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Enumerative formulae for ruled cubic surfaces and rational quintic curves

DANIEL F. CORAY and ISRAEL VAINSENCHER

1. Introduction

Our aim is to calculate the degrees of certain strata of the variety parametrizing the family of ruled cubic surfaces and to obtain a few related enumerative formulae. The link with the rational quintic curves is as follows. Such a curve c has a unique quadrisecant d, in general ([LB], p. 190). Now there are ∞^9 cubic surfaces doubled along d. Choosing 8 general points on c, there remain ∞^1 such surfaces containing them. Each of these has $2 \cdot 4 + 8 = 16$ intersections with c, and therefore contains c. Thus c may be got as the residual intersection of two cubic surfaces f, f' doubled along d, i.e., $f \cdot f' = c + 4d$. Conversely, every irreducible quintic obtained in this way is rational, since its points are parametrized by the planes through d.

It is well-known that a ruled cubic surface is doubled along a line. There are four projectively distinct types (Abhyankar [A]):

0) non special: $x^2w + y^2z = 0$;
1) special: $x(xw + yz) + y^3 = 0$;
2) cone over nodal cubic: $zy^2 + zx^2 + x^3 = 0$, and
3) cone over cuspidal cubic: $zy^2 + x^3 = 0$.

THEOREM A. *Let $X_i \subset \mathbb{P}_{19}$ be the closure of the set of cubic surfaces of type i ($i = 0, \ldots, 3$). Then we have*: $\deg X_0 = 504$, $\deg X_1 = 1552$, $\deg X_2 = 900$, *and* $\deg X_3 = 960$.

We observe that $\dim X_i = 13 - i$ and $\deg X_i$ is the number of cubic surfaces of type i passing through $13 - i$ general points. We also get a few other formulae by imposing Schubert conditions on the double line or on the tangent cones along it. As a bonus of the main construction, we obtain the following:

THEOREM B. *There are* 105 *rational quintics through* 10 *general points of* $\mathbb{P}_3$.

This result was anticipated in [C] via a different reasoning and assuming a conjecture. Here the calculation takes place in the Chow ring of a smooth compactification for the parameter space of the family of rational quintic curves. This compactification is the Grassmann bundle of pencils of cubic surfaces doubled along a common line. Boundary elements include pencils with a fixed component. It would be nice to obtain a good description of the loci of such pencils together with their normal bundles, hopefully confirming our guess stated in (3.2.1) below. Presumably, blowing up these loci should produce a compactification endowed with a universal curve on it. This might be suitable for obtaining other characteristic numbers ([FKM], [F]) for the family of rational quintics.

2. The variety of ruled cubic surfaces

2.1 NOTATION. We denote by V a 4-dimensional vector space over the ground field and write $\mathbb{P}_3 = \mathbb{P}(V)$ for the associated projective space of rank 1 subspaces of V. For a locally free sheaf F, we write $S_m F$ for m-th symmetric power. Thus $\mathbb{P}_{19} = \mathbb{P}(S_3(V^*))$ is the complete linear system of cubic surfaces; its hyperplane class will be denoted by $h = c_1\mathcal{O}_{\mathbb{P}}(1)$. We set

$$G = \text{Grass}\,(2, V) = \{\text{lines in } \mathbb{P}_3\},$$

with tautological exact sequence

$$K \rightarrowtail V \twoheadrightarrow Q \quad (\text{rank}\, K = \text{rank}\, Q = 2).$$

We recall that the universal line,

$$D = \mathbb{P}(K) = \{(P, \ell) \in \mathbb{P}_3 \times G \mid P \in \ell\},$$

is the scheme of zeros of a regular section of $Q(1)$. Hence its normal sheaf is isomorphic to $Q(1)_{|D}$. The Schubert cycles on G will be written:

$\lambda = c_1 Q$ (lines meeting a given line),

$\theta = c_2 K$ (lines contained in a given plane),

$\pi = c_2 Q$ (lines containing a given point)

([F], Ex. 14.7.2). A cycle $\lambda^n \pi^p \theta^q$ corresponds to the condition that a line meet n

others, pass through p points and lie in q planes. $\int_Y z$ denotes the degree of the zero-dimensional part of a cycle z on a complete variety Y. We denote by the same letter a class (or sheaf) and its pullback if no confusion is likely.

2.2 PROPOSITION. *There is an exact sequence of locally free $\mathcal{O}_G$-Modules*

$$E \rightarrowtail S_3V^* \twoheadrightarrow E'$$

such that:

1) rank $E = $ rank $E' = 10$;
2) *for each d in G the fibre E_d consists of cubic forms doubled along d*;
3) *the projective subbundle*

$$Y := \mathbb{P}(E) \subset G \times \mathbb{P}_{19}$$

maps onto the variety $X := X_0 \subset \mathbb{P}_{19}$ of cubic surfaces doubled along a line. Furthermore, the induced map $Y \to X$ is a desingularization in characteristic $\neq 2$ and is purely inseparable in characteristic 2.

