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Properties of the scattering map

THOMAS KAPPELER and EUGENE TRUBOWITZ

§1. Introduction

To define the scattering function and the scattering map, let us consider the Schrödinger equation on the whole line

$$-\frac{d^2}{dx^2}y(x)+q(x)y(x)=k^2y(x) \tag{1.1}$$

where $q(x)$ is a complex valued potential with $\int_{-\infty}^{\infty}|q(x)|\,(1+x^2)\,dx<\infty$. Denote by $f_1(x, k)$ and $f_2(x, k)$ ($\operatorname{Im} k \geq 0$) the so called Jost functions, i.e. the solutions of (1.1) with $f_1(x, k) \sim e^{ikx}$ for $x \to +\infty$ and $f_2(x, k) \sim e^{-ikx}$ for $x \to -\infty$ respectively. For k in $\mathbb{R}\backslash\{0\}$, $f_1(x, k)$ and $f_1(x, -k)$ are solutions of the same differential equation (1.1) but with different boundary conditions at $+\infty$, so they must be linearly independent. The function $f_2(x, k)$ is also a solution of (1.1) and can thus be represented as a linear combination of $f_1(x, k)$ and $f_1(x, -k)$: for k in $\mathbb{R}\backslash\{0\}$

$$f_2(x, k)=\frac{W(k)}{2ik}f_1(x, -k)+\frac{S(k)}{2ik}f_1(x, k).$$

Similarly one finds the corresponding expression for $f_1(x, k)$ (k in $\mathbb{R}\backslash\{0\}$)

$$f_1(x, k)=\frac{W(k)}{2ik}f_2(x, -k)+\frac{S(-k)}{2ik}f_2(x, k).$$

$S(k):=S(k, q)$ is called the scattering function associated to q. The functions $W(k)$ and $S(k)$ are related to the more often used reflection coefficients $R_1(k)$ and $R_2(k)$ and the transmission coefficient $T(k)$ in the following way:

$$R_1(k)=\frac{S(k)}{W(k)}, \qquad R_2(k)=\frac{S(-k)}{W(k)}, \qquad T(k)=\frac{2ik}{W(k)}.$$

The matrix

$$\begin{pmatrix} T(k) & R_2(k) \\ R_1(k) & T(k) \end{pmatrix} \quad (k \text{ in } \mathbb{R}\backslash\{0\})$$

is called the scattering matrix [3, 4]. Its coefficients have a straightforward physical interpretation: $T(k)f_1(x, k) = R_2(k)f_2(x, k) + f_2(x, -k)$ is a solution of the time independent Schrödinger equation (1.1) which corresponds to a particle sent in freely from $-\infty$. The appropriate asymptotic form of the wave function describing the particle at $x = -\infty$ is given by $R_2(k)e^{-ikx} + e^{ikx}$ where $R_2(k)e^{-ikx}$ is the part of the wave function which is reflected by the potential. The asymptotic form of the wave function describing the particle at $x = +\infty$ consists of the transmitted part and is given by $T(k)e^{ikx}$.

The aim of this paper is to analyse the so-called scattering map S associating to a potential q its scattering function $S(q) := S(\cdot, q)$. First we will show that the map S can be defined between certain function spaces as a holomorphic map (Section 2). In Section 3 we compute the derivative d_qS of S and prove that d_qS is boundedly invertible on the corresponding tangent spaces. In Section 4 the global behavior of S is studied.

We investigate S rather than the map R_1 associating to a potential q its reflection coefficient $R_1(q)$ because S can be defined on nicer function spaces. Let us point out that in delicate investigations of scattering theory the map S was useful to other people as well [7, 8].

The map R_1 has been studied by many authors [e.g. 2, 3, 4, 9]. Their main interest was to characterize the properties of the reflection coefficients corresponding to potentials in certain spaces, e.g. in a weighted L^1-space, and to solve the inverse problem. It is for example known that a real valued potential q in L_1^1 with no bound states is completely determined by its reflection coefficient $R_1(q)$ [3, 4].

One motivation to study the scattering map in greater detail comes from the well-known fact [5] that $S(q)$ linearizes the Korteweg–deVries equation (KdV), i.e. if $u(x, t)$ is a solution of $\partial_t u(x, t) - 6u(x, t)\partial_x u(x, t) + \partial_x^3 u(x, t) = 0$ then $\partial_t S(k, u(\cdot, t)) + k^3 S(k, u(\cdot, t)) = 0$. To solve the initial value problem of KdV one can proceed in the following way: to $u(\cdot, 0)$ one associates $S(k, u(\cdot, 0))$, solves the linear differential equation for S to get $S(k, t)$ and, provided one can solve the inverse problem, associates to $S(k, t)$ the potential $u(\cdot, t)$. This shows that the scattering map can be used to solve certain non-linear differential equations in an analogous way as the Fourier transform on the line is used to solve certain linear differential equations. The analogy between the Fourier transform and the scattering map has been observed before [e.g. 1, 2, 3] but primarily on a formal level. In the course of our investigations we will make this analogy more precise.

The first problem arises in the choice of the spaces where the scattering map should be set up. If we look at the asymptotics of $S(k, q)$ as $k \to \pm\infty$ we get

$$S(k, q) = \int_{-\infty}^{\infty} e^{2ikt} q(t)\, dt + O\left(\frac{1}{k}\right).$$

So in first approximation $S(\cdot, q)$ is the Fourier transform of q and a natural choice of spaces would be L^2. However for q in L^2 the scattering function $S(k, q)$ is not defined and it is this fact that complicates the situation. Another choice could be the Schwartz space of rapidly decreasing functions which is no longer a Banach space. As is well-known it is difficult to do analysis in Fréchet spaces. We consider here weighted Sobolev spaces. Let us introduce the following notation:

$$H_{n,\alpha} := \{f:\mathbb{R} \to \mathbb{C}: x^{\beta}\partial_x^j f(x) \in L^2, 0 \le \beta \le \alpha, 0 \le j \le n\}$$

and

$$H^{\#}_{n,\alpha} := \{f \in H_{n,\alpha}: x^{\beta}\partial_x^{n+1} f(x) \in L^2, 1 \le \beta \le \alpha\}.$$

In Section 2 we prove that for $N \ge 3$ S maps $H_{N,N}$ into $H^{\#}_{N-1,N}$. So for q in $H_{N,N}$, $S(\cdot, q)$ is almost in $H_{N,N}$ except that the N-th derivative $\partial_k^N S(k, q)$ is not in L^2_{loc} around $k = 0$, however, $k^{\beta}\partial_k^N S(k, q)$ is in L^2 for $1 \le \beta \le N$. This property makes the scattering map again very similar to the Fourier transform which maps $H_{N,N}$ into $H_{N,N}$.

Clearly unlike the Fourier transform S will no longer be linear. However, we will show (Section 2) that S is not only continuous but also differentiable and thus holomorphic.

In a next step (Section 3) we investigate on the local properties of S. First we show that the derivative $d_q S[v]$ of S in direction $v (v \in H_{N,N})$ can be computed explicitly

$$d_q S[v](k) = \int_{-\infty}^{\infty} dx v(x) f_1(x, -k, q) f_2(x, k, q).$$

For $q = 0$ the Jost functions are given by $f_1(x, -k) = e^{-ikx}$ and $f_2(x, k) = e^{ikx}$, thus the derivative $d_q S$ at $q = 0$ is the Fourier transform

$$d_{q=0} S[v](k) = \int_{-\infty}^{\infty} dx v(x) e^{-2ikx}.$$

One could be tempted to guess that for $q \ne 0$ $d_q S$ is a compact perturbation of the Fourier transform, but this is not the case. We prove that $d_q S: H_{N,N} \to H^{\#}_{N-1,N}$ is $1-1$ and onto for $N \ge 3$ and q in $Q_N(\mathbb{C}) := \{q \in H_{N,N}: W(k, q) \ne 0 \text{ in } \operatorname{Im} k \ge 0\}$. To show it we use the following fundamental orthogonality relation, simple to verify

$$\delta(k - l) = \frac{1}{\pi} \frac{T(l)T(-l)}{2il} \int_{-\infty}^{\infty} dx f_1(x, -k) f_2(x, k) \frac{d}{dx} f_1(x, l) f_2(x, -l),$$

where δ denotes as usual Dirac's delta function and where the integral on the right hand side is a distribution which has to be integrated against functions $\sigma(l)$ in $L^1(\mathbb{R})$. It turns out that the inverse $(d_qS)^{-1}: H^{\#}_{N-1,N} \to H_{N,N}$ of d_qS again can be computed explicitly and is given by $(\sigma \in H^{\#}_{N-1,N})$

$$(d_qS)^{-1}(\sigma)(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} dl\sigma(l)\frac{T(l)T(-l)}{2il}\frac{\partial}{\partial x}f_1(x, l)f_2(x, -l).$$

Now let us turn to the global properties of S. We restrict ourselves to real valued potentials which are contained in the set $Q_N(\mathbb{R}) := \{q:\mathbb{R} \to \mathbb{R} \mid q \in H_{N,N}; W(k, q) \neq 0 \text{ in } \operatorname{Im} k \geq 0\}$. Let us denote by $\mathcal{S}_N$ the set $\{\sigma \in H^{\#}_{N-1,N}: \sigma(-k) = \sigma(k)^*; \sigma(0) > 0\}$ where * means complex conjugation. In Section 4 we prove that $S: Q_N(\mathbb{R}) \to \mathcal{S}_N$ is $1-1$ and onto.

In order to summarize our results we have to introduce still another space, namely $H^{\#r}_{N-1,N} := \{f \in H^{\#}_{N-1,N}: f(k)^* = f(-k) \quad (k \text{ in } \mathbb{R})\}$.

THEOREM. *If* $N \geq 3$ *then*

(1) $S: Q_N(\mathbb{R}) \to \mathcal{S}_N$ *is a real analytic isomorphism.*

(2) *In particular at every* q *in* $Q_N(\mathbb{R})$ the Jacobian $d_qS: H_{N,N}(\mathbb{R}, \mathbb{R}) \to H^{\#r}_{N-1,N}$ *is boundedly invertible. Moreover* d_qS *is an integral operator given by*

$$d_qS(v)(k) = \int_{-\infty}^{\infty} dxf_1(x, -k, q)f_2(x, k, q)v(x) \quad (v \in H_{N,N}(\mathbb{R}, \mathbb{R})).$$

The inverse $(d_qS)^{-1}$ *is also an integral operator and given by*

$$(d_qS)^{-1}(\sigma)(x) = \int_{-\infty}^{\infty} dk\frac{1}{\pi}\frac{T(k)T(-k)}{2ik}\frac{\partial}{\partial x}[f_1(x, k)f_2(x, -k)]\sigma(k) \quad (\sigma \in H^{\#r}_{N-1,N}).$$

The notation is mostly standard. Besides the spaces already introduced we define for an arbitrary Banach space E the Banach space $C^0([a, \infty[, E)$ of all continuous functions $f:[a, \infty[\to E$ with $\lim_{t\to\infty} f(t) = 0$ (a in $\mathbb{R}$). $C^0_+(\mathbb{R}, E)$ denotes the intersection of all the spaces $C^0([a, \infty[, E)$ (a in $\mathbb{R}$). Similarly one defines $C^0(]-\infty, a], E)$ and $C^0_-(\mathbb{R}, E)$. $L^2_N(]a, \infty[, E)$ $(a \geq -\infty)$ denotes the Banach space of all Bochner measurable functions $f:]a, \infty[\to E$ such that $\int_a^\infty dx(1 + |x|^N)^2 \|f(x)\|^2 < \infty$. $L^2_+(\mathbb{R}, E)$ denotes the intersection of all spaces $L^2(]a, \infty[, E)$ with $a > -\infty$. Similarly one defines $L^2_-(\mathbb{R}, E)$. For two Banach spaces E and F, $\mathscr{L}(E, F)$ denotes the Banach space of all linear bounded operators. $\mathscr{L}(E, E)$ is denoted simply by $\mathscr{L}(E)$.

Let us recall from [6] the notion of differentiability. Let U be an open subset of a Banach space E and let F be another Banach space. $f: U \to F$ is called differentiable at x in U if there exists a bounded linear map $d_x f: E \to F$ such that $\|f(x+k) - f(x) - d_x f(k))\| = O(r)$ uniformly in $\|k\| < r$. $d_x f$ is called the derivative of f at x. f is called continuously differentiable on U if f is differentiable at each point x in U, and if the map $x \mapsto d_x f$ is continuous from U to $\mathscr{L}(E, F)$. In the same way one defines higher order derivatives. We will use freely well-known properties of differentiable maps (e.g. chain rule, inverse function theorem). Let v be in E. We say that the directional derivative $d_x f[v]$ of f at x in direction v exists, if the limit $\lim_{\varepsilon \to 0} (f(x + \varepsilon v) - f(x))/\varepsilon$, denoted by $d_x f[v]$, exists. If f is differentiable on U then f is differentiable at each point x in U in any direction v and $d_x f(v) = d_x f[v]$. Now assume that E and F are *complex* Banach spaces. $f: U \to F$ is called holomorphic on U if it is continuously differentiable on U. If f is holomorphic on U it is infinitely often differentiable on U and is represented by a Taylor series in a neighborhood of each point in U. One can show that f is holomorphic iff it is weakly holomorphic, i.e. if f is locally bounded and $z \to Lf(x + zk)$ is holomorphic on the disc $|z| < r$ in the usual sense of one complex variable for x in U, k in E and L in the dual of F, provided r is small enough.

Finally we introduce the notion of real analytic map. Let E, F be real Banach spaces, denote by $\mathbb{C}E$ and $\mathbb{C}F$ their complexifications and let $U \subseteq E$ be open. $f: U \to F$ is called real analytic on U if for each point x in U there exists a neighborhood V of x in $\mathbb{C}E$ and a holomorphic map $g: V \to \mathbb{C}F$ such that $f = g$ on $U \cap V$.

