $K > 0$

$$|s(\omega_1) - s(\omega_2)| < K\,|\zeta_1 - \zeta_2| = K\,|f^{*-1}(\omega_1) - f^{*-1}(\omega_2)| \tag{3.1}$$

The Hölder continuity of f^{*-1} yields, for some $K_1 > 0$,

$$|f^{*-1}(\omega_1) - f^{*-1}(\omega_2)| < K_1\,|\omega_1 - \omega_2|^\alpha \tag{3.2}$$

for any $\alpha < 1$. But from the rectifiability of Γ it follows that f^{-1} is absolutely continuous on Γ and

$$\begin{aligned} |f^{-1}(\omega_1) - f^{-1}(\omega_2)| &= \left| \int_{s(\omega_1)}^{s(\omega_2)} f^{-1\prime}(s)\, ds \right| \\ &\leq K_2\,|s(\omega_1) - s(\omega_2)| \end{aligned}$$

where $|f^{-1\prime}| \leq K_2$. This together with (3.1) and (3.2) yields the Hölder continuity of f^{-1} for any exponent $\alpha < 1$.

This example is also of interest in connection with the Muckenhoupt A_∞

condition for $|f'|$, which is equivalent to the existence of $\delta > 0$ and $M > 0$ such that for any interval $I \subset \partial D$,

$$\begin{aligned}\frac{1}{M}\left(\frac{1}{|I|}\int_I |f'(z)|^{1+\delta}\,|dz|\right)^{1/1+\delta} &\leq \frac{1}{|I|}\int_I |f'(z)|\,|dz| \\ &\leq M\left(\frac{1}{|I|}\int |f'(z)|^{-\delta}\,|dz|\right)^{-1/\delta}.\end{aligned} \tag{3.3}$$

This of course implies that $|f'| \in L^{1+\delta}$ and $|1/f'| \in L^{\delta}$; it also implies that $\log f'$ is of bounded mean oscillation.

We shall say that Γ has bounded arclength – interior distance ratio if there exists a constant $M > 0$ such that,

$$\frac{l(C(\omega_1, \omega_2))}{d_\Omega(\omega_1, \omega_2)} < M \tag{3.4}$$

where $l(\ )$ denotes arclength. A corresponding definition holds for bounded arclength – exterior distance ratio. If (3.4) holds then Γ is an interior quasicircle. Pommerenke [9] has shown that (3.4) is equivalent to the condition that Ω is a Smirnov domain ($\log|f'| \in H^1$) and $|f'|$ satisfies the A_∞ condition.

By a result of Jerison [3], the rectifiability of our Γ and the fact that $1/f'$ is bounded imply that Ω is a Smirnov domain, so that Pommerenke's theorem implies that $|f'|$ does not satisfy the A_∞ condition (since (3.4) fails for our Γ). Thus, our example yields a function $|f'|$ which is exponentially integrable, with $1/|f'|$ bounded, but for which (3.3) fails. Furthermore $\log|f'|$ is of bounded mean oscillation, since $\arg f'$ is bounded on ∂D.

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