4) *we have the formula for the total Chern classes*:

$$c(E)^{-1} = c(E') = 1 + 9\lambda + 48\theta + 33\pi + 232\lambda\theta + 504\theta^2.$$

Proof. A surface f is doubled along a line d if and only if its equation lies in the square of the ideal of d. Denote by I the Ideal of $D = \mathbb{P}(K)$ in $\mathbb{P}_3 \times G$ and let $2D$ stand for the subscheme defined by I^2. Tensoring the exact sequence

$$I^2 \hookrightarrow \mathcal{O}_{\mathbb{P}_3 \times G} \twoheadrightarrow \mathcal{O}_{2D}$$

by the invertible sheaf

$$L := \mathcal{O}_{\mathbb{P}_{19}}(1) \otimes \mathcal{O}_{\mathbb{P}_3}(3)$$

yields the diagram of sheaves over $\mathbb{P}_{19} \times \mathbb{P}_3 \times G$:

$$\begin{tikzcd} & \mathcal{O} \arrow[d] \arrow[dr, "s"] & \\ L \otimes I^2 \arrow[r, hook] & L \arrow[r, two heads] & L \otimes \mathcal{O}_{2D}. \end{tikzcd} \tag{2.2.1}$$

The vertical arrow comes from the natural section of $\mathcal{O}_{\mathbb{P}_{19}}(1) \otimes \mathcal{O}_{\mathbb{P}_3}(3)$ defining the universal cubic surface in $\mathbb{P}_{19} \times \mathbb{P}_3$.

Let p denote the projection $\mathbb{P}_3 \times G \to G$ or $\mathbb{P}_{19} \times \mathbb{P}_3 \times G \to \mathbb{P}_{19} \times G$. Set

$$E' := p_*(\mathcal{O}_{2D}(3)).$$

The section s in (2.2.1) vanishes on the fibre over (f, d) in $\mathbb{P}_{19} \times G$ if and only if f is in $I(d)^2$. Hence the set

$$Y := \{(f, d) \mid f \supset 2d\}$$

has a natural subscheme structure as the zeros of the section $p_* s$ of $E' \otimes \mathcal{O}_{\mathbb{P}_{19}}(1)$ (cf. [AK], prop. (2.3)). Now E' fits into the exact sequences:

$$\begin{array}{ccccc} & & p_*(\mathcal{O}_{\mathbb{P}_3}(3) \otimes \mathcal{O}_G) & & \\ & & \| & & \\ E := p_*(I^2 \otimes \mathcal{O}_{\mathbb{P}_3}(3)) & \rightarrowtail & S_3 V^* & \twoheadrightarrow & E', \end{array} \tag{2.2.2}$$

$$p_*(I/I^2 \otimes \mathcal{O}_D(3)) = Q^* \otimes S_2(K^*) \rightarrowtail E' \twoheadrightarrow p_* \mathcal{O}_D(3) = S_3(K^*). \tag{2.2.3}$$

Indeed $I/I^2 = Q^* \otimes \mathcal{O}_D(-1)$, as we recalled in (2.1), and we may use Serre's formulae ([H], p. 253) for the cohomology of a projective bundle. One checks that $p_* s$ is equal to the composition of the tautological section of $S_3 V^* \otimes \mathcal{O}_{\mathbb{P}_{19}}(1)$ with the map $(S_3 V^*)(1) \twoheadrightarrow E'(1)$ obtained from (2.2.2). It follows that Y is equal to the subbundle $\mathbb{P}(E)$ of $\mathbb{P}_{19} \times G$. An explicit calculation of tangent maps (which we'd rather omit) shows that $Y \to X$ is generically unramified if and only if the characteristic is not equal to 2, whence birational because the double line is unique for almost all f in X.

The formula for Chern classes follows from (2.2.2, 3) plus standard formal properties (cf. [F], Examples 3.2.3, 3.2.6). □

2.2.4 *Remark.* Put $f := x^2 w + y^2 z$. The tangent cone to f at a point on the double line consists of two distinct movable planes (i.e., neither of which is fixed) (char. $\neq 2$!) with two exceptions A, B, where the tangent cones consist of planes a, b counted twice. The calculation of tangent maps referred to at the end of the proof above shows that f is a smooth point on X and the tangent space there consists of all cubic surfaces tangent to a at A and to b at B.