§2. Investigation on the scattering function and the reflexion and transmission coefficients

The main results of this section are contained in Theorem 2.15, 2.18 and 2.19. In order to prove these theorems we need a number of rather technical results.

In the whole section let N be an integer with $N \geq 2$ if not otherwise stated. In this section only we use for derivatives the following convenient notation: Let f be a function of n variables $x_1, \ldots, x_n$. Then $D^j f(x_1, \ldots, x_n)$ denotes the j-th partial derivative of f with respect to x_1. $D^{\alpha,\beta} f(x_1, \ldots, x_n)$ denotes the mixed derivative $(\partial^\alpha / \partial x_1^\alpha)(\partial^\beta / \partial x_2^\beta) f(x_1, \ldots, x_n)$.

For x, k in $\mathbb{R}$, y in $\mathbb{R}^+$ and q in $H_{N,N}$ we want to study the following integral equations

$$B_1(x, y, q) = \int_{x+y}^{\infty} q(z)\,dz + \int_0^y dz \int_{x+y-z}^{\infty} B_1(s, z, q)q(s)\,ds \quad (y \geq 0) \tag{2.1}$$

$$m_1(x, k, q) = 1 + \int_x^{\infty} D_k(t-x)m_1(t, k, q)q(t)\,dt \tag{2.2}$$

$$B_2(x, y, q) = \int_{-\infty}^{x+y} q(z)\,dz - \int_0^y dz \int_{-\infty}^{x+y-z} B_2(s, z, q)q(s)\,ds \quad (y \leq 0) \tag{2.3}$$

$$m_2(x, k, q) = 1 + \int_{-\infty}^{x} D_k(x-t)m_2(t, k, q)q(t)\,dt \tag{2.4}$$

where $D_k(y) := \int_0^y e^{2ikt}\,dt$.

We restrict ourselves to investigate on (2.1) and (2.2) alone. Equations (2.3) and (2.4) are treated similarly. We will often write $B(x, y, q)$ and $m(x, k, q)$ instead of $B_1(x, y, q)$ and $m_1(x, k, q)$ respectively.

We start investigating on equation (2.1). For that reason we introduce the following family of Banach spaces $E = E(x_0, n)$ (x_0 in $\mathbb{R}$, $n \geq 1$) of functions $f \in C^0([x_0, \infty[\times\mathbb{R}^+, \mathbb{C})$ with the norm

$$\begin{aligned}\|f\|_E &:= \sup\{|f(u, v)| : (u, v) \in [x_0, \infty[\times\mathbb{R}^+\} \\ &+ \int_0^{\infty} dy(1+y)^{2n} \sup\{|f(t, z)|^2 : (t, z) \in G(x_0, y)\} \\ &+ \int_{x_0}^{\infty} dx \int_0^{\infty} dy(1+y)^{2n-2} \sup\{|f(t, z)|^2 : (t, z) \in G(x, y)\} < \infty\end{aligned}$$

where

$$G(x, y) = \{(t, z) := 0 \leq z \leq y,\ t \geq x + y - z\}.$$

For q in $H_{N,N}$ let $P_1(q)$ denote the operator defined on $E(x_0, n)$ in the following way ($g \in E(x_0, n)$)

$$P_1(q)[g](x, y) := \int_0^y dz \int_{x+y-z}^{\infty} g(t, z)q(t)\,dt.$$

LEMMA 2.1. *Let x_0 be in $\mathbb{R}$ and $n \geq 1$. Then*
(1) *$P_1(q)$ is a linear bounded operator on $E(x_0, n)$.*

(2) $\|P_1(q)^m\| \le \frac{1}{m!}\left(\int_{x_0}^{\infty} |q(t)|\,(t-x_0)\,dt\right)^m \quad (m \ge 1)$

where $\|\cdot\|$ *denotes the operator norm in* $\mathscr{L}(E(x_0, n))$

(3) *The map* $P_1 : H_{N,N} \to \mathscr{L}(E(x_0, n))$ *is linear and bounded.* $(Id - P_1(q))^{-1}$ *is holomorphic in* q.

Proof: (1) For $x \ge x_0$, $y \ge 0$

$$|P_1(q)[g](x, y)| \le \sup_{(t,z)\in G(x,y)} |g(t, z)| \int_x^{\infty} |q(t)|\,(t-x)\,dt. \tag{2.5}$$

This implies that $P_1(q)[g]$ is in $E(x_0, n)$ and (1) follows. Moreover

$$\|P_1(q)\| \le \int_{x_0}^{\infty} |q(t)|\,(t-x_0)\,dt.$$

(2) follows by an induction argument on (2.5). We claim that

$$|P_1(q)^m[g](x, y)| \le \sup_{(t,z)\in G(x,y)} |g(t, z)| \frac{1}{m!}\left(\int_x^{\infty} dt\,|q(t)|\,(t-x_0)\right)^m.$$

It remains to prove the induction step from m to $m+1$.

$$\begin{aligned}
&|P_1(q)^{m+1}[g](x, y)| \\
&\le \int_0^y dz \int_{x+y-z}^{\infty} dt\,|q(t)| \sup_{(u,v)\in G(t,z)} |g(u, v)| \frac{1}{m!}\left(\int_t^{\infty} dw\,|q(w)|\,(w-x_0)\right)^m \\
&\le \sup_{(u,v)\in G(x,y)} |g(u, v)| \int_x^{\infty} dt \frac{-1}{m+1}\frac{d}{dt}\frac{1}{m!}\left(\int_t^{\infty} dw\,|q(w)|\,(w-x_0)\right)^{m+1}
\end{aligned}$$

(3) By (2) the von Neumann series $(Id - P_1(q))^{-1} = \sum_{m\ge 0} P_1(q)^m$ converges absolutely in $\mathscr{L}(E(x_0, n))$. As $P_1(q)$ is linear in q the analyticity of $(Id - P_1(q))^{-1}$ follows. Looking at equation (2.1) one sees that

$$B(x, y, q) = (Id - P_1(q))^{-1}\left(\int_{x+y}^{\infty} q(t)\,dt\right).$$

For later reference we state the following lemma easy to prove

LEMMA 2.2. *If* $x^\alpha f(x)$ *is in* L^2_+ *for* $0 \le \alpha \le n$ *with* $n \ge 1$, *then* $x^\alpha \int_x^\infty |f(y)|\, dy$ *is in* L^2_+ *for* $0 \le \alpha \le n-1$.

With this result one concludes that $\int_{x+y}^\infty q(t)\, dt$ is in $E(x_0, N-1)$ for any x_0 in $\mathbb{R}$ and is holomorphic in q.

Let us summarize our results in the following

LEMMA 2.3. *There exists a unique solution* $B(x, y, q)$ *in* $C^0_+(\mathbb{R}, L^2_{N-1}(\mathbb{R}^+)) \cap L^2_+(\mathbb{R}, L^2_{N-2}(\mathbb{R}^+))$ *of equation* (2.1), *continuous in x and y. B is holomorphic as a map of q from* $H_{N,N}$ *to* $C^0_+(\mathbb{R}, L^2_{N-1}(\mathbb{R}^+)) \cap L^2_+(\mathbb{R}, L^2_{N-2}(\mathbb{R}^+))$.

In order to prove that $D^{\alpha,\beta}B(x, y, q)$ is in $C^0_+(\mathbb{R}, L^2_{N-1}(\mathbb{R}^+))$ for $0 \le \alpha + \beta \le N+1$ one writes formally for $0 \le \alpha \le N$

$$D^{\alpha+1,0}B(x, y, q) = \sum_{j=0}^{\alpha} \binom{\alpha}{j} \int_0^y dz D^j q(x+y-z) D^{\alpha-j,0} B(x+y-z, z, q) \quad (2.6)$$

and

$$D^{0,1}B(x, y, q) = D^{1,0}B(x, y, q) + \int_x^\infty B(s, y, q) q(s)\, ds. \quad (2.7)$$

So it is convenient to introduce for f in $L^2_N(\mathbb{R})$ the following linear operator $P_2(f)$ on $C^0([x_0, \infty[, L^2_{N-1}(\mathbb{R}^+)) \cap L^2([x_0, \infty[, L^2_{N-2}(\mathbb{R}^+))$

$$P_2(f)[g](x, y) := \int_0^\infty dz f(x+y-z) g(x+y-z, z) \quad (x \ge x_0, y \ge 0).$$

LEMMA 2.4.

(1) $P_2(f)[g] \in C^0([x_0, \infty[, L^2_{N-1}(\mathbb{R}^+)) \cap L^2([x_0, \infty[, L^2_{N-2}(\mathbb{R}^+))$

(2) *The map* $P_2 : L^2_N(\mathbb{R}) \to \mathscr{L}(C^0([x_0, \infty[, L^2_{N-1}(\mathbb{R}^+)) \cap L^2([x_0, \infty[, L^2_{N-2}(\mathbb{R}^+)))$ *is linear and bounded.*

Proof. Let h be in $L^2(\mathbb{R}^+)$. Then

$$\left| \int_0^\infty h(y)(1+y)^{N-1} P_2(f)[g](x, y)\, dy \right|$$

$$= \left| \int_0^\infty dz f(x+z) \int_z^\infty dy h(y)(1+y)^{N-1} g(x+z, y-z) \right|$$

$$\le \int_0^\infty dz\, |f(x+z)|\, \|h\| \left(\int_0^\infty dy (1+y+z)^{2N-2} |g(x+z, y)|^2 \right)^{1/2}$$

and thus we get

$$\begin{aligned}
&\|(1+y)^{N-1}P_2(f)[g](x,y)\|_{L^2(\mathbb{R}^+)}\\
&\quad\le 2^{N-1}\int_0^\infty dz\,|f(x+z)|\,(1+z)^{N-1}\Big(\int_0^\infty dy\,|g(x+z,y)|^2\Big)^{1/2}\\
&\qquad+2^{N-1}\int_0^\infty dz\,|f(x+z)|\,\Big(\int_0^\infty dy\,|g(x+z,y)^2\,(1+y)^{2N-2}\Big)^{1/2}\\
&\quad\le 2^N\Big(\int_0^\infty dz\,|f(x+z)|^2(1+z)^{2N}\Big)^{1/2}\sup_{t\ge x}\Big(\int_0^\infty dy\,|g(t,y)|^2\,(1+y)^{2N-2}\Big)^{1/2}.
\end{aligned}$$

Using this inequality we get

$$\begin{aligned}
&\int_{x_0}^\infty \|(1+y)^{N-2}P_2(f)[g](x,y)\|^2_{L^2(\mathbb{R}^+)}\,dx\\
&\quad\le 2^{2N}\int_{x_0}^\infty dz\int_0^\infty dy\,|g(z,y)|^2\int_{x_0}^\infty dz\,|f(z)|^2\int_{x_0}^z(1+z-x)^{2N-2}\,dx\\
&\qquad+2^{2N}\int_{x_0}^\infty dz\,|f(z)|^2\,(z-x_0)\cdot\int_{x_0}^\infty dz\int_0^\infty dy\,|g(z,y)|^2\,(1+y)^{2(N-2)}.
\end{aligned}$$

Now (1) follows by Lebesgue's dominated convergence theorem.

LEMMA 2.5. *If* $0\le\alpha\le N$ *then*

(1) $D^{\alpha+1,0}B(x,y,q)\in C^0_+(\mathbb{R},L^2_{N-1}(\mathbb{R}^+))\cap L^2_+(\mathbb{R},L^2_{N-2}(\mathbb{R}^+))$.

(2) $D^{\alpha+1,0}B(x,y,q)$ *is holomorphic as a map of* q *from* $H_{N,N}$ *to* $C^0_+(\mathbb{R},L^2_{N-1}(\mathbb{R}^+))\cap L^2_+(\mathbb{R},L^2_{N-2}(\mathbb{R}^+))$.

Proof. One argues inductively, using formula (2.6) which can be written as

$$D^{\alpha+1,0}B(x,y,q)=\sum_{j=0}^{\alpha}\binom{\alpha}{j}P_2(D^{\alpha-j}q)[D^{j,0}B(\cdot,\cdot,q)](x,y). \tag{2.8}$$

For $\alpha=0$, $D^{1,0}B(x,y,q)=P_2(q)[B(\cdot,\cdot,q)](x,y)\in C^0_+(\mathbb{R},L^2_{N-1}(\mathbb{R}^+))\cap L^2_+(\mathbb{R},L^2_{N-2}(\mathbb{R}^+))$ and as $B(\cdot,\cdot,q)$ is holomorphic in q the analyticity of $P_2(q)[B(\cdot,\cdot,q)]$ in q on $H_{N,N}$ follows by the product rule for differentiation. Now let us prove the induction step from α to $\alpha+1$. On the right hand side of formula (2.8) only terms of the form $D^{j,0}B(\cdot,\cdot,q)$ with $j\le\alpha$ are involved. Using Lemma 2.4 and the induction hypothesis one concludes that $D^{\alpha+1,0}B(x,y,q)$ satisfies (1) and (2).

To discuss the derivatives $D^{\alpha,\beta}B(x,y,q)$ for $\beta\ge1$ we still need another lemma.

