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Extremal length and Hölder continuity of conformal mappings

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1. Introduction

Let A be a set in the complex plane $\mathbf{C}$ and let $0 < \alpha \leq 1$. A complex-valued function f defined on A is said to belong to $\mathrm{Lip}_\alpha(A)$, the *Lipschitz class in A with exponent α*, if there is a constant $M > 0$ such that

$$|f(z) - f(w)| \leq M\,|z - w|^\alpha \tag{1}$$

for all z and w in A, i.e., if f satisfies a *uniform Hölder condition with exponent α* on A.

Consider a simply connected proper subdomain D of $\mathbf{C}$ and a conformal mapping f of D onto the open unit disk $B = \{z : |z| < 1\}$. Of late it has become a matter of some interest to identify geometric criteria under which either the mapping f or its inverse belongs to some Lipschitz class. Sufficient conditions for uniform Hölder continuity on the part of f or f^{-1} are to be found, among other places, in [7], [8], [9], [13] and [15], while [10] contains a description of certain necessary conditions for such behavior. The conditions referred to are all euclidean geometric in nature. If, on the other hand, one actually desires to characterize the domains D for which either f or f^{-1} is a member of some Lipschitz class, evidence would suggest that one is compelled to abandon a euclidean perspective and to enter the realm of conformal invariants. Thus, for example, Becker and Pommerenke [1] and the authors [11] have exhibited necessary and sufficient conditions for Hölder continuity of conformal mappings in terms of hyperbolic geometry. In the present article we employ a different conformal invariant, extremal length, to provide alternative characterizations of the domains D for which f, as well as those for which f^{-1}, belongs to a specific Lipschitz class. Our characterizations are subsequently applied to derive euclidean geometric criteria for Hölder continuity. These applications include simplified proofs and extensions of two recent results due to Lesley [7].

* Part of this research was done while the author was visiting The University of Texas at Austin in 1984–85.

2. Modulus of an arc family

Let Γ be a family of arcs in $\mathbf{C}$. Consider non-negative extended real-valued Borel functions ρ in $\mathbf{C}$ such that

$$\int_\gamma \rho\,|dz| \geq 1$$

for each rectifiable arc γ in Γ. Define

$$M(\Gamma) = \inf_\rho \int_{\mathbf{C}} \rho^2\,dx\,dy.$$

The quantity $M(\Gamma)$ is termed the *modulus* of Γ, while its reciprocal,

$$\lambda(\Gamma) = \frac{1}{M(\Gamma)},$$

is referred to as the *extremal length* of Γ. We prefer to work with the modulus rather than with extremal length, although the latter is perhaps a more common term of reference in the literature. Both the modulus and the extremal length of an arc family are conformal invariants.

For future reference we compile here some more or less standard modulus estimates.

The Grötzsch ring domain. For $0<r<1$ let

$$R_G(r) = B\setminus\{z = x+iy : 0 \leq x \leq r,\, y = 0\}.$$

The domain $R_G(r)$ is called the *Grötzsch ring domain* corresponding to r. Let $\mu_G(r)$ denote the modulus of the family of arcs joining the boundary components of $R_G(r)$. Then

$$\frac{2\pi}{\log\frac{4}{r}} \leq \mu_G(r) \leq \frac{2\pi}{\log\frac{1}{r}}. \tag{2}$$

See [6, p. 64]. Using the identity

$$\mu_G(r)\mu_G\left(\frac{1-r}{1+r}\right) = 8$$

[6, p. 63], we obtain from (2)

$$\frac{4}{\pi}\log\frac{1+r}{1-r} \le \mu_G(r) \le \frac{4}{\pi}\log\frac{4(1+r)}{1-r}, \tag{3}$$

with the result that

$$\frac{4}{\pi}\log\frac{1}{1-r} \le \mu_G(r) \le \frac{4}{\pi}\log\frac{8}{1-r}. \tag{4}$$

The estimates for $\mu_G(r)$ in (2) are most useful for small $r>0$; in fact, the lower bound in (2) is asymptotically sharp as $r \to 0$. For r near 1, however, the estimates in (3) are better than those in (2).

The Teichmüller ring domain. For $r>0$ let

$$R_T(r) = \mathbf{C} \setminus \{z = x+iy : -1 \le x \le 0 \text{ or } r \le x < \infty,\ y=0\}.$$

The domain $R_T(r)$ is called the *Teichmüller ring domain* corresponding to r. Let $\mu_T(r)$ denote the modulus of the family of arcs joining the boundary components of $R_T(r)$. Then

$$\mu_T(r) = \tfrac{1}{2}\mu_G\left(\sqrt{\frac{1}{1+r}}\right). \tag{5}$$

See [6, p. 63]. Combining (2) and (5) we obtain

$$\frac{2\pi}{\log 16(1+r)} \le \mu_T(r) \le \frac{2\pi}{\log(1+r)}. \tag{6}$$

Arcs joining connected sets in B. Given three sets, E, F and G, we let $\Delta(E, F : G)$ denote the family of all arcs joining E to F through G. Now let E and F be nondegenerate connected sets in the closed unit disk $\bar{B}$. Then [3]

$$\begin{aligned} M[\Delta(E, F : B)] &= M[\Delta(\bar{E}, \bar{F} : B)] \\ &\ge \tfrac{1}{2}M[\Delta(\bar{E}, \bar{F} : \bar{\mathbf{C}})] \\ &\ge \tfrac{1}{2}\mu_T\left(\frac{q(a,c)q(b,d)}{q(a,b)q(c,d)}\right), \end{aligned}$$

where $a, b \in \bar{E}$ and $c, d \in \bar{F}$ and where q designates the chordal metric in the extended complex plane $\bar{\mathbf{C}}$ defined by

$$q(z, w) = \frac{|z - w|}{(1 + |z|^2)^{1/2}(1 + |w|^2)^{1/2}}$$

for points z and w in $\mathbf{C}$ and by

$$q(z, \infty) = \frac{1}{(1 + |z|^2)^{1/2}}.$$

Since $|z - w|/2 \le q(z, w) \le |z - w|$ for z and w in $\bar{B}$, we obtain using (6)

$$M[\Delta(E, F : B)] \ge \frac{\pi}{\log \dfrac{128}{\operatorname{dia}(E) \operatorname{dia}(F)}}. \tag{7}$$

Inequality (7) is valid for all connected sets E and F in $\bar{B}$.