2.3. THEOREM. *The number of ruled cubic surfaces through m general points and with double line satisfying Schubert conditions $\lambda^n \pi^p \theta^q$ ($m + n + 2p + 2q = 13$) is equal to the degree of the zero-cycle $h^m \lambda^n \pi^p \theta^q$. This number is zero if*

$m<9$ or if $pq \neq 0$, and the remaining values are given in the table below (blanks $=0$):

m	9	9	9	9	9	10	10	10	11	11	11	12	13
n	4	2	2			3	1	1	2			1	
p		1		2			1			1			
q			1		2			1			1		
$\int$	2	1	1	1	1	18	9	9	81	33	48	232	504

Proof. Let $p: Y = \mathbb{P}(E) \to G$ be the structure map. We have

$$\int_Y h^m \lambda^n \pi^p \theta^q = \int_G (p_* h^m) \lambda^n \pi^p \theta^q.$$

The latter may be computed from the formulae:

$$p_*(\sum h^m) = c(E)^{-1} \qquad \text{([F], Ex. 3.2.1)}$$

and

$$\lambda^4 = 2, \qquad \pi^2 = \theta^2 = \lambda^2 \pi = \lambda^2 \theta = 1, \qquad \pi\theta = 0$$

([F], Ex. 14.7.2). That the degree of the zero-cycle accurately gives the weighted number of surfaces fulfilling the conditions imposed follows from standard counting constants arguments. Indeed, set

$$S := \mathbb{P}_3^m \times G^n \times \mathbb{P}_3^p \times \mathbb{P}_3^{*q}.$$

Consider the subvariety Z of $S \times Y$ defined by

$$Z := \{((A_i), (d_j), (B_r), (p_s), f, d) \mid A_i \in f,\ d_j \cap d \neq \varnothing,\ B_r \in d \subset p_s \text{ for all } i, j, r, s\}.$$

The fibre of Z over (f, d) in Y is $f^m \times d'^n \times d^p \times d^{*q}$, where d' (resp. d^*) is the variety of lines (resp. planes) incident to (resp. containing) d. Therefore Z is irreducible of dimension $2m + 3n + p + q + 13 = \dim S$, provided $m + n + 2p + 2q = 13$. In this case, a general fibre of Z over S is finite and its degree is readily seen (2.4(1) below) to be that of the zero-cycle under consideration. $\square$

2.4 *Remarks.* (1) Each fibre of Z over S is an intersection of Cohen–Macaulay subschemes of Y. Thus, whenever such an intersection is proper, the intersection multiplicity at each point is the length of the corresponding local ring ([F], Ex. 8.2.7). In this case, the cycle-class of the intersection subscheme is equal to the product of the cycle-classes of the intersectands and, in particular, the degrees are the same and equal to the degree of the extension of rational function fields; moreover, the intersection multiplicity is the inseparable degree ([V_2]).

(2) Since in characteristic 2 the map $Y \to X$ is everywhere ramified, the degree of X is 504 divided by 2, 4 or 8.

(3) In characteristic $\neq 2$ the dual surface of $x^2w + y^2z$ is a non-special cubic surface with a double line. Hence there are 504 ruled cubic surfaces tangent to 13 general planes. How many are there tangent to m planes and passing through $13 - m$ points?

3. Compactification of the space of rational quintics

As mentioned in the introduction, we have the following:

3.1 PROPOSITION. *A general rational quintic is given by a unique pencil of cubic surfaces with a common double line, and conversely.*

Proof. We still have to prove the uniqueness of the pencil. First we claim that if two cubic surfaces with a common double line d contain a rational quintic c then d is a quadrisecant of c. Indeed, a general plane through d meets c in only one other point. This is because the intersection of the plane with the first cubic surface f_1 is of the form $2d + \ell$; and ℓ meets f_2 twice on d and once elsewhere.

On the other hand, c has a unique quadrisecant. Here is a direct argument for uniqueness: suppose there are two quadrisecants d, d'. Let d'' be a trisecant. Then the quadric through d, d', d'' would contain c entirely. This is impossible since a general c does not lie on a quadric (cf. [C], lemma 2.5).

Finally, observe that 8 general points on c impose 8 independent conditions on cubics with d as a double line. Indeed, 7 points do not suffice, since there exists a quadric through d and 6 of the points; the union of this quadric with the plane through d and the seventh point does not contain c. □

Notation being as in (2.2), set

$T := \text{Grass}\,(2, E)$,

the Grassmann bundle$/G$ with tautological sequence

$$A \rightarrowtail E \twoheadrightarrow B, \tag{3.1.1}$$

where rank $A = 2$, rank $B = 8$. For a point t in T lying over d in G, the fibre $A_t = t$ is a 2-dimensional vector space of cubic forms doubled along d, or, what is the same, a pencil of cubic surfaces doubled along d. Notice T is of dimension 20 as expected: a rational quintic curve in $\mathbb{P}_3$ is given by 4 forms of degree 5 in 2 variables, up to scalar multiple and projective change of the two variables (cf. [C], lemma 2.4).

3.2 PROPOSITION. (1) *There is a regular section of* $A^* \otimes \mathcal{O}_{\mathbb{P}_3}(3)$ *over* $T \times \mathbb{P}_3$ *which vanishes precisely on*

$$\mathcal{S} := \{(t, P) \mid P \text{ lies in the base locus of } t\}.$$

(2) *We have the equality of cycles of dimension* 21,

$$[\mathcal{S}] = 4[D_T] + [C],$$

where $D_T = D \times_G T$ *and the fibre of* C *over a general point* t *in* T *is the rational quintic curve defined by the pencil* t.