LEMMA 2.6. *q induces a linear operator $P_3(q)$ on $C^0([x_0, \infty[, L^2_{N-1}(\mathbb{R}^+)) \cap L^2([x_0, \infty[, L^2_{N-2}(\mathbb{R}^+))$ in the following way ($x \geq x_0$)*

$$P_3(q)[g](x, y) := \int_x^\infty q(s)g(s, y)\, ds.$$

The following properties hold:

(1) $P_3(q)[g](x, y) \in C^0([x_0, \infty[, L^2_{N-1}(\mathbb{R}^+)) \cap L^2([x_0, \infty[, L^2_{N-2}(\mathbb{R}^+))$

(2) $\|P_3(q)\| \leq \int_{x_0}^\infty |q(s)|^2(1 + s - x_0)^2\, ds$ *where $\|\cdot\|$ denotes the operator norm on*

$$\mathscr{L}(C^0([x_0, \infty[, L^2_{N-1}(\mathbb{R}^+)) \cap L^2([x_0, \infty[, L^2_{N-2}(\mathbb{R}^+))).$$

Proof. Let h be in $L^2(\mathbb{R}^+)$. Then

$$\left| \int_0^\infty dy h(y)(1+y)^{N-1} \int_x^\infty ds q(s)g(s, y) \right|$$
$$\leq \int_x^\infty ds\, |q(s)|\, \|h\| \left(\sup_{t \geq x} \int_0^\infty dy\, |g(t, y)|^2\, (1+y)^{2N-2} \right)^{1/2}.$$

On the other side

$$\int_{x_0}^\infty dx \left\| \int_x^\infty ds q(s)g(s, y) \right\|^2_{L^2_{N-2}(\mathbb{R}^+)}$$
$$\leq \int_{x_0}^\infty dx \int_x^\infty ds\, |q(s)|^2 \int_{x_0}^\infty ds \int_0^\infty dy(1+y)^{2(N-2)}\, |g(s, y)|^2.$$

So (1) and (2) follow.

LEMMA 2.7.

(1) $D^{\alpha,\beta}B(x, y, q) \in C^0_+(\mathbb{R}, L^2_{N-1}(\mathbb{R}^+)) \cap L^2_+(\mathbb{R}, L^2_{N-2}(\mathbb{R}^+))$ *for* $0 \leq \alpha + \beta \leq N + 1$.

(2) $D^{\alpha,\beta}B(\cdot, \cdot, q)$ *is holomorphic as a map of q from $H_{N,N}$ to* $C^0_+(\mathbb{R}, L^2_{N-1}(\mathbb{R}^+)) \cap L^2_+(\mathbb{R}, L^2_{N-2}(\mathbb{R}^+))$.

Proof. The proof proceeds by induction on $m = \alpha + \beta$. For $m = 0$ statements (1) and (2) are already proved. In view of Lemma 2.5 let us assume that $D^{\alpha,\beta}B(x, y, q) \in C^0_+(\mathbb{R}, L^2_{N-1}(\mathbb{R}^+)) \cap L^2_+(\mathbb{R}, L^2_{N-2}(\mathbb{R}^+))$ for $0 \leq \alpha + \beta \leq m$ and $\alpha + \beta = m + 1$ with $\beta \leq \beta_0(\leq m)$ and $\alpha \geq \alpha_0 := m + 1 - \beta_0(\geq 1)$. One has to prove that

$$D^{\alpha_0-1,\beta_0+1}B(x, y, q) \in C^0_+(\mathbb{R}, L^2_{N-1}(\mathbb{R}^+)) \cap L^2_+(\mathbb{R}, L^2_{N-2}(\mathbb{R}^+))$$

and is holomorphic in q. Using (2.7) one gets formally

$$D^{\alpha_0-1,\beta_0+1}B(x, y, q) = D^{\alpha_0,\beta_0}B(x, y, q) + D^{\alpha_0-1,0}\int_x^\infty q(s)D^{0,\beta_0}B(s, y, q)\, ds.$$

By the induction hypothesis $D^{\alpha_0,\beta_0}B(x, y, q)$ is in $C^0_+(\mathbb{R}, L^2_{N-1}(\mathbb{R}^+)) \cap L^2_+(\mathbb{R}, L^2_{N-2}(\mathbb{R}^+))$ and is holomorphic in q. If $\alpha_0 = 1$ then $\int_x^\infty q(s)D^{0,\beta_0}B(s, y, q)\, ds$ is in $C^0_+(\mathbb{R}, L^2_{N-1}(\mathbb{R}^+)) \cap L^2_+(L^2_{N-2}(\mathbb{R}^+))$ by Lemma 2.6 and the induction hypothesis. If $\alpha_0 \geq 2$ then

$$D^{\alpha_0-1,0}\int_x^\infty q(s)D^{0,\beta_0}B(s, y, q)\, ds = -\sum_{j=0}^{\alpha_0-2}\binom{\alpha_0-2}{j}D^{j,\beta_0}B(x, y, q)D^{\alpha_0-2-j}q(x).$$

But $0 \leq j + \beta_0 \leq \alpha_0 - 2 + \beta_0 \leq m$ so one can apply the induction hypothesis on $D^{j,\beta_0}B(x, y, q)$. This implies that $D^{\alpha_0-1,0}\int_x^\infty q(s)D^{0,\beta_0}B(s, y, q)\, ds$ is in $C^0_+(\mathbb{R}, L^2_{N-1}(\mathbb{R}^+)) \cap L^2_+(\mathbb{R}, L^2_{N-2}(\mathbb{R}^+))$.

It is a well-known fact that $m(x, k, q) - 1 = \int_0^\infty B(x, y, q)e^{2iky}\, dy$ where $m(x, k, q)$ is the solution of (2.2).

Thus one gets the following

COROLLARY 2.8. *If $0 \leq j \leq N - 1$, $0 \leq \alpha \leq j$ and $\beta \geq 0$ such that $0 \leq \alpha + \beta \leq N + 1$ then*

(1) *$k^\alpha D^{\beta,j}(m(x, k, q) - 1)$ is in $C^0_+(\mathbb{R}, L^2(\mathbb{R}))$ and is holomorphic as a map of q from $H_{N,N}$ to $C^0_+(\mathbb{R}, L^2(\mathbb{R}))$.*

(2) *If in addition $j \leq N - 2$ then $k^\alpha D^{\beta,j}(m(x, k, q) - 1)$ is in $L^2_+(\mathbb{R}, L^2(\mathbb{R}))$ and is holomorphic as a map of q from $H_{N,N}$ to $L^2_+(\mathbb{R}, L^2(\mathbb{R}))$.*

Proof. One has $D^{\beta,j}(m(x, k, q) - 1) = \int_0^\infty (2iy)^j D^{\beta,0}B(x, y, q)e^{2iky}\, dy$. As $\alpha \leq j$ one can integrate α times by parts without getting boundary terms to give

$$(2ik)^\alpha D^{\beta,j}(m(x, k, q) - 1) = (-1)^\alpha \int_0^\infty D^{0,\alpha}[(2iy)^j D^{\beta,0}B(x, y, q)]e^{2iky}\, dy.$$

Next we will prove that $m(x, k, q)$ has better differentiability properties than indicated in Corollary 2.8. For this reason we introduce the following linear operator $P_4(q)$ on $C^0([x_0, \infty[, L^2(\mathbb{R}))$

$$P_4(q)[g](x, k) := \int_x^\infty D_k(t-x)g(t, k)q(t)\,dt$$

where $x \geq x_0$, k in $\mathbb{R}$ and g in $C^0([x_0, \infty[, L^2(\mathbb{R}))$.

This definition has to be understood in the following way: For x in $\mathbb{R}$, $P_4(q)[g](x, \cdot)$ defines a linear bounded functional on $L^2(\mathbb{R})$,

$$\int_{-\infty}^{\infty} P_4(q)[g](x, k)h(k)\,dk = \int_x^\infty dtq(t) \int_0^{t-x} dz \int_{-\infty}^{\infty} dkh(k)e^{2ikz}g(t, k)$$

for h in $L^2(\mathbb{R})$. The right hand side is well defined and

$$\left| \int_{-\infty}^{\infty} P_4(q)[g](x, k)h(k)\,dk \right| \leq \int_{x_0}^{\infty} dt\,|q(t)|\,(t-x_0)\,\|g\|_{x_0}\,\|h\|$$

where $\|g\|_{x_0} = \sup_{x \geq x_0} \|g(x, \cdot)\|_{L^2(\mathbb{R})}$.

In order to have $P_4(q)[g]$ in $C^0([x_0, \infty[, L^2(\mathbb{R}))$, $P_4(q)[g](x, \cdot)$ must be continuous in x. But this can be easily verified using Lebesgue's convergence theorem. So we get the following

LEMMA 2.9.

(1) *$P_4(q)$ defines a bounded linear operator on $C^0([x_0, \infty[, L^2(\mathbb{R}))$ with*

$$\|P_4(q)^m\| \leq \frac{1}{m!}\left(\int_{x_0}^{\infty} |q(t)|\,(t-x_0)\,dt\right)^m$$

where the norm $\|\cdot\|$ is the operator norm in $\mathscr{L}(C^0([x_0, \infty[, L^2(\mathbb{R})))$.

(2) *$(Id - P_4(q))^{-1}$ is holomorphic as a map of q from $H_{N,N}$ to*

$$\mathscr{L}(C^0([x_0, \infty[, L^2(\mathbb{R}))) \quad \text{and} \quad \|(Id - P_4(q))^{-1}\| \leq \exp\left\{\int_{x_0}^{\infty} |q(t)|\,(t-x_0)\,dt\right\}.$$

Proof. (2) follows from (1) immediately as $P_4(q)$ is linear in q. So let us prove (1). It remains to prove the estimate for the iterated map.

It suffices to show that ($x \geq x_0$, $g \in C^0([x_0, \infty[, L^2(\mathbb{R}))$)

$$\|P_4(q)^m[g]\|_x \leq \frac{1}{m!}\left(\int_x^\infty |q(t)|\,(t-x_0)\,dt\right)^m \|g\|_x.$$

One does prove it by induction. For $m = 1$ the estimate has already been proved. So it remains to prove the induction step from m to $m+1$. Let h be in $L^2(\mathbb{R})$. Then

$$\begin{aligned}
&\left| \int_{-\infty}^{\infty} dk h(k) P_4(q)^{m+1}[g](x, k) \right| \\
&= \left| \int_{-\infty}^{\infty} dk h(k) \int_x^{\infty} D_k(t-x) P_4(q)^m[g](t, k) q(t)\, dt \right| \\
&= \left| \int_x^{\infty} dt q(t) \int_0^{t-x} dz \int_{-\infty}^{\infty} dk h(k) e^{2ikz} P_4(q)^m[g](t, k) \right| \\
&\le \int_x^{\infty} dt\, |q(t)|\, (t-x)\, \|h\| \frac{1}{m!} \left(\int_t^{\infty} |q(z)|\, (z - x_0)\, dz \right)^m \|g\|_x \\
&\le \|h\|\, \|g\|_x \int_x^{\infty} dt\, |q(t)|\, (t - x_0) \frac{1}{m!} \left(\int_t^{\infty} |q(z)|\, (z - x_0)\, dz \right)^m \\
&\le \|h\|\, \|g\|_x \frac{1}{(m+1)!} \left(\int_x^{\infty} |q(z)|\, (z - x_0)\, dz \right)^{m+1}.
\end{aligned}$$

Now let us look at the integral equation (2.2)

$$m(x, k, q) = 1 + \int_x^{\infty} D_k(t-x) m(t, k, q) q(t)\, dt.$$

Then formally for $1 \le \alpha \le N$

$$(Id - P_4(q))[k^{\alpha} D^{0,N} m(x, k, q)] = k^{\alpha} r(x, k, q)$$

where

$$k^{\alpha} r(x, k, q) = \sum_{n=1}^{N} \binom{N}{n} k^{\alpha} r_n(x, k, q) + k^{\alpha} r_{N+1}(x, k, q)$$

and

$$k^{\alpha} r_n(x, k, q) := k^{\alpha} \int_x^{\infty} \partial_k^n D_k(t-x) q(t) D^{0,N-n}(m(t, k, q) - 1)\, dt \quad (1 \le n \le N)$$

$$k^{\alpha} r_{N+1}(x, k, q) := k^{\alpha} \int_x^{\infty} \partial_k^N D_k(t-x) q(t)\, dt.$$

So formally $k^\alpha D^{0,N}m(x, k, q) = (Id - P_4(q))^{-1}[k^\alpha r(\cdot, \cdot, q)](x, k)$.

It remains to show that $k^\alpha r_n(x, k, q) \in C^0_+(\mathbb{R}, L^2(\mathbb{R}))$ for $1 \le \alpha \le N$ and $1 \le n \le N+1$ and is holomorphic as a map of q from $H_{N,N}$ to $C^0_+(\mathbb{R}, L^2(\mathbb{R}))$. This will be done in the following lemmata.

LEMMA 2.10. *For $1 \le \alpha \le N$, $k^\alpha r_{N+1}(x, k, q)$ is in $C^0_+(\mathbb{R}, L^2(\mathbb{R}))$ and is holomorphic as a map of q from $H_{N,N}$ to $C^0_+(\mathbb{R}, L^2(\mathbb{R}))$.*

Proof. By partial integration $|k\, \partial_k^N D_k(t-x)| \le (2(t-x))^N$ $(t \ge x)$, so $(2ik)^\alpha r_{N+1}(x, k, q)$ can be written as

$$(2ik)^\alpha \int_x^\infty \partial_k^N D_k(t-x) q(t)\, dt = (2ik)^{\alpha-1} \int_x^\infty dt e^{2ik(t-x)} (2i(t-x))^N q(t)$$

$$-(2ik)^{\alpha-1} 2iN \int_0^\infty dz e^{2ikz} (2iz)^{N-1} \int_{x+z}^\infty q(t)\, dt.$$

Integration by parts furnishes for $1 \le \alpha \le N$

$$(2ik)^\alpha r_{N+1}(x, k, q) = (-1)^{\alpha-1} \int_x^\infty dt e^{2ik(t-x)}\, \partial_t^{\alpha-1}([2i(t-x)]^N q(t))$$

$$+(-1)^\alpha 2iN \int_0^\infty dz e^{2ikz}\, \partial_z^{\alpha-1}\left((2iz)^{N-1} \int_{x+z}^\infty q(t)\, dt\right)$$

and thus the lemma follows.