Cross-cuts separating interior points of B. Fix a point $z \neq 0$ in B and consider the family Γ of cross-cuts of B separating 0 from z in B. Elementary considerations reveal that

$$M(\Gamma) \le \tfrac{1}{4}\mu_G(r),$$

where $r = |z|$. Invoking (4) and a standard lower bound for $M(\Gamma)$ we thus obtain

$$\frac{1}{\pi} \log \frac{1}{1 - r} \le M(\Gamma) \le \frac{1}{\pi} \log \frac{8}{1 - r}. \tag{8}$$

Next, let Γ_c denote the subfamily of Γ consisting of circular cross-cuts centered at $z/|z|$. A direct computation shows that

$$M(\Gamma_c) = \int_{1-r}^{1} \frac{dt}{t\left(\pi - 2 \arcsin \dfrac{t}{2}\right)},$$

from which it can be inferred that

$$\frac{1}{\pi} \log \frac{1}{1 - r} \le M(\Gamma_c) \le \frac{1}{\pi} \log \frac{3/2}{1 - r}. \tag{9}$$

Finally, let Γ_c^* designate the family of circular arcs in $\mathbf{C}\backslash B$ complementary to the arcs in Γ_c. Again, an elementary calculation yields

$$M(\Gamma_c^*) = \int_{1-r}^{1} \frac{dt}{t\left(\pi + 2 \arcsin \frac{t}{2}\right)},$$

which gives rise to the estimates

$$\frac{1}{\pi} \log \frac{1 - r/4}{1 - r} \le M(\Gamma_c^*) \le \frac{1}{\pi} \log \frac{1}{1 - r}. \tag{10}$$

Cross-cuts separating boundary points of B. Let z and w be two distinct points on ∂B and let A_1 and A_2 be the arcs into which ∂B is partitioned by z and w, labeled so that $\operatorname{dia}(A_1) \le \operatorname{dia}(A_2)$. For $i = 1, 2$ denote by Γ_i the family of cross-cuts of B which separate A_i from the origin. By [12, p. 196],

$$M(\Gamma_1) = \tfrac{1}{8}\mu_G\left(\cos \frac{l(A_1)}{4}\right),$$

$$M(\Gamma_2) = \tfrac{1}{8}\mu_G\left(\sin \frac{l(A_1)}{4}\right),$$

where $l(A_1)$ designates the length of A_1. Since

$$\cos \frac{l(A_1)}{4} = \tfrac{1}{2}[2 + (4 - |z - w|^2)^{1/2}]^{1/2},$$

$$\sin \frac{l(A_1)}{4} = \tfrac{1}{2}[2 - (4 - |z - w|^2)^{1/2}]^{1/2},$$

we deduce from (3) that

$$\frac{1}{\pi} \log \frac{2(1 + \sqrt{2})}{|z - w|} \le M(\Gamma_1) \le \frac{1}{\pi} \log \frac{16}{|z - w|}, \tag{11}$$

$$\frac{1}{\pi} \log \frac{(2 + \sqrt{2})^2}{|z - w|} \le M(\Gamma_1) + M(\Gamma_2) \le \frac{1}{\pi} \log \frac{32}{|z - w|}.$$

Consequently, if Γ designates the family of all cross-cuts of B separating a pair of boundary points z and w of B from the origin, then

$$\frac{1}{\pi} \log \frac{2(1 + \sqrt{2})}{|z - w|} \le M(\Gamma) \le \frac{1}{\pi} \log \frac{32}{|z - w|}. \tag{12}$$

3. Extremal length characterizations for Hölder continuity

In this section we characterize those plane domains D for which a conformal mapping f of D onto B belongs to $\mathrm{Lip}_\alpha(D)$ and those for which f^{-1} is a member of $\mathrm{Lip}_\beta(B)$. We advise the reader of one notational convention: throughout this article the notations $\bar{D}$ and ∂D are used to designate the closure and boundary of D relative to $\mathbf{C}$, not relative to the extended complex plane.

Let A be a set in a simply connected domain D and let z and w be points on ∂D. A cross-cut γ of D is said to *separate A from z and w* if A lies in one of the two components of $D\setminus\gamma$ and if the closure of this component contains neither z nor w.

THEOREM 1. *Let f be a conformal mapping of a domain D onto B and let $0<\alpha\le 1$. Then f belongs to* $\mathrm{Lip}_\alpha(D)$ *if and only if, corresponding to some* (*each*) *continuum A in D, there exists a constant $a>0$ such that each pair of points z and w on ∂D with $|z-w|<a$ can be separated from A by a cross-cut γ of D satisfying*

$$M[\Delta(A,\gamma:D)]\le\frac{\pi}{\alpha\log\dfrac{a}{|z-w|}}. \tag{13}$$

Proof. For the necessity, fix a continuum A in D. We may assume that f is defined and continuous on $\bar{D}$ and that f belongs to $\mathrm{Lip}_\alpha(\bar{D})$. Let $M>0$ be a Lipschitz constant for f corresponding to the exponent α and let $d=\operatorname{dist}[f(A),\partial B]$. We verify (13) with $a=(d/M)^{1/\alpha}$.

Fix a pair of points z and w on ∂D with $|z-w|<a$. Since

$$|f(z)-f(w)|\le M\,|z-w|^\alpha<Ma^\alpha=d$$

and since f^{-1} has angular limits almost everywhere in ∂B, we can separate $A'=f(A)$ from $f(z)$ and $f(w)$ by a cross-cut γ' of B which is an arc of a circle centered at $f(z)$ and at the endpoints of which f^{-1} has angular limits. The set $\gamma=f^{-1}(\gamma')$ is a cross-cut of D which separates A from z and w. Elementary properties of the modulus yield

$$M[\Delta(A,\gamma:D)]=M[\Delta(A',\gamma':B)]<\frac{\pi}{\log\dfrac{d}{|f(z)-f(w)|}}\le\frac{\pi}{\alpha\log\dfrac{a}{|z-w|}},$$

where $a=(d/M)^{1/\alpha}$. This establishes (13).

For the sufficiency, fix a continuum A in D and choose $a > 0$ for which (13) is valid. We begin by demonstrating that f can be extended to a continuous mapping of $\bar{D}$.

Consider a point z of ∂D and suppose that f fails to have a limit at z. Then we can choose sequences $\langle z_k \rangle$ and $\langle w_k \rangle$ in D such that $z_k \to z$ and $w_k \to z$, while $f(z_k) \to z'$ and $f(w_k) \to w'$, where $z' \neq w'$. An elementary geometric argument establishes the existence of end-cuts E_k and F_k of D joining z_k and w_k, respectively, to distinct points z_k^* and w_k^* on ∂D and satisfying $\operatorname{dia}(E_k) \to 0$ and $\operatorname{dia}(F_k) \to 0$, as $k \to \infty$. Obviously

$$\lim_{k\to\infty} M[\Delta(A, E_k : D)] = 0 = \lim_{k\to\infty} M[\Delta(A, F_k : D)]. \tag{14}$$

Since B satisfies the modulus condition (7), it follows that $\operatorname{dia}[f(E_k)] \to 0$ and $\operatorname{dia}[f(F_k)] \to 0$. Furthermore, a classical theorem of Koebe asserts that $f(E_k)$ and $f(F_k)$ are end-cuts of B terminating at certain points z_k' and w_k' on ∂B, respectively. Since $z_k^* \to z$ and $w_k^* \to z$, there is, in view of (13), a sequence $\langle \gamma_k \rangle$ of cross-cuts of D such that γ_k separates z_k^* and w_k^* from A and such that the component G_k of $D \setminus \gamma_k$ not containing A satisfies

$$M[\Delta(A, G_k : D)] \leq M[\Delta(A, \gamma_k : D)] \to 0.$$

The set $\gamma_k' = f(\gamma_k)$ is a cross-cut of B separating $A' = f(A)$ from $G_k' = f(G_k)$. Clearly $\bar{G}_k'$ contains z_k' and w_k'. But since $z_k' \to z'$ and $w_k' \to w'$, where $z' \neq w'$, it follows that $\operatorname{dia}(G_k') \not\to 0$. The modulus condition (7) then implies that

$$M[\Delta(A', G_k' : B)] \not\to 0,$$

as $k \to \infty$. This contradiction to the conformal invariance of the modulus shows that f must have a limit at z, an arbitrary point of ∂D. We conclude that f admits an extension to a continuous mapping of $\bar{D}$. The notation f will be retained for the extended mapping.