Proof. Putting together (3.1.1) and (2.2.2) yields the diagram over $T \times \mathbb{P}_3$, with bottom arrow defined by evaluation at a (variable) point,

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, dashed] & E \arrow[d, tail] \\ \mathcal{O}_{\mathbb{P}_3}(3) & S_3 V^*. \arrow[l, two heads] \end{tikzcd}$$

The resulting section of $A^*(3)$ is regular because the codimension of $\mathcal{S}$ is 2 (and, of course, $T \times \mathbb{P}_3$ is Cohen–Macaulay). Indeed, call F the locus of pencils with a fixed component. Each component of $\mathcal{S}$ is of codimension $\leq 2 = \text{rank}\, A$. Since all the fibres of $\mathcal{S}$ over $T - F$ are curves, we'll be done if we show $\mathcal{S}$ has no component lying over F. Now, for t in F, $\mathcal{S}_t$ is of dimension 2, so it remains to see that dim F is less than 18. Here are the possibilities for t in F:

F_1 – fixed component a general plane; residual pencil made up of quadrics with a common double line. Dimension count:
3(for the plane) + 4(line) + 2(pencil) = 9;

F_2 – fixed component a plane containing the distinguished line d (over which t lies); residual pencil of quadrics through d.
Dimension: 3(plane) + 2(line) + 10(pencil) = 15;

F_3 – fixed component a quadric surface containing d; residual pencil of planes through d. Dimension: 4(line) + 6(quadric) = 10.

Finally, assertion (2) follows because the generic fibre of $\mathcal{S}$ over T is a rational quintic together with its unique quadrisecant counted 4 times. □

3.2.1 Remark. We describe what seem to be the normal directions to each of the three components of F just described, at least at a *general* point. The general philosophy is that, as a pencil acquires a fixed component f, the base locus loses some components inside f. These ghost components account for the normal directions. We write F_i' for the set of normal directions at a general pencil F_i. We should have:

F_1': linear system of quintic curves in the fixed plane with a 4-fold point at the intersection with the distinguished line;

F_2': linear system of conics in the fixed plane passing through the third point of intersection of the twisted cubic determined by the pencil of quadrics and for which the distinguished line is a chord;

F_3': complete system $|d + 4d'|$ in the fixed quadric, where d' is a line transversal to d.

3.2.2 NOTATION. For a point P in $\mathbb{P}_3$, write C_P for the fibre of C (3.2(2)) over P. Notice that a general point of C_P is a rational quintic through P.

3.3 THEOREM. *There are* 105 *rational quintic curves through* 10 *points in general position. Moreover, if the characteristic does not divide* 105, *these curves are all distinct.*

Proof. We use the fact that a general rational quintic determines a unique pencil of cubic surfaces with a common double line and conversely. Notation being as in (3.2), let W be the closure in $T \times \mathbb{P}_3^{10}$ of the set

$$\begin{aligned} W^0 &:= \{(t, (P_i)) \in (T - F) \times \mathbb{P}_3^{10} \mid C_t \ni P_i,\ i = 1, \ldots, 10\} \\ &= \bigcap C_{P_i} \cap (T - F) \times \mathbb{P}_3^{10}. \end{aligned}$$

The fibre W_t is $t \times C_t^{10}$ for t in $T - F$, whence $\dim W = 30$ and W is irreducible because its generic fibre over T is so. The map $W \to \mathbb{P}_3^{10}$ is easily seen to be generically finite, with fibre

$$W_{(P_i)} = \bigcap C_{P_i}$$

for (P_i) sufficiently general, off the image of $W - W^0$. Its degree is the degree of the zero-cycle $[C_P]^{10}$ for any P in $\mathbb{P}_3$. It gives the number of rational quintics through 10 general points counted with multiplicity (equal to the inseparable degree of that map, as in 2.4(1)).

To compute the degree, let $p: C \to T$, $q: C \to \mathbb{P}_3$ be the projections. We have

$$[C_P] = p_*(q^*u^3)$$

where u stands for the hyperplane class of $\mathbb{P}_3$. By 3.2(2), the class of C in the Chow ring of $T \times \mathbb{P}_3$ is readily seen to be ([F], prop. 14.1):

$$[C] = c_2(A^* \otimes \mathcal{O}_{\mathbb{P}_3}(3)) - 4c_2(Q \otimes \mathcal{O}_{\mathbb{P}_3}(1));$$

whence we get

$$[C_P] = c_2A - 4c_2Q,$$

and therefore,

$$[C_P]^{10} = (c_2A)^{10} - 40(c_2A)^9c_2Q + 720(c_2A)^8(c_2Q)^2.$$

Here is an indication for the actual computation. Notation being as in (2.2), consider the tautological sequence on $\mathbb{P}(E)$,

$$\mathcal{O}_E(-1) \rightarrowtail E \twoheadrightarrow R.$$

One checks that $\mathbb{P}(R) = \mathbb{P}(A)$ as bundles over G, whence we have the commutative diagram:

$$\begin{tikzcd}
\mathbb{P}(R) \arrow[rr, "q"] \arrow[d, equal] & & \mathbb{P}(E) \arrow[d, "p"] \\
\mathbb{P}(A) \arrow[r, "\eta"] & T \arrow[r, "\psi"] & G.
\end{tikzcd}$$