LEMMA 2.11. *For $1 \le n \le N$ and $1 \le \alpha \le N$, $k^\alpha r_n(x, k, q)$ is in $C^0_+(\mathbb{R}, L^2(\mathbb{R}))$ and is holomorphic as a map of q from $H_{N,N}$ to $C^0_+(\mathbb{R}, L^2(\mathbb{R}))$.*

Proof.

CASE 1. $1 \le n \le N-1$ and $1 \le \alpha \le N-n$. We introduce the linear operators defined on $C^0([x_0, \infty[, L^2(\mathbb{R}))$ (x_0 in $\mathbb{R}$)

$$P_{5,n}(q)[g](x, k) = \int_x^\infty k\, \partial_k^n D_k(t-x) q(t) g(t, k)\, dt.$$

We use $|k\, \partial_k^n D_k(t-x)| \le (2(t-x))^n$ to get for

$$\int_{-\infty}^\infty h(k) P_{5,n}(q)[g](x, k)\, dk = \int_x^\infty dt q(t) \int_{-\infty}^\infty dk h(k) g(t, k) k\, \partial_k^n D_k(t-x)$$

the estimate

$$\left|\int_{-\infty}^{\infty} h(k)P_{5,n}(q)[g](x,k)\,dk\right|$$
$$\leq \int_x^{\infty} dt\,|q(t)|\,(2(t-x))^n\,\|h\|\sup_{s\geq x}\|g(s,\cdot)\|_{L^2(\mathbb{R})}<\infty.$$

This estimate shows that $P_{5,n}(q)[g]\in C^0([x_0,\infty[,L^2(\mathbb{R}))$ and that $P_{5,n}:H_{N,N}\to\mathcal{L}(C^0([x_0,\infty[,L^2(\mathbb{R})))$ is a bounded linear operator. From Corollary 2.8 one knows that $k^{\alpha-1}D^{0,N-n}(m(t,k,q)-1)$ is in $C^0_+(\mathbb{R},L^2(\mathbb{R}))$ for $1\leq\alpha\leq N-n$, $1\leq N-n\leq N-1$ and is holomorphic in q. Thus in case 1 the lemma follows.

CASE 2. $1\leq n\leq N-1$ and $N-n+1\leq\alpha$. We introduce $\beta:=\alpha-(N-n)$. Then $1\leq\beta\leq n$. By partial integration we get

$$(2ik)^\beta\,\partial_k^n D_k(t-x)=\sum_{\nu=0}^{\alpha-1}(-1)^\nu(2i(t-x))^{n-\nu}\left(\prod_{j=0}^{\nu-1}2i(n-j)\right)\partial_t^{\alpha-1-\nu}e^{2ik(t-x)}$$
$$+(-1)^\alpha\prod_{j=0}^{\alpha-1}2i(n-j)\int_0^{t-x}(2iz)^{n-\alpha}e^{2ikz}\,dz \tag{2.9}$$

where by convention $\prod_{j=0}^{-1}\cdots=1$. Thus

$$(2ik)^\alpha\int_x^{\infty}\partial_k^n D_k(t-x)D^{0,N-n}m(t,k,q)q(t)\,dt$$
$$=\sum_{\nu=0}^{\beta-1}(-1)^\nu\prod_{j=0}^{\nu-1}2i(n-j)\int_x^{\infty}dt(2i(t-x))^{n-\nu}q(t)$$
$$\times(2ik)^{N-n}D^{0,N-n}m(t,k,q)\,\partial_t^{\beta-1-\nu}e^{2ik(t-x)}$$
$$+(-1)^\beta\prod_{j=0}^{\beta-1}2i(n-j)\int_x^{\infty}\partial_k^{n-\beta}D_k(t-x)(2ik)^{N-n}(D^{0,N-n}m(t,k,q))q(t)\,dt.$$

In the following we will discuss the terms appearing in the expression above separately.

First let us look at the term $\int_x^\infty k\,\partial_k^{n-\beta}D_k(t-x)k^{N-n-1}(D^{0,N-n}m(t,k,q))q(t)\,dt$ ($0\leq n\leq N-1$ and $1\leq n-\beta\leq n-1\leq N-1$). One can apply case 1 and conclude that this term is in $C^0_+(\mathbb{R},L^2(\mathbb{R}))$ and is holomorphic in q.

For $\gamma+\delta=\beta-1$ partial integration yields

$$\int_x^{\infty}dt(\partial_t^\gamma e^{2ik(t-x)})[(2i(t-x))^{n-\delta}q(t)(2ik)^{N-n}D^{0,N-n}(m(t,k,q)]$$
$$=(-1)^\gamma\int_x^{\infty}dte^{2ik(t-x)}\left(\sum_{\varepsilon+\eta=\gamma}\partial_t^\varepsilon[(2i(t-x))^{n-\delta}q(t)](2ik)^{N-n}D^{\eta,N-n}m(t,k,q)\right).$$

From Corollary 2.8 we know that $(2ik)^{N-n}D^{\eta,N-n}m(t,k,q)$ is in $C^0_+(\mathbb{R}, L^2(\mathbb{R}))$ and is holomorphic in q. In view of Lemma 2.6 we conclude that in case 2 the lemma holds.

CASE 3. $n=N$ and $1\le\alpha\le N$. Again we apply formula (2.9) to get

$$\begin{aligned}(2ik)^\alpha\int_x^\infty &\partial_k^N D_k(t-x)(m(t,k,q)-1)q(t)\,dt\\ &=\sum_{\nu=0}^{\alpha-1}(-1)^\nu\Big(\prod_{j=0}^{\nu-1}2i(N-j)\Big)(-1)^{\alpha-1-\nu}\\ &\quad\times\int_x^\infty dte^{2ik(t-x)}\,\partial_t^{\alpha-1-\nu}[(2i(t-x))^{N-\nu}q(t)(m(t,k,q)-1)]\\ &\quad+(-1)^\alpha\prod_{j=0}^{\alpha-1}2i(N-j)\int_x^\infty \partial_k^{N-\alpha}D_k(t-x)(m(t,k,q)-1)q(t)\,dt.\end{aligned}$$

Again we have to discuss these terms separately. For this reason let us introduce operators $P_{6,\beta,j}(q)$ defined on $C^0([x_0,\infty[, L^2(\mathbb{R}))\cap L^2([x_0,\infty[, L^2(\mathbb{R}))$

$$P_{6,\beta,j}(q)[g](x,k)=\int_x^\infty dte^{2ik(t-x)}(t-x)^\beta(D^jq(t))g(t,k)$$

where $0\le\beta$, $j\le N$.

One derives in a by now familiar way the estimate

$$\|P_{6,\beta,j}(q)[g](x,\cdot)\|^2_{L^2(\mathbb{R})}\le\int_x^\infty dt\,|D^jq(t)|^2\,(t-x)^{2\beta}\int_x^\infty dt\,\|g(t,\cdot)\|^2_{L^2(\mathbb{R})}.$$

So $P_{6,\beta,j}(q)[g]$ is in $C^0([x_0,\infty[, L^2(\mathbb{R}))$ $(0\le\beta, j\le N)$. Clearly $P_{6,\beta,j}(q)$ is bounded and linear in q. From Corollary 2.8 we know that $D^{l,0}(m(t,k,q)-1)$ is in $C^0_+(\mathbb{R}, L^2(\mathbb{R}))\cap L^2_+(\mathbb{R}, L^2(\mathbb{R}))$ and holomorphic in q. then one concludes that $\int_x^\infty dte^{2ik(t-x)}(t-x)^\beta D^jq(t)D^{l,0}(m(t,k,q)-1)$ is in $C^0_+(\mathbb{R}, L^2(\mathbb{R}))$ and holomorphic in q for $0\le\beta\le N$, $0\le j\le N$, $0\le l\le N$.

It remains to look at terms of the form $\int_x^\infty dt\,\partial_k D_k(t-x)q(t)(m(t,k,q)-1)$ $(0\le\gamma\le N-1)$. For this reason we introduce a linear operator $P_{7,\gamma}(q)$ defined on $C^0([x_0,\infty[, L^2(\mathbb{R}))\cap L^2([x_0,\infty[, L^2(\mathbb{R}))$ in the following way

$$P_{7,\gamma}(q)[g](x,k):=\int_x^\infty dtq(t)\,\partial_k^\gamma D_k(t-x)g(t,k).$$

One derives easily the following estimate

$$\|P_{7,\gamma}(q)[g](x,\cdot)\|^2_{L^2(\mathbb{R})} \le \int_x^\infty dt\, |q(t)|^2 (2(t-x))^{2\gamma} \int_x^\infty dt\, \|g(t,\cdot)\|^2_{L^2(\mathbb{R})}.$$

So $P_{7,\gamma}(q)[g]$ is in $C^0([x_0,\infty[, L^2(\mathbb{R}))$ and is bounded as an operator from

$$C^0([x_0,\infty[, L^2(\mathbb{R})) \cap L^2([x_0,\infty[, L^2(\mathbb{R})) \quad \text{to} \quad C^0([x_0,\infty[, L^2(\mathbb{R})).$$

Again with Corollary 2.8 we conclude that

$$\int_x^\infty dt\, \partial_k^\gamma D_k(t-x)(m(t,k,q)-1)q(t) \quad \text{is in} \quad C^0_+(\mathbb{R}, L^2(\mathbb{R}))$$

and is holomorphic in q.

LEMMA 2.12. *For $1 \le \alpha \le N$, $k^\alpha D^{1,N} m(x,k,q)$ is in $C^0_+(\mathbb{R}, L^2(\mathbb{R}))$ and is holomorphic as a map of q from $H_{N,N}$ to $C^0_+(\mathbb{R}, L^2(\mathbb{R}))$.*

Proof. From (2.2) we obtain the formula $D^{1,0}m(x,k,q) = -\int_x^\infty e^{2ik(t-x)} m(t,k,q)\,dt$. From Corollary 2.8 we know that $k^\alpha D^{1,j} m(x,k,q)$ is in $C^0_+(\mathbb{R}, L^2(\mathbb{R}))$ $(0 \le \alpha \le j,\ 0 \le j \le N-1)$ and is holomorphic in q. For $1 \le \alpha \le N$ we write then formally

$$\begin{aligned} k^\alpha D^{1,N} m(x,k,q) &= -\sum_{n=0}^{N} k^\alpha \binom{N}{n} \int_x^\infty dt (2i(t-x))^n q(t) D^{0,N-n}(m(t,k,q)-1) e^{2ik(t-x)} \\ &\quad - k^\alpha \int_x^\infty dt (2i(t-x))^N q(t) e^{2ik(t-x)}. \end{aligned}$$

Similarly as in Lemma 2.11 one shows (1) and (2).

We summarize our results in the following theorem using

$$B_1(x,y,q) = \frac{1}{\pi} \int_{-\infty}^\infty (m_1(x,k,q)-1) e^{-2iky}\, dy.$$

THEOREM 2.13. *Let $N \ge 2$. Then*

(1) *$D^{j,k} B_1(x,y,q)$ is holomorphic as a map of q from $H_{N,N}$ to $C^0_+(\mathbb{R}, L^2_{N-1}(\mathbb{R}^+))$ for $0 \le j+k \le N+1$.*

(2) $D^{j,k}B_1(x, y, q)$ *is holomorphic as a map of* q *from* $H_{N,N}$ *to* $C^0_+(\mathbb{R}, L^2_N(\mathbb{R}^+))$ *for* $j = 0,1$ *and* $1 \le k \le N$.

Remark: Analogous results hold for $B_2(x, y, q)$.

As usual one introduces the Jost functions $f_1(x, k, q) = e^{ikx}m_1(x, k, q)$ and $f_2(x, k, q) = e^{-ikx}m_2(x, k, q)$ and then defines the Wronskian

$$W(k, q) := W[f_2(x, k, q), f_1(x, k, q)].$$

Clearly $W(k, q)$ is defined in $\operatorname{Im} k \ge 0$ and is holomorphic in $\operatorname{Im} k > 0$. For $\operatorname{Im} k = 0$ one introduces the scattering function $S(k, q)$

$$S(k, q) = W[f_1(x, -k, q), f_2(x, k, q)].$$

We now want to study the regularity and decay properties of W and S. For this reason we introduce the following representations for S and W given by Faddeev [4]

$$S(k, q) = \int_{-\infty}^{\infty} \pi_1(y, q) e^{-2iky}\, dy$$

where for $y > 0$

$$\pi_1(y, q) := -D^{1,0}B_1(0, y, q) + \int_{-\infty}^{0} D^{1,0}B_2(0, z, q)B_1(0, y - z, q)\, dz - \int_{y}^{\infty} D^{1,0}B_1(0, z, q)B_2(0, y - z, q)\, dz$$

and for $y < 0$

$$\pi_1(y, q) := D^{1,0}B_2(0, y, q) + \int_{-\infty}^{y} D^{1,0}B_2(0, z, q)B_1(0, y - z, q)\, dz - \int_{0}^{\infty} D^{1,0}B_1(0, z, q)B_2(0, y - z, q)\, dz.$$

Similarly one can write

$$W(k, q) = 2ik - \int_{-\infty}^{\infty} q(z)\, dz + \int_{0}^{\infty} \pi_2(y, q) e^{2iky}\, dy$$

where for $y \geq 0$

$$\begin{aligned}\pi_2(y, q) = D^{1,0}B_1(0, y, q) + D^{0,1}B_2(0, -y, q) - D^{1,0}B_2(0, -y, q)\\ - D^{0,1}B_1(0, y, q) + \int_0^\infty \{D^{1,0}B_1(0, z, q) - D^{0,1}B_1(0, z, q)\}B_2(0, z-y, q)\, dz\\ - B_1(0, 0, q)B_2(0, -y, q) - \int_0^\infty B_1(0, z, q)D^{1,0}B_2(0, z-y, q)\, dz.\end{aligned}$$

Let us discuss $\pi_1(y, q)$ first.