We are now in a position to demonstrate that f belongs to $\operatorname{Lip}_\alpha(D)$. For this, fix a pair of points z and w on ∂D. Assume that $|z - w| < a$ and that $f(z) \neq f(w)$. Let γ be a cross-cut of D separating A from z and w and satisfying (13). Next let G be the component of $D \setminus \gamma$ not containing A. Then by (13)

$$M[\Delta(A, G : D)] \leq \frac{\pi}{\alpha \log \dfrac{a}{|z - w|}}. \tag{15}$$

We denote $A' = f(A)$ and $G' = f(G)$ and use (7) to estimate

$$M[\Delta(A', G' : B)] \geq \frac{\pi}{\log \dfrac{128}{\operatorname{dia}(A') \operatorname{dia}(G')}} \geq \frac{\pi}{\log \dfrac{b}{|f(z) - f(w)|}}, \tag{16}$$

where $b = 128/\operatorname{dia}(A')$. Since the modulus is a conformal invariant, we infer from (15) and (16) that

$$|f(z) - f(w)| \leq \frac{b}{a^\alpha} |z - w|^\alpha.$$

The above estimate holds trivially for boundary points z and w of D satisfying $|z - w| \geq a$ or $f(z) = f(w)$. Hence the boundary mapping $f \mid \partial D$ belongs to $\operatorname{Lip}_\alpha(\partial D)$ and, consequently, f belongs to $\operatorname{Lip}_\alpha(D)$. (See [5] or [14].)

An analogue to the preceding theorem for mappings from B into the complex plane is:

THEOREM 2. *Let f be a conformal mapping of B onto a domain D and let $0 < \beta \leq 1$. Then f belongs to* $\operatorname{Lip}_\beta(B)$ *if and only if, corresponding to some* (*each*) *continuum A in D, there exists a constant $b > 0$ such that*

$$M[\Delta(A, \gamma : D)] \geq \frac{\beta\pi}{\log \dfrac{b}{|z - w|}}, \tag{17}$$

whenever z and w are distinct points on ∂D and γ is a cross-cut of D terminating in z and w.

Proof. For the necessity, fix a continuum A in D. We may again assume that f is defined and continuous on $\bar{B}$ and that f belongs to $\operatorname{Lip}_\beta(\bar{B})$. Let $M > 0$ be a Lipschitz constant for f corresponding to the exponent β. Next, let γ be a cross-cut of D terminating in distinct points z and w. The set $f^{-1}(\gamma)$ is a cross-cut of B with distinct endpoints ζ and ω. Using the estimate (7) we obtain

$$M[\Delta(A, \gamma : D)] = M[\Delta(f^{-1}(A), f^{-1}(\gamma) : B)] \geq \frac{\pi}{\log \dfrac{128}{\operatorname{dia}[f^{-1}(A)] \operatorname{dia}[f^{-1}(\gamma)]}}$$

$$\geq \frac{\pi}{\log \dfrac{c}{|\zeta - \omega|}} \geq \frac{\beta\pi}{\log \dfrac{b}{|z - w|}},$$

where $c = 128/\operatorname{dia}[f^{-1}(A)]$ and $b = c^\beta M$. This establishes (17).

For the sufficiency, fix a continuum A in D and choose a number $b>0$ for which (17) is valid. It is an elementary consequence of (17) that D is bounded. We show first that f can be extended to a continuous mapping of $\bar{B}$. To do so, we modify slightly the argument used in the proof of Theorem 1.

Fix a point z on ∂B and suppose that f fails to have a limit at z. Then we can choose sequences $\langle z_k \rangle$ and $\langle w_k \rangle$ in B such that $z_k \to z$ and $w_k \to z$, while $f(z_k) \to z'$ and $f(w_k) \to w'$, where $z' \neq w'$ and $|z'-w'|<b$. Select end-cuts E_k and F_k of D joining $f(z_k)$ and $f(w_k)$, respectively, to distinct points z'_k and w'_k on ∂D and satisfying $\operatorname{dia}(E_k) \to 0$ and $\operatorname{dia}(F_k) \to 0$, as $k \to \infty$. Since (14) holds again and since B satisfies the modulus condition (7), it follows that $\operatorname{dia}[f^{-1}(E_k)] \to 0$ and $\operatorname{dia}[f^{-1}(F_k)] \to 0$. Furthermore, classical theorems of Koebe and Lindelöf assert that $f^{-1}(E_k)$ and $f^{-1}(F_k)$ are end-cuts of B terminating at certain points z_k^* and w_k^*, respectively, on the unit circle and that f has angular limits z'_k at z_k^* and w'_k at w_k^*. Let γ_k^* be the line segment with endpoints z_k^* and w_k^*. Then

$$M[\Delta(f^{-1}(A), \gamma_k^*:B)] \to 0$$

as $k \to \infty$, since $z_k^* \to z$ and $w_k^* \to z$. On the other hand, $f(\gamma_k^*)$ is a cross-cut of D with terminal points z'_k and w'_k. In light of (17),

$$M[\Delta(A, f(\gamma_k^*):D)] \geq \frac{\beta\pi}{\log \dfrac{b}{|z'_k - w'_k|}} \to \frac{\beta\pi}{\log \dfrac{b}{|z'-w'|}} > 0,$$

since $z'_k \to z'$ and $w'_k \to w'$. This contradiction to the conformal invariance of the modulus shows that f must have a limit at z, an arbitrary point of ∂B. We conclude that f admits an extension to a continuous mapping of $\bar{B}$. We retain the notation f for this extension.

We next verify that f belongs to $\mathrm{Lip}_\beta\,(B)$. Fix a pair of points z and w on ∂B for which $f(z) \neq f(w)$. Assume first that $|z-w|<d$, where $d = \operatorname{dist}[f^{-1}(A), \partial B]$. Join z to w by a line segment γ_0 and consider the arc family $\Gamma = \Delta(f^{-1}(A), \gamma_0:B)$. By virtue of (17),

$$\frac{\beta\pi}{\log \dfrac{b}{|f(z)-f(w)|}} \leq M[f(\Gamma)] = M(\Gamma) \leq \frac{\pi}{\log \dfrac{d}{|z-w|}},$$

from which we infer that

$$|f(z)-f(w)| \leq \frac{b}{d^\beta}|z-w|^\beta. \tag{18}$$

If $|z-w| \geq d$ or if $f(z)=f(w)$, then

$$|f(z)-f(w)| \leq \frac{\operatorname{dia}(D)}{d^{\beta}}|z-w|^{\beta}. \tag{19}$$

Combining (18) and (19) we see that (1) holds, with $M=d^{-\beta}\max\{b, \operatorname{dia}(D)\}$, for all z and w on ∂B. Thus the boundary mapping $f \mid \partial B$ belongs to $\operatorname{Lip}_{\beta}(\partial B)$. A classical result due to Hardy and Littlewood allows us to conclude that f is a member of $\operatorname{Lip}_{\beta}(B)$.

4. Hölder continuity and separating cross-cut families

In the section at hand we discuss alternative formulations of Theorems 1 and 2.