In fact, the identification above comes from the natural diagram of sheaves:

$$\begin{tikzcd}
\mathcal{O}_A(-1) \arrow[r, equal] \arrow[d, tail] & \mathcal{O}_E(-1) \arrow[d, tail] & \\
A \arrow[r, tail] \arrow[d, two heads] & E \arrow[r, two heads] \arrow[d, two heads] & B \arrow[d, equal] \\
\mathcal{O}_R(-1) \arrow[r, tail] & R \arrow[r, two heads] & B.
\end{tikzcd}$$

Now the trick is to lift calculations on T over to $\mathbb{P}(A)$, then to descend via the projective bundles q, p. Precisely, setting

$$v = c_1\mathcal{O}_R(1)$$

and recalling

$$h = c_1\mathcal{O}_E(1) = c_1\mathcal{O}_A(1),$$

we get

$$c_2\varphi^*A = hv.$$

Now we may compute:

$$\begin{aligned}
\int_T (c_2A)^{10} &= \int_T \varphi_*(h\varphi^*(c_2A)^{10}) && \text{([F], thm. 3.2(c))}\\
&= \int_{\mathbb{P}(E)} q_*(h(hv)^{10}) && \text{([F], p. 13)}\\
&= \int_G p_*(h^{11}s_2R) && \text{([F], p. 47)}
\end{aligned}$$

Further $sR = c(E)^{-1}(1-h)$, and hence:

$$\begin{aligned}
\int_T (c_2A)^{10} &= \int_G p_*(h^{11}(33\pi + 48\theta - 9\lambda h)) && \text{(2.2(4))}\\
&= \int_G (33\pi + 48\theta)s_2E - 9\lambda s_3E\\
&= \int_G (33\pi + 48\theta)^2 - 9\lambda(232\lambda\theta) = 1305.
\end{aligned}$$

Similarly, we get:

$$\int_T (c_2A)^9c_2Q = 48 \quad \text{and} \quad \int_T (c_2A)^8(c_2Q)^2 = 1,$$

which yields, at last,

$$\int_T [C_P]^{10} = 1305 - 40.48 + 720.1 = 105. \quad \square$$

Remark. The same method applies to the $(6r+8)$-dimensional family of curves of degree $2r+1$ with a $2r$-fold secant ($r \geq 2$). These curves are rational, and they are all obtained as the intersection of a pencil of surfaces of degree $r+1$ with a common r-tuple line.

4. Special ruled cubics

What distinguishes a special ruled cubic from the general one is the nature of the family of tangent cones along the double line. For the non-special cubic $(x^2w+y^2z=0)$, the tangent cones form a pencil of plane pairs without fixed component, whereas for the special cubic $(x(xw+yz)+y^3=0)$ the pencil acquires a fixed plane.

4.1. Recall (2.2(3))

$$Y = \{(f, d) \in \mathbb{P}_{19} \times G \mid f \supset 2d\}.$$

Set

$$D_Y = Y \times_G \mathbb{P}(K) = \{(f, d, P) \in Y \times \mathbb{P}_3 \mid P \in d\},$$

and put

$$F = \{(f, d, P) \in Y \times \mathbb{P}_3 \mid P \in f\}.$$

Consider the blowup of $Y \times \mathbb{P}_3$ along D_Y,

$$b : B \to Y \times \mathbb{P}_3,$$

and take the total transforms $\tilde{D} := b^{-1}(D_Y)$, $\tilde{F} := b^{-1}(F)$. Then $\tilde{F} - 2\tilde{D}$ is an effective divisor on B. We recall that $\tilde{D}$ is the projectivized normal bundle, i.e., $\tilde{D} = D_Y \times_G \mathbb{P}(Q \otimes \mathcal{O}_K(1))$. Next, we set

$$\varsigma = \tilde{D} \cap (\tilde{F} - 2\tilde{D}). \tag{4.1.1}$$

The fibre of ς over $(f, d, P) \in D_Y$ is the projectivized tangent cone $\varsigma(f, P)$ provided P is not a triple point on f. Notice ς is a divisor with associated line bundle

$$\begin{aligned} \mathcal{O}_{\tilde{D}}(\varsigma) &= \mathcal{O}_B(\tilde{F}) \otimes \mathcal{O}_{\tilde{D}}(2) \\ &= \mathcal{O}_E(1) \otimes \mathcal{O}_K(3) \otimes \mathcal{O}_{\tilde{D}}(2). \end{aligned} \tag{4.1.2}$$

4.2 LEMMA. *There is a natural isomorphism of bundles over* D_Y,

$$i: D_Y \times_G \mathbb{P}(Q^*) \xrightarrow{\sim} D_Y \times_G \mathbb{P}(Q \otimes \mathcal{O}_K(1)),$$

such that

(1) *the following diagram commutes*:

$$\begin{array}{ccc} i^*\mathcal{O}_{Q(1)}(1) & \rightarrowtail & i^*(Q(1)) = Q \otimes \mathcal{O}_K(1) \\ \| & & \wr\| \\ \mathcal{O}_{Q^*}(-1) \otimes \overset{2}{\Lambda} Q \otimes \mathcal{O}_K(1) & \rightarrowtail & Q^* \otimes \overset{2}{\Lambda} Q \otimes \mathcal{O}_K(1) \end{array}$$

(2) $i^*\mathcal{O}_{\tilde{D}}(\varsigma) = \mathcal{O}_E(1) \otimes \mathcal{O}_K(1) \otimes \mathcal{O}_{Q^*}(2) \otimes (\overset{2}{\Lambda} Q^*)^2$.