LEMMA 2.14.
(1) *$\pi_1(\cdot, q)$ is holomorphic as a map of q from $H_{N,N}$ to $L^2_{N-1}(\mathbb{R})$.*
(2) *$D^j\pi_1(\cdot, q)$ is holomorphic as a map of q from $H_{N,N}$ to $L^2_N(\mathbb{R})$ $(1 \leq j \leq N)$.*

Proof. The proof is divided into 5 steps:

STEP 1 [2]. $\pi_1(\cdot, q)$ is holomorphic as a map of q from $H_{N,N}$ to $L^2_{N-1}(\mathbb{R}^+)[L^2_{N-1}(\mathbb{R}^-)]$.

STEP 3 [4]. $D^j\pi_1(y, q)$ is holomorphic as a map of q from $H_{N,N}$ to $L^2_N(\mathbb{R}^+)[L^2_N(\mathbb{R}^-)]$ $(1 \leq j \leq N)$.

STEP 5. $D^j\pi_1(y, q) - D^jq(y)$ is continuous in y for $0 \leq j \leq N$.

It will suffice to show Step 1 and Step 3. Clearly Step 2 and Step 4 can be proved similarly. As concerns Step 5 one remarks that only the continuity at $y = 0$ is in question. Writing out the expression for $D^j\pi_1(y, q)$ in the cases $y > 0$ and $y < 0$ and letting $y \to 0$ the continuity follows if one can show that

$$(D^j\pi_1(\cdot, q) - D^jq)(0^+) = (D^j\pi_1(\cdot, q) - D^jq)(0^-)$$

or

$$\begin{aligned}D^{1,j}B_1(0, 0, q) + D^jq(0) + \sum_{\alpha+\beta=j-1} D^{1,\alpha}B_1(0, 0, q)D^{0,\beta}B_2(0, 0, q)\\ = D^{1,j}B_2(0, 0, q) - D^jq(0) + \sum_{\alpha+\beta=j-1} D^{1,\alpha}B_2(0, 0, q)D^{0,\beta}B_1(0, 0, q).\end{aligned}$$

It is not difficult to check that our hypothesis on q suffice to derive this equality following the line of arguments of the appendix in [2]. So let us come to the proof of Step 1 and Step 3. For $0 \le j \le N$ and $y \ge 0$

$$D^j\pi_1(y,q) = D^{1,j}B_1(0,y,q) + \int_{-\infty}^{0} D^{1,0}B_2(0,z,q)D^{0,j}B_1(0,y-z,q)\,dz$$
$$+ \sum_{\alpha+\beta=j-1} D^{1,\alpha}B_1(0,y,q)D^{0,\beta}B_2(0,0,q)$$
$$- \int_{y}^{\infty} D^{1,0}B_1(0,z,q)D^{0,j}B_2(0,y-z,q)\,dz.$$

If $j \ge 1$ one gets integrating by parts

$$- \int_{y}^{\infty} D^{1,0}B_1(0,z,q)D^{0,j}B_2(0,y-z,q)\,dz$$
$$= - \int_{y}^{\infty} D^{1,1}B_1(0,z,q)D^{0,j-1}B_2(0,y-z,q)\,dz$$
$$- D^{1,0}B_1(0,y,q)D^{0,j-1}B_2(0,0,q).$$

So for $1 \le j \le N$

$$D^j\pi_1(y,q) = D^{1,j}B_1(0,y,q) + \int_{-\infty}^{0} D^{1,0}B_2(0,z,q)D^{0,j}B_1(0,y-z,q)\,dz$$
$$+ \sum_{\substack{\gamma+\beta=j-1\\ \gamma\ge 1}} D^{1,\gamma}B_1(0,y,q)D^{0,\beta}B_2(0,0,q)$$
$$- \int_{y}^{\infty} D^{1,1}B_1(0,z,q)D^{0,j-1}B_2(0,y-z,q)\,dz.$$

From Theorem 2.13 Step 1 and Step 3 now follow.

As an application we get the following

THEOREM 2.15. *S is holomorphic as a map of q from $H_{N,N}$ to $H^{\#}_{N-1,N}$.*

Now let us turn towards $\pi_2(y,q)$ in order to analyze $W(k,q)$.

LEMMA 2.16.
(1) *$\pi_2(y,q)$ and $\partial_q^N\pi_2(y,q)$ are holomorphic as a map of q from $H_{N,N}$ to $L^2_{N-1}(\mathbb{R}^+)$.*
(2) *$D^j\pi_2(y,q)$ is holomorphic as a map of q from $H_{N,N}$ to $L^2_N(\mathbb{R}^+)$ $(1 \le j \le N-1)$.*

Proof. For $1 \leq j \leq N$ the integrals appearing in the expression for $D^j\pi_2(y, q)$ are splitted up, $\int_0^y = \int_0^{y/2} + \int_{y/2}^y$, and in $\int_{y/2}^y$ one integrates by parts to get

$$\begin{aligned}
D^j\pi_2(y, q) = {} & D^{1,j}B_1(0, y, q) + (-1)^j D^{0,j+1}B_2(0, -y, q) \\
& + (-1)^{j+1}D^{1,j}B_2(0, -y, q) - D^{0,j+1}B_1(0, y, q) \\
& - (-1)^j B_1(0, 0, q)D^{0,j}B_2(0, -y, q) \\
& + \sum_{\substack{\alpha+\beta=j-1 \\ \alpha\geq 1}} (-1)^\beta\{D^{1,\alpha}B_1(0, y, q) - D^{0,\alpha+1}B_1(0, y, q)\}D^{0,\beta}B_2(0, 0, q) \\
& + \int_0^{y/2} \{D^{1,0}B_1(0, z, q) - D^{0,1}B_1(0, z, q)\}D^{0,j}B_2(0, z-y, q)(-1)^j\, dz \\
& + \int_{y/2}^{y} \{D^{1,1}B_1(0, z, q) - D^{0,2}B_1(0, z, q)\}D^{0,j-1}B_2(0, z-y, q)(-1)^{j+1}\, dz \\
& + (-1)^{j+1}\{D^{1,0}B_1(0, y/2, q) - D^{0,1}B_1(0, y/2, q)\}D^{0,j-1}B_2(0, -y, q) \\
& + \sum_{\alpha+\beta=j-1} (-1)^\beta D^{0,\alpha}B_1(0, y, q)D^{1,\beta}B_2(0, 0, q) \\
& + (-1)^{j+1}\int_0^{y/2} B_1(0, z, q)D^{1,j}B_2(0, z-y, q)\, dz \\
& + (-1)^j\int_{y/2}^{y} D^{0,1}B_1(0, z, q)D^{1,j-1}B_2(0, z-y, q)\, dz \\
& (-1)^j B_1(0, y/2, q)D^{1,j-1}B(0, -y/2, q).
\end{aligned}$$

The results now follow easily from Theorem 2.13.

As an application we get

THEOREM 2.17. *Let $N \geq 2$. Then*
(1) *$W(k, q) - 2ik$ is holomorphic as a map of q from $H_{N,N}$ to $L^\infty(\mathbb{R})$.*
(2) *$k^\alpha(\partial^j/\partial k^j)(W(k, q) - 2ik)$ is holomorphic as a map of q from $H_{N,N}$ to $L^2(\mathbb{R})$ for (α, j) in $\{(\alpha, j): 0 \leq \alpha \leq j$ and $1 \leq j \leq N-1\} \cup \{(\alpha, N): 1 \leq \alpha \leq N-1\}$.*

Now let us recall from the introduction the definition

$$Q_N(\mathbb{C}) := \{q \in H_{N,N} : W(k, q) \neq 0,\ k \text{ in } \mathbb{R}\}.$$

Clearly $Q_N(\mathbb{C})$ is open in $H_{N,N}$.

Let us define as usual the reflection coefficients

$$R_1(k, q) := \frac{S(k, q)}{W(k, q)} \quad \text{and} \quad R_2(k, q) = +\frac{S(-k, q)}{W(k, q)}.$$

THEOREM 2.18. *Let $N \geq 3$. Then R_1 and R_2 are holomorphic as a map of q from $Q_N(\mathbb{C})$* to $H^{\#}_{N-1,N+1}$.

Proof. Clearly it suffices to prove the theorem for R_1. Writing $W(k, q)R_1(k, q) = S(k, q)$ one gets formally

$$W(k, q)D^jR_1(k, q) = D^jS(k, q) - \sum_{\nu=0}^{j-1} \binom{j}{\nu} D^\nu R_1(k, q)D^{j-\nu}W(k, q). \tag{2.10}$$

Now (1) and (2) follow using induction on j and the results of Theorem 2.15 and 2.17.

As usual one defines the transmission coefficient $T(k, q) = 2ik/W(k, q)$ (q in $Q_N(\mathbb{C})$). Using induction and Theorem 2.17 one obtains the following

THEOREM 2.19. *Let $N \geq 3$. Then*

(1) *$D^j(T(k, q) - 1)$ is holomorphic as a map of q from $Q_N(\mathbb{C})$ to $L^2_N(\mathbb{R})$.*

(2) *$D^j \dfrac{1}{W(k, q)}$ is holomorphic as a map of q from $Q_N(\mathbb{C})$ to $L^2_{N-1}(\mathbb{R})$.*

(3) *$k^\alpha D^N \dfrac{1}{W(k, q)}$ is holomorphic as a map of q from $Q_N(\mathbb{C})$ to $L^2(\mathbb{R})$ $(1 \leq \alpha \leq N)$.*

Remark. Theorem 2.18 and 2.19 show that the range of the reflection coefficient $R_1(\cdot, q)$ *for q in $Q_N(\mathbb{C})$ is not open in* $H^{\#}_{N-1,N+1}$ in general: From $T(k, q)T(-k, q) = 1 - R_1(k, q)R_1(-k, q)$ one concludes that

$$\frac{\partial^N}{\partial k^N} \frac{1 - R_1(k, q)R_1(-k, q)}{k^\alpha} \in L^2(\mathbb{R}) \quad \text{for} \quad \alpha = 0, 1.$$

§3. The Jacobian of the scattering map

In this section we will first compute explicitly the directional derivative of S and determine the kernel of the Jacobian d_qS. This kernel is used to show that

d_qS is boundedly invertible. Moreover we compute the kernel of the inverse $(d_qS)^{-1}$. By abuse of notation we denote the kernel of d_qS by $\partial S(k, q)/\partial q(x)$. We also compute the kernels of the derivatives of the reflection and transmission coefficients. The main results are summarized in Theorem 3.12.

In the whole section we assume $N \geq 3$ if not otherwise stated. Whenever possible we will not indicate explicitly the dependence on q of the functions considered. Let q in $Q_N(\mathbb{C})$ be a fixed given potential. Now let us look at the Jost functions $f_i(x, k) := f_i(x, k, q)$ $(i = 1,2)$ and compute the directional derivatives $d_qf_1[v]$ and $d_qf_2[v]$ for v in $H_{N,N}$. Clearly $d_qf_1[v]$ satisfies the Schrödinger equation

$$-\frac{\partial^2}{\partial x^2} d_qf_1[v](x, k) + q(x)d_qf_1[v](x, k) = k^2 d_qf_1[v](x, k) - v(x)f_1(x, k)$$

with the boundary conditions

(i) $\lim_{x \to +\infty} d_qf_1[v](x, k) = 0$ and

(ii) $\lim_{x \to +\infty} \frac{\partial}{\partial x} d_qf_1[v](x, k) = 0.$

One gets

$$d_qf_1[v](x, k) = \frac{f_2(x, k)}{W(k)} \int_x^\infty f_1^2(t, k)v(t)\, dt - \frac{f_1(x, k)}{W(k)} \int_x^\infty f_1(t, k)f_2(t, k)v(t)\, dt \tag{3.1}$$

and similarly

$$d_qf_2[v](x, k) = \frac{f_1(x, k)}{W(k)} \int_{-\infty}^x f_2^2(t, k)v(t)\, dt - \frac{f_2(x, k)}{W(k)} \int_{-\infty}^x f_2(t, k)f_1(t, k)v(t)\, dt. \tag{3.2}$$

THEOREM 3.1. *If* $N \geq 3$ *and* q *is in* $H_{N,N}$ *then*

(1) $\dfrac{\partial S(k, q)}{\partial q(x)} = f_1(x, -k, q)f_2(x, k, q)$

(2) $\dfrac{\partial W(k, q)}{\partial q(x)} = -f_1(x, k, q)f_2(x, k, q).$

Proof. (1) and (2) are proved similarly, so let us concentrate on (2) only. It suffices to prove (2) for q in $Q_N(\mathbb{C})$, as $Q_N(\mathbb{C})$ is dense in $H_{N,N}$ due to the asymptotic properties of $W(k, q)$ for $|k| \to \infty$. From

$$W(k, q) = f_2(x, k, q)\frac{\partial}{\partial x}f_1(x, k, q) - \frac{\partial}{\partial x}f_2(x, k, q)f_1(x, k, q)$$

one obtains after interchanging differentiation with respect to q and x, for v in $H_{N,N}$

$$d_qW[v](k, q) = d_qf_2[v](x, k)\frac{\partial}{\partial x}f_1(x, k) - \frac{\partial}{\partial x}d_qf_2[v](x, k)f_1(x, k)$$
$$+ f_2(x, k)\frac{\partial}{\partial x}d_qf_1[v](x, k) - \frac{\partial}{\partial x}f_2(x, k)d_qf_1[v](x, k).$$

Using (3.1) and (3.2) one gets

$$d_qW[v](k) = -\int_{-\infty}^{\infty} f_1(t, k)f_2(t, k)v(t)\,dt$$

and (2) follows.