THEOREM 3. *Let f be a conformal mapping of a domain D onto B and let $0<\alpha\leq 1$. Then f belongs to* $\operatorname{Lip}_{\alpha}(D)$ *if and only if, corresponding to some* (*each*) *point z_0 in D, there exists a constant $a>0$ such that, for each pair of points z and w on ∂D, the family Γ of cross-cuts of D which separate z_0 from z and w satisfies*

$$M(\Gamma) \geq \frac{\alpha}{\pi}\log\frac{a}{|z-w|}. \tag{20}$$

Proof. To establish the necessity of condition (20), fix a point z_0 in D. We may assume that $f(z_0)=0$. We may further assume that f is defined and continuous on $\bar{D}$ and that f belongs to $\operatorname{Lip}_{\alpha}(\bar{D})$. Let $M>0$ be a Lipschitz constant for f corresponding to the exponent α. Fix a pair of points z and w on ∂D. Let Γ denote the family of cross-cuts of D which separate z_0 from z and w. Next, let Δ denote the family of cross-cuts of B which separate 0 from the shorter of the arcs into which ∂B is partitioned by the points $f(z)$ and $f(w)$. (If $f(z)=f(w)$, the arc degenerates to a single point.) Finally, let Δ_0 designate the family of those cross-cuts in Δ whose images under f^{-1} are rectifiable and are, therefore, cross-cuts of D. Write $\Gamma_0=f^{-1}(\Delta_0)$. Since f is continuous on $\bar{D}$, each cross-cut in Γ_0 separates z_0 from z and w. The conformal invariance of the modulus, in combination with (11) and other elementary properties of the modulus, implies

$$M(\Gamma) \geq M(\Gamma_0) = M(\Delta_0) = M(\Delta) \geq \frac{1}{\pi}\log\frac{2(1+\sqrt{2})}{|f(z)-f(w)|} \geq \frac{\alpha}{\pi}\log\frac{a}{|z-w|},$$

where $a=(2(1+\sqrt{2})/M)^{1/\alpha}$. This establishes (20).

Conversely, assume that (20) is satisfied, with $z_0 = f^{-1}(0)$. We first demonstrate that f can be extended to a continuous mapping of $\bar{D}$.

Fix a point z in ∂D. Choose a sequence $\langle z_k \rangle$ of distinct points in ∂D converging to z and choose, for each k, a cross-cut γ_k of D which separates z and z_k from z_0 and whose chordal diameter $q(\gamma_k)$ satisfies

$$q(\gamma_k) \to 0, \tag{21}$$

as $k \to \infty$. The existence of such a cross-cut γ_k is a straightforward consequence of inequality (20). Let D_k denote the component of $D \setminus \gamma_k$ which contains z_0. Since $\operatorname{dist}(z, D_k)$ is positive, we can find a sequence $\langle U_k \rangle$ of neighborhoods of z such that $U_k \cap \bar{D}_k = \varnothing$ for each k. Combining (21) with standard conformal modulus considerations yields

$$\operatorname{dia}[f(\gamma_k)] \to 0. \tag{22}$$

Since $f(\gamma_k)$ separates $f(D \cap U_k)$ from 0 in B, we infer from (22) that $\operatorname{dia}[f(D \cap U_k)] \to 0$ and, thus, that the cluster set of f at z reduces to a single point. In other words, f has a limit at z, an arbitrary point of ∂D. Therefore, f admits a continuous extension to $\bar{D}$. We continue to denote the extended mapping by f.

We now verify that f belongs to $\mathrm{Lip}_\alpha(D)$. For this, fix a pair of points z and w on ∂D such that $f(z) \neq f(w)$. Consider the family Γ of cross-cuts of D separating z_0 from z and w in D. By virtue of (12) and (20),

$$\frac{\alpha}{\pi} \log \frac{a}{|z-w|} \leq M(\Gamma) = M[f(\Gamma)] \leq \frac{1}{\pi} \log \frac{32}{|f(z)-f(w)|},$$

which implies

$$|f(z) - f(w)| \leq M\,|z-w|^\alpha,$$

where $M = 32/a^\alpha$. This estimate holds trivially for points z and w on ∂D with $f(z) = f(w)$. We infer that the boundary mapping $f \mid \partial D$ belongs to $\mathrm{Lip}_\alpha(\partial D)$. As before, this fact permits us to conclude that f belongs to $\mathrm{Lip}_\alpha(D)$.

Theorem 2 has the following counterpart in the present setting.

THEOREM 4. *Let f be a conformal mapping of B onto a domain D and let $0 < \beta \leq 1$. Then f belongs to* $\mathrm{Lip}_\beta(B)$ *if and only if, corresponding to some (each) point z_0 in D, there exists a constant $b > 0$ such that, for each cross-cut γ_0 of D*

with distinct endpoints z and w, the family Γ of cross-cuts of D which separate z_0 from γ_0 in D satisfies

$$M(\Gamma) \leq \frac{1}{\beta\pi} \log \frac{b}{|z-w|}. \tag{23}$$

Proof. To establish the necessity of condition (23), fix a point z_0 in D. Once again we make the assumption that $f(0)=z_0$. In addition, we may assume that f is defined on $\bar{B}$ and that f belongs to $\mathrm{Lip}_\beta(\bar{B})$. Let $M>0$ be a Lipschitz constant for f corresponding to the exponent β. Fix a cross-cut γ_0 of D with distinct endpoints z and w. Let Γ denote the family of cross-cuts of D which separate z_0 from γ_0. By a classical theorem of Koebe, $f^{-1}(\gamma_0)$, as well as each member of the family $f^{-1}(\Gamma)$, is a cross-cut of B. Let z^* and w^* be the endpoints of $f^{-1}(\gamma_0)$. Each arc in $f^{-1}(\Gamma)$ separates 0 from $f^{-1}(\gamma_0)$ and, by virtue of (12),

$$M(\Gamma) = M[f^{-1}(\Gamma)] \leq \frac{1}{\pi} \log \frac{32}{|z^*-w^*|} \leq \frac{1}{\beta\pi} \log \frac{b}{|z-w|},$$

where $b = M32^\beta$. This establishes (23).

Conversely, assume that (23) is satisfied, with $z_0 = f(0)$. We observe, as an elementary consequence of (23), that $\mathrm{dia}(D) \leq b$. Once more we begin by demonstrating that f can be extended to a continuous mapping of $\bar{B}$.

Fix a point z in ∂B and suppose that f fails to have a limit at z. Then we can choose sequences $\langle z_k \rangle$ and $\langle w_k \rangle$ in B such that $z_k \to z$ and $w_k \to z$, while $f(z_k) \to z'$ and $f(w_k) \to w'$, where $z' \neq w'$. As in the proof of Theorem 2, we find distinct points z'_k and w'_k on ∂D such that $z'_k \to z'$, $w'_k \to w'$ and such that z'_k and w'_k, respectively, are angular limits of f at certain points z^*_k and w^*_k on ∂B, with $z^*_k \to z$ and $w^*_k \to z$. Let γ^*_k be the line segment with endpoints z^*_k and w^*_k. We may assume that γ^*_k does not pass through the origin. Denote by Γ^*_k the family of cross-cuts of B separating γ^*_k from the origin. It is easily seen that

$$M(\Gamma^*_k) \to \infty,$$

as $k \to \infty$. On the other hand, $f(\gamma^*_k)$ is a cross-cut of D with terminal points z'_k and w'_k. Let Γ_k be the family of cross-cuts of D separating z_0 from $f(\gamma^*_k)$. Since the paths in the family $f(\Gamma^*_k) \setminus \Gamma_k$ are non-rectifiable, we can use (23) to estimate

$$M[f(\Gamma^*_k)] = M(\Gamma_k) \leq \frac{1}{\beta\pi} \log \frac{b}{|z'_k - w'_k|} \to \frac{1}{\beta\pi} \log \frac{b}{|z'-w'|} < \infty.$$

This contradiction to the conformal invariance of the modulus shows that f must have a limit at z, an arbitrary point of ∂B. We conclude that f admits an extension to a continuous mapping of $\bar{B}$. The notation f will be retained for the extended mapping.