Proof. (1) follows from the universal property of projective bundles in view of the natural isomorphism of rank 2 bundles, $Q^* \cong Q \otimes \overset{2}{\Lambda} Q^*$. Then (2) follows from (4.1.2). □

Recall that $\mathbb{P}(Q^*) \subset G \times \mathbb{P}(V^*)$ is the subbundle of pairs (d, v) such that the line d is contained in the plane v. Thus, under the above identification, we may write a point of $\tilde{D}$ as a quadruple (f, d, P, v) such that f is a cubic surface doubled along d, P is a point of d, and v contains d.

4.3 PROPOSITION. *Set*

$$Y' := \{(f, d, v) \in Y \times_G \mathbb{P}(Q^*) \mid v \in \varsigma(f, P) \quad \forall P \in d\}$$

and set

$$K' := K^* \oplus (\overset{2}{\Lambda} Q^*)^2 \otimes \mathcal{O}_E(1) \otimes \mathcal{O}_{Q^*}(2),$$

$$\kappa := c_1 \mathcal{O}_{Q^*}(1).$$

Then

(1) $Y' \to \mathbb{P}(Q^*)$ *is a projective subbundle of* $Y \times_G \mathbb{P}(Q^*)$ *of codimension* 2;

(2) *we have the formula*:

$$[Y'] = c_2(K') = 3\theta + 2\pi - 3\lambda h + h^2 + (4h - 6\lambda)\kappa + 4\kappa^2.$$

Proof. (1) Pick homogeneous coordinates so that v is the plane $x = 0$, d is the line $x = y = 0$. Then a cubic form doubled along d may be written

$$f = a(x, y, z, w)x^2 + b(y, z, w)xy + c(y, z, w)y^2,$$

with a, b, c linear. The tangent cone at $(0:0:z':w')$ is given by the form

$$(a_z z' + a_w w')x^2 + (b_z z' + b_w w')xy + (c_z z' + c_w w')y^2.$$

This is divisible by x for any z', w' if and only if $c_z = c_w = 0$, which gives two linearly independent conditions.

(2) We show Y' is the scheme of zeros of a regular section of K'. Construct the diagram:

$$\begin{tikzcd} \text{\c{C}}' := \text{\c{C}} \times_G \mathbb{P}(Q^*) \subset Y \times_G \mathbb{P}(K) \times_G \mathbb{P}(Q^*) \times_G \mathbb{P}(Q^*) = \{(f, d, P, u, v) \mid P \in d \subset u \cap v,\ 2d \subset f\} \arrow[dr] & \\ Y \times_G \mathbb{P}(K) \times_G \mathbb{P}(Q^*) = \Delta \quad (u = v) \arrow[u, hook] & Y \times_G \mathbb{P}(Q^*) \end{tikzcd}$$

We see that a point $x = (f, d, v)$ in $Y \times_G \mathbb{P}(Q^*)$ lies in Y' if and only if the fibre $\text{\c{C}}'_x$ contains Δ_x, i.e., if and only if the natural section $\mathcal{O}_\Delta \to \mathcal{O}_\Delta(\text{\c{C}}')$ vanishes over Δ_x. It follows that Y' is the scheme of zeros of a section of the direct image of $\mathcal{O}_\Delta(\text{\c{C}}')$ in $Y \times_G \mathbb{P}(Q^*)$ (cf. [AK], prop. (2.3)). In view of (4.2), the said direct image is K'. Assertion (2) now follows from [F], p. 61.

We may add that the formula for $c_2(K')$ can be written in several other ways, since there are some relations. For instance, from the canonical exact sequence on $\mathbb{P}(Q^*)$,

$$S \rightarrowtail Q \twoheadrightarrow \mathcal{O}_{Q^*}(1),$$

one gets:

$$1 + \lambda + \pi = c(Q) = (1 + c_1(S))(1 + \kappa),$$

whence

$$\pi = \kappa(\lambda - \kappa). \quad \square$$

4.4 Remarks. (1) Let Y_1 (resp. X_1) be the image of Y' in Y (resp. in X). Then the maps $Y' \to Y_1$ and $Y_1 \to X_1$ are generically finite with separable degree 1, hence birational in characteristic 0. Notice X_1 is the closure of the set of special ruled cubic surfaces.