THEOREM 3.2. *If $N \geq 3$ and q is in $Q_N(\mathbb{C})$ then*

(1) $$\frac{\partial R_1(k, q)}{\partial q(x)} = \frac{T(k, q)}{W(k, q)}f_2^2(x, k, q)$$

(2) $$\frac{\partial R_2(k, q)}{\partial q(x)} = \frac{T(k, q)}{W(k, q)}f_1^2(t, k, q).$$

Proof. It suffices to prove (1). One writes $W(k)R_1(k) = S(k)$ and applies the product rule for differentiation to get

$$W(k)\frac{\partial R_1(k)}{\partial q(x)} = \frac{\partial S(k)}{\partial q(x)} - \frac{\partial W(k)}{\partial q(x)}R_1(k)$$
$$= f_1(x, -k)f_2(x, k) + R_1(k)f_1(x, k)f_2(x, k)$$
$$= T(k)f_2(x, k)^2$$

where we used that $f_1(x, -k) + R_1(k)f_1(x, k) = T(k)f_2(x, k)$. So (1) follows.

Similarly one proves the following

THEOREM 3.3. *If $N \geq 3$ and q is in $Q_N(\mathbb{C})$ then*

$$\frac{\partial T(k, q)}{\partial q(x)} = \frac{T(k, q)}{W(k, q)}f_1(x, k, q)f_2(x, k, q) \qquad (k \text{ in } \mathbb{R}).$$

Again let us fix a q in $Q_N(\mathbb{C})$ and study the operator B on $H_{1,1}$ given by the kernel

$$\frac{1}{\pi}\frac{T(-l)T(l)}{2il}\frac{d}{dx}f_1(x, l)f_2(x, -l).$$

First we want to show the following

PROPOSITION 3.4. *If $N \geq 2$ and q is in $Q_N(\mathbb{C})$ then*

(1) $\displaystyle\int_{-\infty}^{\infty} dl\sigma(l)\frac{T(-l)T(l)}{2il}\frac{d}{dx}(f_1(x, l)f_2(x, -l)) \in H_{1,1}$ *for* $\sigma \in H_{1,1}$.

(2) $\displaystyle\int_{-\infty}^{\infty} dl\sigma(l)\frac{T(-l)T(l)}{2il}\frac{d}{dx}(f_1(x, l)f_2(x, -l)) \in H_{N,N}$ *for* $\sigma \in H^{\#}_{N-1,N}$.

Before giving the proof let us make the following observations. (1) and (2) are proved in the same way, so let us concentrate on (2) only. Now let us introduce functions $\zeta_1(x)$ and $\zeta_2(x)$ in $C^{\infty}(\mathbb{R})$ with $0 \leq \zeta_i(x) \leq 1$ $(i = 1,2)$, $\zeta_1(x) + \zeta_2(x) = 1$ and $\zeta_2(x) = 1$ for $x \leq 0$ as well as $\zeta_2(x) = 0$ for $x \geq 1$. Obviously it suffices to show that

$$\zeta_1(x)\int_{-\infty}^{\infty} dl\sigma(l)\frac{T(-l)T(l)}{2il}\frac{d}{dx}(f_1(x, l)f_2(x, -l)) \in H_{N,N}$$

for σ in $H^{\#}_{N-1,N}$. We use $T(-l)f_2(x, -l) = R_1(-l)f_1(x, -l) + f_1(x, l)$ to get

$$\begin{aligned}
\zeta_1(x)&\int_{-\infty}^{\infty} dl\sigma(l)\frac{T(-l)T(l)}{2il}\frac{d}{dx}(f_1(x, l)f_2(x, -l))\\
&= \zeta_1(x)\int_{-\infty}^{\infty} dl\sigma(l)\frac{R_1(-l)}{W(l)}(m_1(x, -l)m_1(x, l))'\\
&\quad + \zeta_1(x)\int_{-\infty}^{\infty} dl\sigma(l)\frac{2il}{W(l)}e^{2ilx}\\
&\quad + 2\zeta_1(x)\int_{-\infty}^{\infty} dl\sigma(l)\frac{2il}{W(l)}e^{2ilx}(m_1(x, l) - 1)\\
&\quad + \zeta_1(x)\int_{-\infty}^{\infty} dl\sigma(l)\frac{2il}{W(l)}e^{2ilx}(m_1(x, l) - 1)^2\\
&\quad + \zeta_1(x)\int_{-\infty}^{\infty} dl\sigma(l)\frac{1}{W(l)}e^{2ilx}2m_1(x, l)m_1'(x, l).
\end{aligned}$$

Clearly $\int_{-\infty}^{\infty} dl\sigma(l)(2il/W(l))e^{2ilx} \in H_{N,N}$ if σ is in $H^{\#}_{N-1,N}$. As the other terms are treated similarly it suffices to show that

$$x^j \frac{d^k}{dx^k} \int_{-\infty}^{\infty} dl\sigma(l)T(l)e^{2ilx}(m_1(x, l) - 1) \in L^2(\mathbb{R}^+) \quad \text{for} \quad 0 \le j, k \le N. \tag{3.3}$$

Using the Schrödinger equation $m_1''(x, k) + 2ikm_1'(x, k) = q(x)m_1(x, k)$ one proves by induction that $(\partial^n/\partial x^n)m_1(x, k)$ can be written as $(3 \le n \le N + 1)$

$$\frac{\partial^n}{\partial x^n} m_1(x, l) = \frac{\partial}{\partial x} m_1(x, l) \cdot P_{n1}\Big((-2il), q(x), \ldots, \frac{d^{n-3}}{dx^{n-3}} q(x)\Big)$$
$$+ m(x, l) \cdot P_{n2}\Big((-2il), q(x), \ldots, \frac{d^{n-2}}{dx^{n-2}} q(x)\Big) \tag{3.5}$$

where $P_{n1} - (-2il)^{n-1}$ is a polynomial in $(-2il)$ of degree $(n - 3)$ with coefficients which are polynomials in $q(x), \ldots, (d^{n-3}/dx^{n-3})q(x)$ of degree ≥ 1 and P_{n2} is a polynomial in $(-2il)$ of degree $(n - 2)$ with coefficients which are polynomials in $q(x), \ldots, (d^{n-2}/dx^{n-2})q(x)$ of degree ≥ 1.

Let us make one more observation. From

$$B_1(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} dl(m_1(x, l) - 1)e^{-2ily}$$

and

$$|B_1(x, y)| \le c \int_{x+y}^{\infty} |q(t)|\, dt$$

one derives

$$x^{\alpha}(m_1(x, l) - 1) \in L^2(\mathbb{R}^+, L^2(\mathbb{R})) \quad 0 \le \alpha \le N - 3/2. \tag{3.6}$$

Proof (of Proposition 3.4). With the above observations it suffices to prove (3.3). So one has to prove that

$$x^j \int_{-\infty}^{\infty} dl(2il)^{\alpha} e^{2ilx} \frac{\partial^{\beta}}{\partial x^{\beta}} (m_1(x, l) - 1)\sigma(l)T(l) \in L^2(\mathbb{R}^+)$$

where $0 \le \alpha + \beta \le N$, $0 \le j \le N$.

CASE 1. $\beta = 0$. Use $m_1(x, l) - 1 = \int_x^{\infty} D_l(t - x)q(t)m_1(t, l)\, dt$ and prove first that for $0 \leq j \leq N$

$$x^j \int_{-\infty}^{\infty} dl (2il)^\alpha e^{2ilx} \int_x^\infty D_l(t-x) q(t)\, dt \sigma(l) T(l) \in L^2(\mathbb{R}^+).$$

From the definition $D_l(t) = \int_0^t e^{2ilz}\, dz$ one obtains that

$$x^j \int_{-\infty}^{\infty} dl (2il)^\alpha e^{2ilx} \int_x^\infty D_l(t-x) q(t)\, dt \sigma(l) T(l)$$
$$= x^j \int_0^\infty dz \int_{z+x}^\infty dt q(t) \int_{-\infty}^{\infty} dl e^{2il(z+x)} \sigma(l) T(l) (2il)^\alpha$$

is in $L^2(\mathbb{R}^+)$ for $0 \leq j \leq N$. Second one proves that

$$x^j \int_{-\infty}^{\infty} dl (2il)^\alpha e^{2ilx} \int_x^\infty D_l(t-x) q(t)(m_1(t, l) - 1)\, dt \sigma(l) T(l)$$

is in $L^2(\mathbb{R}^+)$ for $0 \leq j \leq N$.

Now the claim follows from (3.6) and the following estimate

$$\left| \int_{-\infty}^{\infty} dl (2il)^\alpha e^{2ilx} \int_x^\infty D_l(t-x) q(t)(m_1(t, l) - 1)\, dt \sigma(l) T(l) \right|$$
$$\leq \|(2il)^\alpha \sigma(l) T(l)\|_{L^2(\mathbb{R})} \int_x^\infty dt\, |q(t)|\, (t-x)\, \|m_1(t, l) - 1\|_{L^2(\mathbb{R})}.$$

CASE 2. $\beta \geq 1$. Then write $\beta + 1$ instead of β. We have to show that

$$x^j \int_{-\infty}^{\infty} dl e^{2ilx} (2il)^\alpha \sigma(l) T(l) \frac{\partial^{\beta+1}}{\partial x^{\beta+1}} m_1(x, l) \in L^2(\mathbb{R}^+)$$

for $0 \leq j \leq N$ and $0 \leq \alpha + \beta \leq N - 1$.

We want to use (3.5). The only term in the expression for $(\partial^{\beta+1}/\partial x^{\beta+1}) m_1(x, l)$ which does not contain $q(x)$ or one of its derivatives is $(-2il)^\beta m_1'(x, l)$, so clearly it suffices to prove that

$$x^j \int_{-\infty}^{\infty} dl e^{2ilx} (2il)^\alpha \sigma(l) T(l) (-2il)^\beta m_1'(x, l) \in L^2(\mathbb{R}^+)$$

for $0 \le j \le N$ and $0 \le \alpha + \beta \le N - 1$. Now

$$m_1'(x, l) = -\int_x^\infty e^{2il(t-x)} q(t) m_1(t, l)\, dt$$
$$= -\int_x^\infty dt q(t) e^{2il(t-x)} - \int_x^\infty dt q(t) e^{2il(t-x)} (m_1(t, l) - 1).$$

So let us first show that for $0 \le j \le N$, $0 \le \alpha + \beta \le N - 1$

$$x^j \int_{-\infty}^\infty dl e^{2ilx} (2il)^{\alpha+\beta} \sigma(l) T(l) \int_x^\infty dt q(t) e^{2il(t-x)} \in L^2(\mathbb{R}^+).$$

Changing the order of integration one gets

$$x^j \int_0^\infty dt q(t + x) \int_{-\infty}^\infty dl e^{2il(t+x)} (2il)^{\alpha+\beta} \sigma(l) T(l)$$

and this expression is clearly in $L^2(\mathbb{R}^+)$ for $0 \le j \le N$ and $0 \le \alpha + \beta \le N - 1$.
Finally it is easy to see, using (3.6), that

$$x^j \int_{-\infty}^\infty dl T(l) \sigma(l) (2il)^{\alpha+\beta} e^{2ilx} \int_x^\infty dt e^{2il(t-x)} q(t) (m_1(t, l) - 1)$$

is in $L^2(\mathbb{R}^+)$ for $0 \le j \le N$ and $0 \le \alpha + \beta \le N - 1$, thus (3.3) is proved.

For a q given in $Q_N(\mathbb{C})$ let us study the operator A on $H_{1,1}$ given by the kernel $f_1(x, -k) f_2(x, k)$. We will need the following

PROPOSITION 3.5. *If $N \ge 3$ and q is in $Q_N(\mathbb{C})$ then*

$$\int_{-\infty}^\infty dx f_1(x, -k) f_2(x, k) v(x) \in H_{1,1} \quad \text{for} \quad v \in H_{1,1}.$$

Proof. It suffices to show that

$$\int_{-\infty}^\infty dx f_1(x, -k) f_2(x, k) v(x) \in H_{1,1}$$

for all $v \in H_{1,1}$ with $\sup v \subseteq]-\infty, a]$ as the case where $\operatorname{supp} v \subseteq [a, \infty[$ is treated similarly and the operator A is linear. Let us introduce $\xi_1 \in C^\infty(\mathbb{R})$, $0 \le \xi_1 \le 1$

such that $\xi_1(k) = 1$ for $|k| \geq 1$ and $\xi_1(k) = 0$ for $|k| \leq \frac{1}{2}$. Define $\xi_2 := 1 - \xi_1$. We first show that $\xi_1(k) \int_{-\infty}^{a} dx f_1(x, k) f_2(x, -k) v(x) \in H_{1,1}$. Clearly this term is equal to

$$\xi_1(k) \frac{R_2(k)}{T(k)} \int_{-\infty}^{a} dx v(x) m_2(x, -k) m_2(x, k) + \xi_1(k) \frac{1}{T(k)} \int_{-\infty}^{a} dx v(x) e^{+2ikx}$$
$$+ \xi_1(k) \frac{1}{T(k)} \int_{-\infty}^{a} dx v(x) e^{2ikx} 2(m_2(x, -k) - 1)$$
$$+ \xi_1(k) \frac{1}{T(k)} \int_{-\infty}^{a} dx v(x) e^{2ikx} (m_2(x, -k) - 1)^2.$$

The first two terms are in $H_{1,1}$ using $v \in H_{1,1}$ and Corollary 2.8, Theorem 2.18 and 2.19. As concerns the last two terms they are seen to be in $H_{1,1}$ by partial integration.