Finally, we verify that f belongs to $\mathrm{Lip}_\beta\,(B)$. Fix a pair of points z and w on ∂B for which $f(z) \neq f(w)$. Assume first that $|z-w|<1$. Join z to w by a line segment γ_0 and let Γ^* denote the family of all cross-cuts of B separating γ_0 from the origin. Then $f(\Gamma^*)$ is contained in Γ, the family of all cross-cuts of D separating z_0 from the cross-cut $f(\gamma_0)$ in D. We employ (9) and (23) to estimate

$$\frac{1}{\pi}\log\frac{1}{|z-w|} \leq M(\Gamma^*) = M[f(\Gamma^*)] \leq M(\Gamma) \leq \frac{1}{\beta\pi}\log\frac{b}{|f(z)-f(w)|}.$$

This permits us to infer that

$$|f(z)-f(w)| \leq b\,|z-w|^\beta.$$

If $f(z)=f(w)$ or if $|z-w|\geq 1$, this inequality holds trivially, because $\mathrm{dia}\,(D)\leq b$. Thus the boundary mapping $f \mid \partial B$ belongs to $\mathrm{Lip}_\beta\,(\partial B)$. A classical result due to Hardy and Littlewood allows us to conclude that f is a member of $\mathrm{Lip}_\beta\,(B)$.

Although the condition described by Theorem 4 might strike one as somewhat awkward, something of this nature is really needed to deal with non-Jordan domains. Anticipating future reference, we record the following variant of Theorem 4 valid in the special case where D is a Jordan domain. The proof, which is a much simplified version of the proof given for Theorem 4, is left to the reader.

THEOREM 5. *Let f be a conformal mapping of B onto a Jordan domain D and let $0<\beta\leq 1$. Then f belongs to* $\mathrm{Lip}_\beta\,(B)$ *if and only if, corresponding to some (each) point z_0 in D, there exists a constant $b>0$ such that, for each pair of points z and w on ∂D, the family Γ of cross-cuts of D which separate z_0 from z and w satisfies*

$$M(\Gamma) \leq \frac{1}{\beta\pi}\log\frac{b}{|z-w|}.$$

A complex-valued function f on a set A in $\mathbf{C}$ is called a *quasi-isometry* if there is a constant $L>0$ such that

$$\frac{|z-w|}{L} \leq |f(z)-f(w)| \leq L\,|z-w|$$

for all z and w in A. We conclude this section by characterizing the plane domains D with the property that conformal mappings between D and B are quasi-isometries.

THEOREM 6. *Let f be a conformal mapping of a domain D onto B. Then f is a quasi-isometry if and only if, corresponding to some (each) point z_0 in D, there exist constants $a > 0$ and $b > 0$ such that, for each pair of points z and w on ∂D, the family Γ of cross-cuts of D which separate z_0 from z and w satisfies*

$$\frac{1}{\pi}\log\frac{a}{|z-w|} \leq M(\Gamma) \leq \frac{1}{\pi}\log\frac{b}{|z-w|}. \tag{24}$$

Proof. The necessity of (24) follows from Theorems 3 and 5. To prove the sufficiency, we first employ Theorem 3 to infer from the left-hand inequality in (24) that f belongs to $\mathrm{Lip}_\alpha(D)$ for $\alpha = 1$. In particular, f has a continuous extension to $\bar{D}$. From the right-hand inequality in (24) we deduce, without difficulty, that D is bounded and that the extension of f is injective. In other words, D is a Jordan domain. Theorem 5 then guarantees that f^{-1} belongs to $\mathrm{Lip}_\beta(B)$ for $\beta = 1$. Consequently, f is a quasi-isometry.

5. Euclidean geometric separation properties

In the remainder of this paper we illustrate how the preceding characterizations can be used to retrieve information on the Hölder continuity of a conformal mapping from certain euclidean geometric data.

Let D be a simply connected domain in $\mathbf{C}$. Fix a point z_0 in D and let $0 < \beta \leq 1$. If there exists a constant $b > 0$ such that any pair of points z and w on ∂D can be separated from z_0 by a cross-cut γ of D satisfying

$$\mathrm{dia}(\gamma) \leq b\,|z-w|^\beta, \tag{25}$$

we will declare D to have the *separation property with exponent β*. In the special case $\beta = 1$ we will say that D has the *linear separation property*. It is not difficult to see that, except for the value of the constant b, (25) does not depend on the choice of z_0.

A class of domains satisfying (25) was considered in [9]: domains which are arcwise connected with exponent β. A domain D in $\mathbf{C}$ is said to be *arcwise connected with exponent β* if there exists a constant $b > 0$ such that each pair of points z and w in D can be joined by an arc γ in D satisfying (25). If D satisfies this condition with $\beta = 1$, we referred to D in [9] as *b-arcwise connected.* All bounded simply connected domains of such types are necessarily Jordan domains. On the other hand, it is not difficult to exhibit bounded simply connected non-Jordan domains which have the linear separation property.

THEOREM 7. *Let f be a conformal mapping of a domain D onto B and let $0 < \beta \leq 1$. If D has the separation property with exponent β, then f belongs to* $\mathrm{Lip}_\alpha(D)$ *for $\alpha = \beta/2$. This Hölder exponent is the best possible for each β,* $0 < \beta < 1$.

Proof. Fix a point z_0 in D. Let $d = \mathrm{dist}(z_0, \partial D)$ and let $b > 0$ be a constant such that (25) holds. Define $a > 0$ by $ba^\beta = d/2$. We show that each pair of points z and w on ∂D with $|z - w| < a$ can be separated from $A = \bar{B}(z_0, d/2)$, the closed disk of radius $d/2$ centered at z_0, by a cross-cut γ of D satisfying

$$M[\Delta(A, \gamma : D)] \leq \frac{2\pi}{\beta \log \dfrac{a}{|z - w|}}. \tag{26}$$

Fix such a pair of points z and w. Select a cross-cut γ of D which separates z and w from z_0 and satisfies (25). Then γ separates z and w from A, because $\mathrm{dia}(\gamma) < d/2$. It is routine to obtain the estimate

$$M[\Delta(A, \gamma : D)] \leq \frac{2\pi}{\log \dfrac{d/2}{\mathrm{dia}(\gamma)}} \leq \frac{2\pi}{\log \dfrac{d}{2b\,|z - w|^\beta}},$$

and (26) follows with a as indicated. Theorem 1 now implies that f belongs to $\mathrm{Lip}_\alpha(D)$ with $\alpha = \beta/2$.

Finally, the sharpness of the Hölder exponent α for fixed β, $0 < \beta < 1$, is demonstrated by a conformal mapping f of D_β onto B, where

$$D_\beta = B \setminus \{z = x + iy : 0 \leq x \leq 1, 0 \leq y \leq x^{1/\beta}\}.$$

COROLLARY 1. *Let f be a conformal mapping of a domain D with the linear separation property onto B. Then f belongs to* $\mathrm{Lip}_\alpha(D)$ *for $\alpha = 1/2$.*

The conclusion of Corollary 1 cannot be improved if D is allowed to vary over the entire class of domains enjoying the linear separation property. It is natural to ask what supplementary information is required concerning a domain D from this class in order to place f in $\mathrm{Lip}_\alpha(D)$, where now $1/2 < \alpha \leq 1$. One possible answer to this question is supplied by the next theorem. In this result the notation h_D designates the hyperbolic distance in D normalized to have curvature -1. As earlier, $\mathrm{dist}(z, \partial D)$ indicates the euclidean distance of a point z from ∂D.