(2) The class κ defined above (4.3) is the class of the condition that a plane contain a given point.

4.5 THEOREM. *The number of special cubic surfaces through m general points and with double line and fixed plane satisfying corresponding Schubert conditions is the degree of the zero-cycle* $h^m\lambda^n\pi^p\theta^q\kappa^r$ *in* Y' (4.3), *a sample of which is given in the table below.*

m	7	7	7	7	7	7	7	8	8	9	9	10	10	10	10	10	11	11	12
n	2		4	1	3		2	4	1	1	1	2			1		1		
p	1			1					1	1			1						
q		2				1								1					
r	1	1	1	2	2	3	3		1		2				1	2		1	
$\int$	1	1	2	1	2	1	1	8	10	34	62	288	114	174	286	172	766	652	1552

Proof. One applies successive push-forwards of the zero-cycle from Y' to $Y \times_G \mathbb{P}(Q^*)$, then to Y and finally to G, employing (4.3) and (2.3). For instance, the degree of X_1 (= variety of special ruled cubics) is calculated as follows:

$$\begin{aligned}
\int_{Y_1} h^{12} &= \int_{Y'} h^{12} \\
&= \int_{Y \times_G \mathbb{P}(Q^*)} h^{12}[Y'] \\
&= \int_{Y \times_G \mathbb{P}(Q^*)} h^{12}(3\theta + \cdots + (4h - 6\lambda)\kappa + 4\kappa^2) \\
&= \int_Y (4h^{13} - 2h^{12}\lambda) = 2016 - 464 = 1552,
\end{aligned}$$

using the fact that $\kappa + \kappa^2 + \kappa^3$ pushes down to $c(Q^*)^{-1} = c(K^*) = 1 + \lambda + \theta$. As a further example we may compute:

$$\begin{aligned}
\int_{Y_1} h^{10}\lambda\kappa &= \int_{Y'} h^{10}\lambda\kappa \\
&= \int_{Y \times_G \mathbb{P}(Q^*)} h^{10}\lambda\kappa[Y']
\end{aligned}$$

$$= \int_{Y \times_G \mathbb{P}(Q^*)} h^{10}\lambda\kappa(3\theta + 2\pi - 3\lambda h + h^2 + (4h - 6\lambda)\kappa + 4\kappa^2)$$

$$= \int_Y h^{10}\lambda(3\theta + 2\pi - 3\lambda h + h^2) + h^{10}\lambda^2(4h - 6\lambda) + 4h^{10}\lambda\theta = 286. \quad \square$$

5. Cones

Consider the tautological sequence on $\mathbb{P}_3 = \mathbb{P}(V)$,

$$H \rightarrowtail V^* \twoheadrightarrow U := \mathcal{O}(1). \tag{5.1}$$

The fibre $W = H_P$ over a point P in $\mathbb{P}_3$ is the subspace of $V^* = H^0(\mathbb{P}_3, U)$ of linear forms vanishing at P. Therefore a cubic cone in $\mathbb{P}_3$ with vertex P has an equation f in $S_3W \subset S_3V^*$. On the other hand, the plane $P^* := \mathbb{P}(W^*)$ imbeds in G as the star of lines through P. Thus, a nonzero f in S_3W yields a base curve for the cone. This allows us to further stratify the family of cones by imposing conditions on the base curve.

5.2 LEMMA. *Notation as in* (5.1) *and* (2.1), *we have*:
(1) $\mathbb{P}(H^*) = \mathbb{P}(K)$
(2) $\Omega_{\mathbb{P}(H^*)/\mathbb{P}_3} = Q^* \otimes M^{-1}$, *where* $M := \mathcal{O}_{H^*}(1)$.
More precisely, there is a commutative diagram with exact rows and columns:

$$\begin{tikzcd}
\mathcal{O}_{H^*}(1) \arrow[r, tail] & H^* \arrow[r, two heads] & Q \\
K \arrow[u, two heads] \arrow[r, tail] & V \arrow[u, two heads] \arrow[r, two heads] & Q \arrow[u, equal] \\
U^* \arrow[u, tail] \arrow[r, equal] & U^* \arrow[u, tail] &
\end{tikzcd}$$

where the top row (*resp. first column*) *is the tautological sequence on* $\mathbb{P}(H^*)$ (*resp.* $\mathbb{P}(K)$). $\square$

We denote by B' the bundle of first order principal parts of M^3 over $\mathbb{P}(H^*) \to \mathbb{P}_3$ (cf. [V_1], p. 403; [EGA], 16.7.1.2). By 5.2(2), it fits into the exact sequence

$$Q^* \otimes M^2 \rightarrowtail B' \twoheadrightarrow M^3. \tag{5.2.1}$$

5.3 PROPOSITION. *There is an exact sequence of vector bundles over* $\mathbb{P}(H^*)$,

$$R \rightarrowtail S_3H \twoheadrightarrow B',$$

such that:

(1) $\mathbb{P}(R) = \{(f, d) \in \mathbb{P}(S_3H) \times_{\mathbb{P}(V)} \mathbb{P}(H^*) \mid d$ *is a singular point of* $f\}$;