Finally we have to show that $\xi_2(k) \int_{-\infty}^{a} dx f_1(x, k) f_2(x, -k) v(x)$ is in $H_{1,0}$. One writes

$$f_1(x, k) = \frac{R_2(k) + 1}{2ik} W(k) f_2(x, k) - W(k) \frac{f_2(x, k) - f_2(x, -k)}{2ik}$$

or

$$f_1(x, k) f_2(x, -k) = \frac{R_2(k) + 1}{2ik} W(k) m_2(x, k) m_2(x, -k)$$
$$- W(k) \frac{m_2(x, k) - m_2(x, -k)}{2ik} m_2(x, -k) + W(k) \frac{e^{2ikx} - 1}{2ik} m_2(x, -k)^2.$$

Using the Hardy–Littlewood–Polya inequality, $N \geq 3$ and $R_2(0) = -1$ we conclude that

$$\frac{R_2(k) + 1}{2ik} \in H_{1,1}, \qquad W(k) \xi_2(k) \in H_{1,1}$$

and

$$\frac{d^j}{dk^j} \frac{m_2(x, -k) - m_2(x, k)}{2ik} \in L^2_-(\mathbb{R}, L^2(\mathbb{R})) \quad \text{for} \quad j = 1,2.$$

This gives that

$$\xi_2(k)\int_{-\infty}^{a} dx f_1(x, k) f_2(x, -k) v(x) \in H_{1,1}.$$

The next step is to show that B is a right inverse of A, where B and A are defined as above

PROPOSITION 3.6. *Let $N \geq 3$ and q be in $Q_N(\mathbb{C})$. Then*

(1) $A \circ B = Id \quad on \quad H_{1,1}$

(2) $A|_{H_{N,N}} \circ B|_{H^{\#}_{N-1,n}} = Id|_{H^{\#}_{N-1,N}}$.

Proof. (2) follows immediately from (1) and Proposition 3.4. The proof of (1) is a calculation which is at the heart of the matter. Let σ be in $H_{1,1}$. Then

$$(A \circ B\sigma)(k) = \int_{-\infty}^{\infty} dy f_1(y, -k) f_2(y, k) \frac{1}{\pi}$$

$$\times \int_{-\infty}^{\infty} dl \frac{T(l)T(-l)}{2il}\left(\frac{d}{dy} f_1(y, l) f_2(y, -l)\right)\sigma(l)$$

$$= \lim_{x\to\infty} \frac{1}{\pi}\int_{-\infty}^{\infty} dl\sigma(l)\frac{T(l)T(-l)}{2il}$$

$$\times \int_{-x}^{x} dy f_1(y, -k) f_2(y, k) \frac{d}{dy}(f_1(y, l) f_2(y, -l)).$$

One shows that $(l \neq \pm k)$

$$2\frac{\partial}{\partial y}\{f_1(y, l) f_2(y, -l)\} f_1(y, -k) f_2(y, k)$$

$$= \frac{\partial}{\partial y}\{f_1(y, l) f_1(y, -k) f_2(y, -l) f_2(y, k)$$

$$+ \frac{1}{k^2 - l^2} W[f_1(y, -k), f_1(y, l)] W[f_2(y, k), f_2(y, -l)]\}.$$

Using the asymptotics of $f_1(y, l)$ and $f_2(y, l)$ as $y \to \pm\infty$ we get for $x \to +\infty$

$$\int_{-x}^{x} \frac{\partial}{\partial y}\{f_1(y, l)f_2(y, -l)\}f_1(y, -k)f_2(y, k)\,dy$$

$$\sim \frac{R_2(-k)}{T(-k)}\frac{1}{T(-l)}e^{2ilx} - \frac{1}{T(l)}\frac{R_2(-k)}{T(-k)}e^{-2ilx}$$

$$+\frac{1}{T(-l)}\frac{1}{T(k)}e^{2ilx}e^{-2ikx} - \frac{1}{T(l)}\frac{1}{T(-k)}e^{2ixk}e^{-2ixl}$$

$$+\tfrac{1}{2}(l+k)\frac{1}{T(-k)}\frac{1}{T(l)}\frac{e^{2ix(k-l)}}{k-l} + \tfrac{1}{2}(l+k)\frac{1}{T(k)}\frac{1}{T(-l)}\left(-\frac{e^{-2ix(k-l)}}{k-l}\right).$$

As $H_{1,1} \subseteq L^1$ the Riemann–Lebesgue lemma can be applied. Together with $\lim_{x\to\infty} pV(e^{2ix(k-l)}/(k-l)) = i\pi\delta(k-l)$ where δ denotes Dirac's δ function one gets

$$\lim_{x\to\infty}\int_{-\infty}^{\infty} dl\sigma(l)\frac{1}{\pi}\frac{T(l)T(-l)}{2il}\int_{-x}^{x} dyf_1(y, -k)f_2(y, k)\frac{d}{dy}(f_1(y, l)f_2(y, -l)) = \sigma(k).$$

Finally we want to prove that the Jacobian d_qS of S is boundedly invertible. It follows from Proposition 3.6 that d_qS is onto.

In order to prove the injectivity of d_qS we need several lemmata. Let us first introduce the following spaces $H^2(a) := H^{2+}(a) \cap H^{2-}(a)$ where

$$H^{2+}(a) := \{f \in L^2(\mathbb{R}) : e^{2ika}f(k) \in H^{2+}\}$$
$$H^{2-}(a) := \{f \in L^2(\mathbb{R}) : e^{2ika}f(-k) \in H^{2+}\}$$
$$H_{1,1}(a) := \{f \in H_{1,1} : \operatorname{supp} f \subseteq [-a, a]\}.$$

Let us recall that H^{2+} denotes the Hardy space of functions $\sigma(k)$ holomorphic in $\operatorname{Im} k > 0$ with $\sup_{b>0}\int_{-\infty}^{\infty}|\sigma(a+ib)|^2\,da < \infty$ and $H^{2-} = \{\sigma(k) : \sigma(-k) \in H^{2+}\}$. Using the equation $m_1(x, k) = 1 + \int_x^\infty dtq(t)D_k(t-x)m_1(t, k)$ it is easy to prove the following

LEMMA 3.7. *Let $b > 0$ be given and q be in $H_{N,N}$ with $N \geq 3$ such that $\operatorname{supp} q \subseteq [-b, b]$. Then for $a \geq b$*

(1) $(e^{2ik(a-x)}(m_1(x, -k) - 1)$ *is holomorphic in* $\operatorname{Im} k > 0$, $|x| \leq a$ *and*

$$|e^{2ik(a-z)}(m_1(x, -k) - 1)| \leq K_1\frac{1 + \max(-x, 0)}{1 + |k|}\ (\operatorname{Im} k \geq 0, x \leq a)$$

where $K_1 = K_1(x)$ *denotes a non-increasing function.*

(2) $m_1(x, k) = 1$ $(x \geq b,\ k \in \mathbb{C})$.

(3) $(m_2(x, -k) - 1)$ *is holomorphic in* $\operatorname{Im} k > 0$, $x \geq -a$ *and*

$$|e^{2ik(x+a)}(m_2(x, -k) - 1)| \leq K_1 \frac{1+|x|}{1+|k|} \quad (\operatorname{Im} k \geq 0,\ x \geq -a).$$

(4) $m_2(x, k) = 1$ $(x \leq -b),\ k \in \mathbb{C})$.

LEMMA 3.8. *Let* q *be in* $H_{N,N}$ *with* $N \geq 3$ *and* $\operatorname{supp} q \subseteq [-b, b]$. *Then for* $a > b$

(1) $A(H_{1,1}(a)) \subseteq H_{1,1} \cap H^{2+}(a)$.

(2) $A(H_{1,1}(a)) \subseteq H_{1,1} \cap H^{2-}(a)$.

Proof. (1) and (2) are proved in a similar way, so let us prove (1) only. Due to Proposition 3.5 we have only to show that for $v \in H_{1,1}$ with $\operatorname{supp} v \subseteq [-a, a]$ $e^{2ika} \int_{-\infty}^{\infty} dx f_1(x, -k) f_2(x, k) v(x) \in H^{2+}$. We write

$$\begin{aligned} e^{2ika} \int_{-\infty}^{\infty} & dx f_1(x, -k) f_2(x, k) v(x) \\ &= \int_{-\infty}^{\infty} e^{2ik(a-x)} m_1(x, -k) m_2(x, k) v(x)\, dx \\ &= \int_{-a}^{a} dx v(x) e^{2ik(a-x)} + \int_{-a}^{a} dx v(x) e^{2ik(a-x)} (m_2(x, k) - 1) \\ &\quad + \int_{-a}^{a} dx v(x) e^{2ik(a-x)} (m_1(x, -k) - 1) m_2(x, k). \end{aligned}$$

Applying Lemma 3.8 (1) follows.

Now let us introduce the set

$$Q_N^0(\mathbb{C}) := \{q \in H_{N,N} : W[f_2(x, k, q), f_1(x, k, q)] \neq 0 \quad \text{for} \quad \operatorname{Im} k \geq 0\}.$$

One sees easily that $Q_N^0(\mathbb{C})$ is open in $H_{N,N}$. For q in $Q_N^0(\mathbb{C})$ we know that $T(k, q)$ is holomorphic in $\operatorname{Im} k > 0$, moreover continuous and bounded in $\operatorname{Im} k \geq 0$.

LEMMA 3.9. *Let* q *be in* $Q_N^0(\mathbb{C})$ *with* $N \geq 3$ *and* $\operatorname{supp} q \subseteq [-b, b]$. *Then for all* $a > b$ $B(H_{1,1} \cap H^2(a)) \subseteq H_{1,1}(a)$.

Proof. Let σ be in $H_{1,1} \cap H^2(a)$ and $x > a$. Then $f_1(x, k) = e^{ikx}$ and $T(-k)f_2(x, -k) = R_1(-k)e^{-ikx} + e^{ikx}$. Thus

$$\frac{1}{2ik}\frac{d}{dx} f_1(x, k)T(-k)f_2(x, -k) = e^{2ikx}$$

and for $x > a$

$$\int_{-\infty}^{\infty} dk\sigma(k)\frac{T(k)T(-k)}{2ik}\frac{d}{dx} f_1(x, k)f_2(x, -k) = \int_{-\infty}^{\infty} dk e^{2ika}\sigma(k)T(k)e^{2ik(x-a)} = 0.$$

Similarly one proves that for $x < -a$ one has

$$\int_{-\infty}^{\infty} dk\sigma(k)\frac{T(k)T(-k)}{2ik}\frac{d}{dx} f_1(x, k)f_2(x, -k) = 0.$$

Together with Proposition 3.4 one concludes that $B(\sigma) \in H_{1,1}(a)$.

LEMMA 3.10. *Let q be in $Q_N^0(\mathbb{C})$ with $N \geq 3$ and* $\operatorname{supp} q \subseteq [-b, b]$. *Then for* $a > b$

(1) $A(H_{1,1}(a)) = H_{1,1} \cap H^2(a)$

(2) $A|_{H_{1,1}(a)}$ *is a compact perturbation of the Fourier transform.*

(3) $B|_{H_{1,1} \cap H^2(a)} \circ A|_{H_{1,1}(a)} = Id|_{H_{1,1}(a)}$

Proof. (1) Follows from Lemma 3.9 and Proposition 3.4. One concludes that

$$A|_{H_{1,1}(a)} \circ B|_{H_{1,1} \cap H^2(a)} = Id|_{H_{1,1} \cap H^2(a)}.$$

(2) One writes

$$f_1(x, -k)f_2(x, k) - e^{-2ikx} = e^{-2ikx}(m_2(x, k) - 1) + e^{2ikx}(m_1(x, -k) - 1)m_2(x, k).$$

It is to show that $e^{-2ikx}(m_2(x, k) - 1)$ and $e^{-2ikx}(m_1(x, -k) - 1)m_2(x, k)$ are kernels of compact operators in $\mathcal{L}(H_{1,1}(a), H_{1,1})$. But this is easily shown using results of Corollary 2.8.

(3) From (2) and (1) and the fact that the Fourier transform is an isomorphism from $H_{1,1}(a)$ onto $H_{1,1} \cap H^2(a)$ we conclude that the Fredholm

alternative holds, so that $A|_{H_{1,1}(a)}$ is $1-1$ and onto. Together with the fact that $A|_{H_{1,1}(a)} \circ B|_{H_{1,1}\cap H^2(a)} = Id|_{H_{1,1}\cap H^2(a)}$ (3) follows.

LEMMA 3.11. *Let q be in $Q_N^0(\mathbb{C})$, with $N \geq 3$ and* $\operatorname{supp} q \subseteq [-b, b]$. *Then A is $1-1$ and $B \circ A = Id$ on $H_{1,1}$.*

Proof. Let $v \in H_{1,1}$. Choose a sequence of cut-off functions ζ_n in C^∞ with $\operatorname{supp} \zeta_n \subseteqq [-n-b, b+n]$ such that $\lim_{n\to\infty} \zeta_n v = v$ in $H_{1,1}$.