THEOREM 8. *Let f be a conformal mapping of a domain D with the linear separation property onto B and let $1/2 < \alpha \leq 1$. Then f belongs to $\mathrm{Lip}_\alpha(D)$ if and only if, corresponding to some (each) point z_0 in D, there exists a constant $a > 0$ such that*

$$h_D(z_0, z) \geq \alpha \log \frac{a}{\mathrm{dist}(z, \partial D)} \tag{27}$$

for all z in D.

Proof. The necessity of condition (27) is immediate. Indeed, if we fix z_0 in D, if we assume that $f(z_0) = 0$, and if we choose a Lipschitz constant $M > 0$ for f corresponding to α, we obtain for z in D

$$\begin{aligned} h_D(z_0, z) = h_B(0, f(z)) &\geq \log \frac{1}{\mathrm{dist}[f(z), \partial B]} \\ &\geq \log \frac{1}{M\, \mathrm{dist}(z, \partial D)^\alpha} = \alpha \log \frac{a}{\mathrm{dist}(z, \partial D)}, \end{aligned}$$

where $a = M^{-1/\alpha}$.

To prove the sufficiency, fix z_0 in D for which (27) holds. We may again assume that $f(z_0) = 0$. By Corollary 1, we may further assume that f is defined on $\bar{D}$ and that f belongs to $\mathrm{Lip}_{1/2}(\bar{D})$. Let $M_1 > 0$ be a Lipschitz constant for f corresponding to the exponent 1/2. By assumption, there exists a constant $b > 0$ such that each pair of points z and w on ∂D can be separated from z_0 by a cross-cut γ of D satisfying

$$\mathrm{dia}(\gamma) \leq b\,|z - w|. \tag{28}$$

Let c be a constant satisfying $0 < c < 1/bM_1^2$.

Consider a pair of points z and w on ∂D for which $|z - w| < c$ and for which $f(z) \neq f(w)$. Select γ as in (28). Then $f(\gamma)$ is a cross-cut of B satisfying

$$\mathrm{dia}[f(\gamma)] \leq M_1\, \mathrm{dia}(\gamma)^{1/2} \leq M_1 b^{1/2} c^{1/2} < 1.$$

It can be inferred from this information that $f(\gamma)$ has two distinct endpoints. Thus

γ must likewise possess two distinct endpoints, say z^* and w^*. Let γ^* be the (necessarily unique) geodesic cross-cut of D with endpoints z^* and w^*. A theorem of Gehring and Hayman [4] guarantees that

$$\mathrm{dia}\,(\gamma^*) \leq k\, \mathrm{dia}\,(\gamma), \tag{29}$$

where $k > 0$ is an absolute constant. If w_0 denotes the point of γ^* at minimal hyperbolic distance from z_0, we employ (27), (28) and (29), along with elementary properties of the hyperbolic distance in B, to compute

$$\log \frac{4}{|f(z) - f(w)|} \geq \log \frac{4}{|f(z^*) - f(w^*)|} \geq h_B(0, f(w_0)) = h_D(z_0, w_0)$$
$$\geq \alpha \log \frac{a}{\mathrm{dist}\,(w_0, \partial D)} \geq \alpha \log \frac{a}{\mathrm{dia}\,(\gamma^*)} \geq \alpha \log \frac{a}{bk\,|z - w|},$$

with the result that

$$|f(z) - f(w)| \leq 4(bk/a)^\alpha\, |z - w|^\alpha.$$

This estimate holds trivially for points z and w on ∂D with $f(z) = f(w)$. Finally, if $|z - w| \geq c$,

$$|f(z) - f(w)| \leq \frac{2}{c^\alpha} |z - w|^\alpha.$$

We conclude that (1) holds, with $M = \max\,\{4(bk/a)^\alpha, 2/c^\alpha\}$, for all points z and w on ∂D. In other words, the boundary mapping $f \mid \partial D$ belongs to $\mathrm{Lip}_\alpha\,(\partial D)$. Thus f is a member of $\mathrm{Lip}_\alpha\,(D)$ [5], [14].

As pointed out in [11], condition (27) by itself affords no guarantee that f will belong to $\mathrm{Lip}_\alpha\,(D)$. The following result describes a more concrete geometric condition which is sufficient to place a conformal mapping of a domain D enjoying the linear separation property onto B in $\mathrm{Lip}_\alpha\,(D)$ with $\alpha > 1/2$. At the same time, it serves to illustrate the use of Theorem 3. (For related results, see [2].)

THEOREM 9. *Let f be a conformal mapping of a domain D with the linear separation property onto B and let $1/2 < \alpha \leq 1$. Suppose there exists an $R > 0$ such that, for each w in ∂D and for $0 < r < R$, no component of $D \cap \partial B(w, r)$ has length exceeding $\pi r/\alpha$. Then f belongs to* $\mathrm{Lip}_\alpha\,(D)$. *This Hölder exponent is the best possible.*

Proof. Let $b>0$ designate a constant corresponding to $z_0=f^{-1}(0)$ for which condition (25) holds with $\beta=1$. We may assume that $0<R<\min\{b, d\}$, where $d=\operatorname{dist}(z_0, \partial D)$. Consider points z and w on ∂D, together with the family Γ of all cross-cuts of D which separate z and w from z_0. We verify that

$$M(\Gamma)\geq\frac{\alpha}{\pi}\log\frac{a}{|z-w|},$$

where $a=R/b$. As this is trivially the case when $b\,|z-w|\geq R$, we may assume that $b\,|z-w|<R$. By hypothesis Γ contains a cross-cut γ_0 of D which satisfies $\operatorname{dia}(\gamma_0)\leq b\,|z-w|$. Let w_0 be an endpoint of γ_0. When $b\,|z-w|<r<R$ we can select a component γ_r of $D\cap\partial B(w_0, r)$ which separates z_0 from γ_0 in D. Such an arc γ_r clearly belongs to Γ and, by assumption, its length does not exceed $\pi r/\alpha$. Letting Γ^* be the family of such γ_r, we have

$$M(\Gamma)\geq M(\Gamma^*)\geq\frac{\alpha}{\pi}\log\frac{R}{b\,|z-w|},$$

as desired. Theorem 3 insures that f is a member of $\operatorname{Lip}_\alpha(D)$. The sharpness of the Hölder exponent is demonstrated by taking $D=\{z:|\arg z|<\pi/2\alpha\}$.

For a simply connected proper subdomain D of $\mathbf{C}$ there is a natural way to formulate a separation condition dual to the linear separation condition: given a point z_0 in D one can simply require the existence of a constant $b>0$ such that

$$\operatorname{dia}(\gamma)\geq b\,|z-w| \tag{30}$$

for every pair of points z and w on ∂D and for every cross-cut γ of D that separates z and w from z_0. It is straightforward to demonstrate that only a bounded domain can possess this property. The suggestion is that a condition such as (30) will insure the uniform Hölder continuity of a conformal mapping of B onto D. If D happens to be a Jordan domain, this is actually the case. In order to accommodate non-Jordan domains, however, condition (30) must be altered somewhat. One possibility is to demand that (30) hold for cross-cuts γ of D separating points z and w inside D from z_0. We prefer a slightly different modification of (30), one which has the added advantage that the Hölder exponent obtained is not far from being optimal. Given a cross-cut γ of D, the notation D_γ will indicate the component of $D\setminus\gamma$ having smaller diameter.