(2) $\mathbb{P}(R)$ *maps birationally onto the variety* $X_2 \subset \mathbb{P}_{19}$ *of cones over singular plane cubics.*

(3) *Set* $u := c_1U$, $m := c_1M$. *Then we have* $m = \lambda - u$ *and*

$$c(R)^{-1} = 1 + 6\lambda + 3u + 12\lambda^2 + 32u\lambda - 8u^2 + 8\lambda^3 + 92u\lambda^2 + 24u^2\lambda - 40u^3 + \\ + 80u\lambda^3 + 260u^2\lambda^2 - 200u^3\lambda + 360u^2\lambda^3 + 180u^3\lambda^2.$$

Proof. The first assertion is a special case of [V_1] (Thm. (9.1)(3)). Assertion (2) follows from two facts: (i) the map which assigns to a singular plane cubic its singular point is rational; (ii) the map which assigns to a cubic cone its vertex is rational and is defined on a neighborhood of $zxy + x^3 + y^3$. These claims amount to verifying that the maps $\mathbb{P}(R) \to \mathbb{P}(S_3H)$ and $\mathbb{P}(S_3H) \to \mathbb{P}_{19}$ are generically injective and unramified. Injectivity is obvious; as for tangent maps, one proceeds by explicit calculation (cf. [AM], p. 340).

For the last assertion, we recall the exact sequence

$$S_3H \rightarrowtail S_3V^* \twoheadrightarrow U \otimes S_2V^*.$$

This yields, in view of (5.2.1) and (5.2),

$$c(R)^{-1} = (1+u)^{10}(1+2m)^4(1+u+2m)^{-1}$$
$$= (1+u)^{10}(1+2m)^3\left(1 + \frac{u}{1+2m}\right)^{-1}.$$

Observing that $u^4 = 0$ and $m = \lambda - u$ (again by 5.2), one obtains (3). $\square$

5.4 Remark. (1) The class u introduced above is the condition that the vertex of a cone lie on a given plane.

(2) The degree of the variety of arbitrary cubic cones is

$$\int_{\mathbb{P}(S_3H)} h^{12} = \int_{\mathbb{P}_3} c(S_3H)^{-1} = 120.$$

5.5 COROLLARY. *The number of cubic cones with a double line satisfying $h^m\lambda^n\pi^p\theta^q u^r$ is given in the table below for a choice of $m, \ldots, r$:*

m	6	6	7	7	8	9	10	10	11
n			3	2			1		
p	1			1	1				
q		2				1			
r	3	1	1		1			1	
$\int$	1	1	18	3	36	116	480	420	900

Proof. The desired degrees are calculated by pushing forward the appropriate zero-cycle from $\mathbb{P}(R)$ to $\mathbb{P}(K)$ and then to G. We recall that $\Sigma_0^5 h^{6+i}$ maps to $c(R)^{-1}$, and $u+u^2+u^3$ to $c(K)^{-1}=1+\lambda+\pi$ (cf. [F], p. 61). We have, for instance,

$$\int_{\mathbb{P}(R)} h^{10}u = \int_{\mathbb{P}(K)} u(80u\lambda^3+260u^2\lambda^2-200u^3\lambda)$$

$$= \int_G (80\lambda^4+260\lambda^2\pi) = 420. \quad \square$$

5.6 PROPOSITION (*Assume char.* $\neq 2$). *The subset Y_3 of $\mathbb{P}(R)$ of singular cubics with a double tangent is a divisor given by the zeros of a section of $(\overset{2}{\Lambda} Q^* \otimes M(1))^2$.*

Proof. Recall that the bundle B'' of second order principal parts fits into the exact sequence

$$(S_2Q^*)M \rightarrowtail B'' \twoheadrightarrow B',$$

in view of 5.2(2). Since $S_3H \rightarrow B'$ factors through B'', and since $\mathbb{P}(R)$ is the scheme of zeros of the section $\mathcal{O}(-1) \rightarrow S_3H \rightarrow B'$, we obtain a section of $(S_2Q^*)M \otimes \mathcal{O}_R(1)$. In characteristic $\neq 2$, S_2Q^* imbeds in $Q^* \otimes Q^* = \underline{\mathrm{Hom}}\,(Q, Q^*)$. Thus, taking determinants, we get a section of $((\overset{2}{\Lambda} Q^* \otimes M(1))^2$. In local coordinates one checks the latter is given by the Hessian of the plane cubic around the singular point. $\square$

5.7 COROLLARY. *The degree of the closure X_3 of the set of cones over a cuspidal plane cubic is* 960 (*up to the inseparable degree of $Y_3 \to X_3$*).

Proof. We have $\int_{X_3} h^{10} = \int_{Y_3} h^{10}$ (up to the inseparable degree). By (5.6), we have $[Y_3] = 2(h + m - \lambda) = 2(h - u)$. The desired degree is therefore $\int_{\mathbb{P}(R)} 2(h - u)h^{10} = 960$. □

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