By Lemma 3.10 we have $\lim_{n\to\infty} B \circ A(\zeta_n v) = \zeta_n v \ \forall n \in \mathbb{N}$. Moreover $B \circ A$ is a continuous bounded operator on $H_{1,1}$ thus

$$B \circ A(v) = \lim_{n\to\infty} B \circ A(\zeta_n v) = \lim_{n\to\infty} \zeta_n v = v.$$

Now let q be in $Q_N^0(\mathbb{C})$ with $N \geq 3$. Choose a sequence of cut-off functions $\zeta_n \in C^\infty$, $0 \leq \zeta_n \leq 1$ such that $\operatorname{supp} \zeta_n$ is compact and $\lim_{n\to\infty} \zeta_n q = q$ in $H_{N,N}$. As $Q_N^0(\mathbb{C})$ is open in $H_{N,N}$ we can and do assume without any loss of generality that $\zeta_n q \in Q_N^0(\mathbb{C})$ for all n in $\mathbb{N}$. Let us denote by A_n and B_n the operators in $\mathscr{L}(H_{1,1})$ generated by the kernels $f_1(x, -k, q\zeta_n)f_2(x, k, q\zeta_n)$ and

$$\frac{T(l)T(-l)}{2il}\frac{d}{dx}f_1(x, l, q\zeta_n)f_2(x, -l, q\zeta_n) \quad \text{respectively.}$$

By Lemma 3.11 and Proposition 3.6 we know that $B_n \circ A_n = Id$ and $A_n \circ B_n = Id$. But $\lim_{n\to\infty} A_n v = Av$ and $\lim_{n\to\infty} B_n \sigma = B\sigma$ for all v and σ in $H_{1,1}$, applying Corollary 2.8. So in all we have proved the following

THEOREM 3.12. *Let q be in $Q_N^0(\mathbb{C})$ with $N \geq 3$. Then*
(1) *The Jacobian $d_qS \in \mathscr{L}(H_{N,N}, H_{N-1,N}^{\#})$ is boundedly invertible.*
(2) *The kernel of d_qS is given by $f_1(x, -k, q)f_2(x, k, q)$.*
(3) *The kernel of the inverse of d_qS is given by*

$$\frac{1}{\pi}\frac{T(l)T(-l)}{2il}\frac{d}{dx}\{f_1(x, l, q)f_2(x, -l, q)\}.$$

We now make some remarks about the Jacobian d_qR_1 of the reflection coefficient R_1. With similar arguments as for S one can show that $d_qR_1 \in$

$\mathscr{L}(H_{N,N}, H^{\#}_{N-1,N+1})$ is $1-1$ and that d_qR_1 has a right inverse given by

$$\frac{1}{\pi}\frac{1}{2il}\frac{d}{dx}f_1(x, l)^2.$$

However the space $d_qR_1(H_{N,N})$ is a rather complicated subspace in $H^{\#}_{N-1,N+1}$ because R_1 has to satisfy the extra condition

$$\frac{\partial^N}{\partial k^N}\frac{1-R_1(k)R_1(-k)}{k^\alpha} \in L^2(\mathbb{R}) \quad \text{for} \quad \alpha = 0,1.$$

(Remark after Theorem 2.19). Thus d_qR_1 is not onto in general.

§4. Global properties of the scattering map

In this whole section we restrict ourselves to real valued potentials. For $N \geq 3$ let us denote by $Q_N(\mathbb{R})$ the set of all real valued potentials in $H_{N,N}$ such that $W(k, q) \neq 0$ in $\operatorname{Im} k \geq 0$.

Let us recall that $\mathscr{S}_N$ denotes the set of functions $\sigma(k) \in L^2(\mathbb{R}, \mathbb{C})$ with the following properties

(1) $\sigma \in H^{\#}_{N-1,N}$
(2) $\sigma(0) > 0$
(3) $\sigma(k)^* = \sigma(-k)$ (k in $\mathbb{R}$)

where $*$ denotes complex conjugation. Q_N and $\mathscr{S}_N$ are given the natural topologies.

First we quickly remark that $S(q) = S(k, q)$ is an element of $\mathscr{S}_N$ if q is in $Q_N(\mathbb{R})$. To see that $S(0) > 0$ recall that $S(0) = -W(0)$. Moreover $W(i\kappa)$ has no zeroes for $\kappa \geq 0$. But for large κ we have $W(i\kappa) \sim 2i(i\kappa) = -2\kappa$. Thus $W(i\kappa) < 0$ for $\kappa \geq 0$. The aim of this section is to show that the scattering map $S: Q_N(\mathbb{R}) \to \mathscr{S}_N$ is $1-1$ and onto. Most of the proof is standard so we will allow ourselves to be rather short.

In order to show that S is $1-1$ one observes that $W(k, q)$ can be represented by the Herglotz formula ($\operatorname{Im} k > 0$)

$$\frac{2i(k+i)}{W(k, q)} = \exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty} dz \log\left(\frac{4(z^2+1)}{4z^2 + S(z, q)S(-z, q)}\right)\frac{1}{z-k}\right\}$$

where we used that $W(k, q)W(-k, q) = 4k^2 + S(k, q)S(-k, q)$ (k in $\mathbb{R}$) and $1/W(k, q)$ is holomorphic in $\operatorname{Im} k > 0$. So for q_1, q_2 in $Q_N(\mathbb{R})$ with $S(k, q_2) =$

$S(k, q_1)$ $(k \in \mathbb{R})$ it follows that $W(k, q_1) = W(k, q_2)$, thus $R_1(k, q_1) = S(k, q_1)/W(k, q_1)$ and $R_1(k, q_2)$ are equal. By [3] it follows that $q_1 = q_2$ and the injectivity of S is proved. The harder part is to show that the map is onto. For σ in $\mathcal{S}_N$ let us define

$$\omega(k) := \begin{cases} \exp\left\{\dfrac{1}{2\pi i}\displaystyle\int_{-\infty}^{\infty} dz \log \dfrac{4(z^2+1)}{4z^2 + \sigma(z)\sigma(-z)} \dfrac{1}{z-k}\right\} & (\operatorname{Im} k > 0) \\ \lim\limits_{\varepsilon \downarrow 0} \omega(k + i\varepsilon) \quad (\operatorname{Im} k = 0) \end{cases}$$

and

$$\frac{1}{w(k)} = \frac{-i/2}{k+i}\,\omega(k), \qquad \tau(k) = \frac{2ik}{w(k)}$$

$$\rho_1(k) = \frac{1}{w(k)}\,\sigma(k), \qquad \rho_2(k) = \frac{1}{w(k)}\,\sigma(-k).$$

The aim is to show that $\rho_1(k)$, $\rho_2(k)$ and $\tau(k)$ are the scattering data of a potential q in $Q_N(\mathbb{R})$.

First we have to discuss the properties of $\omega(k)$. In order to do so we introduce

$$h(k) := \frac{4(k^2+1)}{4k^2 + \sigma(k)\sigma(-k)} \quad (k \text{ in } \mathbb{R}).$$

As $\sigma(0) \neq 0$ and $\sigma(k)\sigma(-k) \geq 0$ there exists $M > 1$ with $1/M \leq h(k) \leq M$ (k in $\mathbb{R}$). For $|k| \geq 1$ one can find a constant $C > 0$ with

$$|\log h(k)| \leq C \frac{1}{k^2}.$$

So one concludes that $\log h(k) \in L^1(\mathbb{R})$ and $k^\alpha \log h(k) \in L^\infty(\mathbb{R})$ for $0 \leq \alpha \leq 2$, moreover one verifies that $\log h(k)$ is uniformly continuous on $\mathbb{R}$.

Now let us introduce for $z := x + iy$ $(y > 0)$

$$u(z) := \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{1}{2}\log h(k) \frac{1}{(x-k)^2 + y^2}\,dk$$

$$v(z) := \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{2}\log h(k) \frac{x-k}{(x-k)^2 + y^2}\,dk.$$

$v(z)$ is then the conjugate function of $u(z)$ and thus $u(z)+iv(z)$ is holomorphic in $\operatorname{Im} z>0$. Clearly $\omega(z)=\exp\{u(z)+iv(z)\}$ in $\operatorname{Im} z>0$. It follows that $\omega(z)$ is an outer function, in H^∞ and continuous in $\operatorname{Im} z\geq 0$. Moreover

$$\omega(-k)\omega(k)=h(k)=\frac{4(k^2+1)}{4k^2+\sigma(k)\sigma(-k)}\quad (k \text{ in } \mathbb{R}).$$

One shows that $\omega(z)$ has no zeroes in the upper half plane $\operatorname{Im} z\geq 0$ and for k in $\mathbb{R}$ one gets

$$\omega(k)=\exp\{\tfrac{1}{2}\log h(k)+i\mathscr{H}(\tfrac{1}{2}\log h)(k)\}$$

where $\mathscr{H}$ denotes the Hilbert transform. It is easy to verify that $\omega(k)^*=\omega(-k)$ (k in $\mathbb{R}$). The asymptotics can be estimated to give

$$\omega(z)=1+0\left(\frac{1}{z}\right)\quad (\operatorname{Im} z\geq 0).$$

From σ in $H^\#_{N-1,N}$ it follows that $h(k)$ is in $H^\#_{N-1,1}$. But then $d/dk \log h=h'/h$ is in L^2 and one shows by induction that $\log h\in H^\#_{N-1,1}$.

As differentation commutes with the Hilbert transform one gets that $\mathscr{H}(\log h)\in H^\#_{N-1,1}$.

Now let us summarize the properties of $\tau(k)=(k/k+i)\omega(k)$ in the following

LEMMA 4.1. *For σ in $\mathscr{S}_N$, $\tau(k)$ is holomorphic in* $\operatorname{Im} k>0$, *continuous in* $\operatorname{Im} k\geq 0$ and satisfies

(1) $\tau(0)=0,\ \tau(k)\neq 0\ \forall k\in\mathbb{R}\backslash\{0\},\ \tau(k)^*=\tau(-k)\quad (k \text{ in } \mathbb{R})$

(2) $\tau(k)=1+0\left(\frac{1}{k}\right)$ *in* $\operatorname{Im} k\geq 0$ for $|k|\to\infty$.

Next let us summarize the properties of $1/w(k):=((-i/2)/(k+i))\omega(k)$ in the following

LEMMA 4.2. *For σ in $\mathscr{S}_N$ we get*

(1) $\dfrac{1}{w(k)}\in H_{N-1,0}$ *and* $\dfrac{d^n}{dk^n}\dfrac{k}{w(k)}\in L^2(\mathbb{R})\quad (1\leq n\leq N)$

(2) $\dfrac{1}{w(-k)}=\dfrac{1}{w(k)^*}$ *and* $\dfrac{1}{w(k)}\neq 0\quad \forall k$ *in* $\mathbb{R}$.

Now let us recall that $\rho_1(k) = \sigma(k)/w(k)$, $\rho_2(k) = \sigma(-k)/w(k)$. Then the following lemma follows from Lemma 4.2 and the assumptions:

LEMMA 4.3. *If σ is in $\mathcal{S}_N$ then* $(i = 1,2)$
(1) $\rho_i \in H^{\#}_{N-1,N+1}$
(2) $\rho_i(k)^* = \rho_i(-k)$ $(k$ *in* $\mathbb{R})$
(3) $\rho_i(k)\rho_i(-k) + \tau(k)\tau(-k) = 1$ $(k$ *in* $\mathbb{R})$
(4) $|\rho_i(k)| < 1$ $\forall k$ *in* $\mathbb{R}\setminus\{0\}$
(5) $\rho_i(0) = -1$.

Introduce

$$F_1(x) := \frac{1}{\pi}\int_{-\infty}^{\infty} \rho_1(k)e^{2ikx}\,dk \quad \text{and} \quad F_2(x) := \frac{1}{\pi}\int_{-\infty}^{\infty} \rho_2(k)e^{-2ikx}\,dk.$$

As $\rho_i \in H^{\#}_{N-1,N+1}$ we conclude that $x^{\alpha}(d/dx)F_i(x) \in L^1(\mathbb{R})$ $(0 \le \alpha \le 2)$. The main theorem in inverse scattering [4] assures that there exists a real valued potential $q:\mathbb{R} \to \mathbb{R}$ such that $\int_{-\infty}^{\infty}(1+x^2)\,|q(x)|\,dx < \infty$ having ρ_1, ρ_2 and τ as its scattering data. It remains to show that q is in $Q_N(\mathbb{R})$. Let us look at the Marchenko equation $(y \ge 0,\ x$ in $\mathbb{R})$

$$0 = B_1(x, y) + F_1(x + y) + \int_0^{\infty} B_1(x, z)F_1(x + y + z)\,dz.$$

With $q(x) = (-d/dx)B_1(x, 0)$. It is well-known that there exists a nonincreasing function $C(x)$ such that for j,k with $1 \le j + k \le N + 1$

$$\left|\frac{\partial^j}{\partial x^j}\frac{\partial^k}{\partial y^k}B_1(x, y) + \frac{\partial^{j+k}}{\partial x^{j+k}}F_1(x + y)\right| \le C(x)\sum_{m=k}^{j+k}\int_{x+y}^{\infty}\left|\frac{d^m}{dz^m}F_1(z)\right|dz.$$

This together with Lemma 4.3 implies that q is in $Q_N(\mathbb{R})$. Thus we have proved.

THEOREM 4.4. *Let $N \ge 3$. Then the scattering map $S: Q_N(\mathbb{R}) \to \mathcal{S}_N$ is $1-1$ and onto.*

§5. Summary

Let us recall the following definitions

$$Q_N(\mathbb{R}) := \{q \in H_{N,N}(\mathbb{R}, \mathbb{R}): W(k, q) \neq 0 \text{ in } \operatorname{Im} k \ge 0\}$$
$$\mathcal{S}_N := \{\sigma \in H^{\#}_{N-1,N}: (1)\ \sigma(0) > 0,\ (2)\ \sigma(-k) = \sigma(k)^*\}.$$

Then by Theorem 2.17 $S: Q_N(\mathbb{R}) \to \mathcal{S}_N$ is real analytic. By Theorem 3.12 the Jacobian d_qS is boundedly invertible for q in $Q_N(\mathbb{R})$. The inverse function theorem thus furnishes together with Theorem 4.4

THEOREM 5.1. *If $N \geq 3$ then the scattering map $S: Q_N(\mathbb{R}) \to \mathcal{S}_N$ is a real analytic isomorphism.*

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