THEOREM 10. *Let f be a conformal mapping of B onto a bounded domain D. Suppose there exists a constant $c>0$ such that*

$$\operatorname{dia}(D_\gamma)\leq c\operatorname{dia}(\gamma) \tag{31}$$

for every cross-cut γ *of* D. *Then* f *belongs to* $\mathrm{Lip}_\beta\,(B)$ *for*

$$\beta = \frac{2}{c^2\pi^2}. \tag{32}$$

Proof. Every cross-cut γ of D satisfies

$$\operatorname{area}(D_\gamma) \le \frac{\pi}{4}[\operatorname{dia}(D_\gamma)]^2 \le \frac{\pi c^2}{4}\operatorname{dia}(\gamma)^2 \le \frac{\pi c^2}{4} l(\gamma)^2 \tag{33}$$

in view of the isodiametric inequality and (31). Lemma 5 in [9], a refined version of Wolff's classical inequality, then implies the existence of a number $\tau > 0$ such that, if z is a point of ∂B and if $0 < r < R \le \tau$, then

$$l(\lambda_\rho^*) \le \left(\frac{R}{\tau}\right)^\beta \left[\frac{\pi \operatorname{area}(D)}{\log \frac{R}{r}}\right]^{1/2} \tag{34}$$

for some ρ in (r, R). Here β is as in (32) and $\lambda_\rho^* = f[B \cap \partial B(z, \rho)]$. The hypothesis (31) combined with (34) – or merely with the classical Wolff's inequality, without the term $(R/\tau)^\beta$ – readily shows that f admits an extension to a continuous mapping of $\bar{B}$ onto $\bar{D}$, which extension we will continue to denote by f.

Choose $t > 0$ such that

$$|f(z) - f(w)| < \tfrac{1}{2}\operatorname{dia}(D) \tag{35}$$

for all z and w in $\bar{B}$ satisfying $|z - w| < 2t$. Next fix distinct points z and w on ∂B. Assume, initially, that

$$2\,|z - w| \le \min\{t, \tau\}. \tag{36}$$

Invoking (34) with $r = |z - w|$ and $R = 2r$, we infer the existence of ρ in (r, R) such that

$$l(\lambda_\rho^*) \le M_1\,|z - w|^\beta, \tag{37}$$

with $M_1 = (4/\tau)^\beta[\pi \operatorname{area}(D)/\log 2]^{1/2}$. The set $\gamma = \lambda_\rho^*$ is a cross-cut of D. The closure of D_γ contains $f(z)$ and $f(w)$ by virtue of (35). Inequalities (31) and (37) yield

$$|f(z) - f(w)| \le \operatorname{dia}(D_\gamma) \le c\,\operatorname{dia}(\gamma) \le M_1 c\,|z - w|^\beta. \tag{38}$$

Finally, if (36) is not satisfied, one evidently obtains

$$|f(z)-f(w)| \leq M_2 |z-w|^{\beta}, \tag{39}$$

by simply taking $M_2 = 2^{\beta} \operatorname{dia}(D) \max\{t^{-\beta}, \tau^{-\beta}\}$. We infer from (38) and (39) that the boundary mapping $f \mid \partial B$ belongs to $\mathrm{Lip}_\beta(\partial B)$. Consequently, f belongs to $\mathrm{Lip}_\beta(B)$.

6. Wedge conditions

In this section we demonstrate how two theorems of Lesley in [7] can be deduced from the results in the present paper.

Let D be a bounded Jordan domain in $\mathbf{C}$ and let $0<\beta\leq 1$. The domain D is said to satisfy an *interior* (respectively, *exterior*) *β-wedge condition* if there exists an $R>0$ such that, for every point w in ∂D, there is a closed circular sector of radius R, angular opening $\beta\pi$ and vertex w which lies in $\bar{D}$ (respectively, in $\mathbf{C}\backslash D$).

Suppose that D satisfies an interior β-wedge condition for some β. Fix a point z_0 in D. Then there exists a constant $c_0>0$ such that each point w of ∂D is the terminal point of an end-cut E of D from z_0 satisfying

$$\operatorname{dia}(E_z) \leq c_0 \operatorname{dist}(z, \partial D)$$

for all points z on E, where E_z denotes the subarc of E with z and w as its endpoints. As observed by Lesley [7], Theorem 1 in [13] then guarantees the existence of a constant $c>0$ such that

$$\operatorname{dia}(\gamma) \geq c \operatorname{dia}[C(z, w)], \tag{40}$$

whenever z and w are distinct points of ∂D and γ is a cross-cut of D terminating in z and w. Here $C(z, w)$ indicates the arc of smaller diameter on ∂D with endpoints z and w. Next, an elementary argument involving (40) shows that any pair of points z and w in the exterior D^* of D can be joined by an arc γ^* in $\bar{D}^*$ with

$$\operatorname{dia}(\gamma^*) \leq \frac{|z-w|}{c}.$$

This, in turn, insures that D^* is b-arcwise connected for any b larger than $1/c$. Similarly, if D satisfies an exterior β-wedge condition for some β, then D is b-arcwise connected for some b.

The following result was established by Lesley [7, Theorem 2] using strip-mapping techniques. In view of the above considerations, it could also be inferred as a corollary of Theorem 9. For reasons that ought to become clear in the closing section of this paper, however, we present a short proof in full detail.

THEOREM 11. *Let f be a conformal mapping of a bounded Jordan domain D onto B. Suppose that D satisfies an exterior β-wedge condition for some β, $0<\beta\leq 1$. Then f belongs to* $\mathrm{Lip}_\alpha(D)$ *for* $\alpha = 1/(2-\beta)$.

Proof. Fix z_0 in D and choose R, $0<R<\operatorname{dist}(z_0, \partial D)$, for which the exterior β-wedge condition is satisfied. Next choose b such that D is b-arcwise connected. Fix a pair of points z and w on ∂D. By Theorem 3 it is sufficient to exhibit a constant $a>0$ such that the family Γ of cross-cuts of D which separate z_0 from z and w satisfies

$$M(\Gamma) \geq \frac{1}{(2-\beta)\pi} \log \frac{a}{|z-w|}. \tag{41}$$

We establish this with $a = R/2b$.

If $|z-w| \geq R/2b$, (41) follows trivially. Assume, therefore, that $|z-w| < R/2b$. The points z and w are the terminal points of a cross-cut γ_0 of D, which can be so chosen that

$$\operatorname{dia}(\gamma_0) \leq 2b\,|z-w|.$$

When $2b\,|z-w| < r < R$, the set $D \cap \partial B(w, r)$ has a component which separates γ_0 from z_0 and which, as a result, belongs to Γ. Let S be a closed sector in $\mathbf{C}\backslash D$ with radius R, with angular opening $\beta\pi$, and with vertex w. It is then apparent that Γ minorizes the family Γ^* composed of the circular arcs $\partial B(w, r)\backslash S$, where $2b\,|z-w| < r < R$. Consequently,

$$M(\Gamma) \geq M(\Gamma^*) = \frac{1}{(2-\beta)\pi} \log \frac{R}{2b\,|z-w|},$$

confirming (41).

We next show that the second major result in [7] can be obtained using Theorem 5 in this paper.

THEOREM 12. *Let f be a conformal mapping of B onto a bounded Jordan domain D. Suppose that D satisfies an interior β-wedge condition for some β, $0<\beta<1$. Then f belongs to* $\operatorname{Lip}_\beta(B)$.

Proof. Fix z_0 in D and choose R, $0<R<\operatorname{dist}(z_0, \partial D)$, for which the interior β-wedge condition is satisfied. Next choose c, $0<c<1$, such that every cross-cut of D satisfies (40). (Should the endpoints z and w of a cross-cut coincide, we interpret $C(z, w)$ to mean this single point.) By Theorem 5, it suffices to find a constant $b>0$ such that, for any pair of points z and w on ∂D, the family Γ of all cross-cuts of D which separate z_0 from z and w satisfies

$$M(\Gamma) \le \frac{1}{\beta\pi} \log \frac{b}{|z-w|}. \tag{42}$$

We establish this with

$$b = \frac{8R}{c} \exp\left[\frac{2\beta\pi^2}{\log 2} + \frac{16\beta\pi \operatorname{area}(D)}{c^2R^2}\right].$$

Fix z and w on ∂D. Let γ be an arc in Γ with endpoints z' and w'. Then either z_0 lies in the subdomain of D bounded by the Jordan curve $\gamma \cup C(z', w')$ or else $C(z', w')$ contains z and w. Thus, by virtue of (40),

$$\min\{R, |z-w|\} \le \operatorname{dia}(\gamma) + \operatorname{dia}[(C(z', w')] \le \left(1+\frac{1}{c}\right) \operatorname{dia}(\gamma) < \frac{2}{c} \operatorname{dia}(\gamma). \tag{43}$$

If $|z-w| \ge R$, we have, accordingly,

$$\operatorname{dia}(\gamma) \ge \frac{cR}{2} \tag{44}$$

for each γ in Γ, whence

$$M(\Gamma) \le \frac{4 \operatorname{area}(D)}{c^2R^2}$$

and (42) follows without difficulty.

We proceed assuming $|z - w| < R$. In light of (43),

$$\operatorname{dia}(\gamma) > \frac{c\,|z - w|}{2} \tag{45}$$

for each γ in Γ. Consider the family

$$\Gamma_1 = \{\gamma \in \Gamma : \gamma \text{ meets } \bar{B}(w, c\,|z - w|/8)\}.$$

By (45), each γ in Γ_1 must meet $\partial B(w, c\,|z - w|/4)$. The minorization property of the modulus gives

$$M(\Gamma_1) \leq \frac{2\pi}{\log 2}. \tag{46}$$

Next, consider the family

$$\Gamma_2 = \left\{\gamma \in \Gamma : \operatorname{dia}(\gamma) \geq \frac{cR}{4}\right\}.$$

Obviously

$$M(\Gamma_2) \leq \frac{16 \operatorname{area}(D)}{c^2 R^2}. \tag{47}$$

The family Γ_2 includes each γ in Γ that meets both $B(w, R/2)$ and $\partial B(w, R)$. It contains, as well, every γ in Γ for which there exists a radius of $B(w, R/2)$ lying in D, but failing to intersect γ. In fact, mimicking the argument used to derive (43), we have under the "omitted radius" condition

$$\frac{R}{2} \leq \operatorname{dia}(\gamma) + \operatorname{dia}[C(z', w')] \leq \left(1 + \frac{1}{c}\right) \operatorname{dia}(\gamma) \leq \frac{2}{c} \operatorname{dia}(\gamma).$$

Finally, consider the family $\Gamma_3 = \Gamma \backslash (\Gamma_1 \cup \Gamma_2)$. Choose a closed sector S in $\bar{D}$ with radius R, with angular opening $\beta\pi$, and with vertex w. Each γ in Γ_3 lies in the annulus $B(w, R) \backslash \bar{B}(w, c\,|z - w|/8)$ and intersects every radius of S, implying that γ must possess a subarc joining the straight sides of S through S. We can once again invoke the minorization property of the modulus and infer that

$$M(\Gamma_3) \leq \frac{1}{\beta\pi} \log \frac{8R}{c\,|z - w|}. \tag{48}$$

As a combination of (46), (47) and (48) we obtain

$$
\begin{aligned}
M(\Gamma) &\le M(\Gamma_1) + M(\Gamma_2) + M(\Gamma_3) \\
&\le \frac{2\pi}{\log 2} + \frac{16\,\text{area}\,(D)}{c^2R^2} + \frac{1}{\beta\pi}\log\frac{8R}{c\,|z-w|} \\
&\le \frac{1}{\beta\pi}\log\frac{b}{|z-w|},
\end{aligned}
$$

provided b is as indicated.

The sharpness of the Hölder exponents in Theorems 11 and 12 is demonstrated by choosing for D an appropriate polygon.

7. Disk conditions

We conclude this article with a brief discussion of a class of domains satisfying a β-wedge condition for all β, $0<\beta<1$.

A bounded Jordan domain D is said to satisfy an *interior* (respectively, *exterior*) *disk condition* if there exists an $R>0$ such that, for every w in ∂D, there is a closed disk of radius R containing w which lies in $\bar{D}$ (respectively, in $\mathbf{C}\setminus D$). The remarks prior to Theorem 11 in Section 6 remain valid for domains satisfying disk conditions.

THEOREM 13. *Let f be a conformal mapping of a bounded Jordan domain D onto B. Suppose that D satisfies an exterior disk condition. Then f belongs to* $\mathrm{Lip}_\alpha\,(D)$ *for* $\alpha = 1$.

Proof. The argument parallels that in the proof of Theorem 11. Under the present hypotheses the estimate (41) can be improved to read

$$M(\Gamma) \ge \frac{1}{\pi}\log\frac{a}{|z-w|}, \tag{49}$$

with $a = 3R/8b$. Indeed, we now take S to be a closed disk in $\mathbf{C}\setminus D$ of radius R containing w, rather than the earlier sector. The corresponding arc family Γ^* can then be estimated below using (10) to arrive at (49).

Our methods yield a new proof for the following natural companion to Theorem 13, a classical result often attributed to Kellogg:

THEOREM 14. *Let f be a conformal mapping of B onto a bounded Jordan domain D. Suppose that D satisfies an interior disk condition. Then f belongs to* $\mathrm{Lip}_\beta(B)$ *for* $\beta = 1$.

Proof. We mimic the proof of Theorem 12. Under the stronger hypothesis of the present theorem estimate (42) can be sharpened to

$$M(\Gamma) \leq \frac{1}{\pi} \log \frac{8b}{|z-w|}, \tag{50}$$

with b as in (42). To see this, merely replace the sector S in the earlier proof with a closed disk of radius R contained in $\bar{D}$ and containing w. The only essential change to be made, then, occurs in inequality (48), which by virtue of (8) can be improved to

$$M(\Gamma_3) \leq \frac{1}{\pi} \log \frac{64R}{c\,|z-w|}.$$

This will yield (50).

We close this paper with a combination of the two preceding results.

COROLLARY 2. *Let f be a conformal mapping of a bounded Jordan domain D onto B. Suppose that D satisfies both an exterior and an interior disk condition. Then f is a quasi-isometry.*

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