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Detecting fibred links in S^3

DAVID GABAI[1]

§0. Introduction

In §1 we describe a simple effective procedure which allows one (in practice) to decide whether or not the oriented link $L = \partial R$ (R a smoothly embedded oriented surface in S^3) is a fibred link with fibre R. As an application in §3 we give an elementary geometric proof of the following fact ($\Leftarrow$ was first proven by Stallings [S2]). Let R be a Murasugi sum of R_1 and R_2, then $L = \partial R$ is a fibred link with fibre R if and only if for $i = 1, 2$ $L_i = \partial R_i$ is a fibred link with fibre R_i. A corollary (worked out with M. Boileau) of the only if proof shows that if R is the unique Seifert surface for the oriented link L, then L_i fibres with fibre R_i for at least one i. As further applications in §4, §5, §6 we show how to decide which oriented links of ≤ 9 crossings, knots of ≤ 10 crossings, oriented alternating links, and oriented pretzel links are fibred links. We either indicate or explicitly exhibit the fibres.

In §7 we give some insight into how essential tori and annuli may arise in the complement of a fibred link where the fibre is a nontrivial plumbing i.e., Murasugi sum along a square. As an application we show how to construct in any closed oriented 3-manifold a fibred knot with pseudo-Anosov monodromy, i.e., a hyperbolic fibred knot. See also Soma [So] for a proof of this last result.

In §8, we discuss an appealing conjecture.

§1. The basic result

We recall the key definitions regarding sutured manifolds.

NOTATION 1.1. If E is a space (resp. set), then $|E|$ denotes the number of components (resp. elements) of E. $\mathring{E}$ denotes the interior of E, and $N(E)$ denotes a regular neighborhood of E in an ambient manifold.

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DEFINITION 1.2. A *sutured manifold* (M, γ) is a compact oriented 3-manifold M together with a set $\gamma \subset \partial M$ of pairwise disjoint annuli $A(\gamma)$ and tori $T(\gamma)$.

The interior of each component of $A(\gamma)$ contains a *suture* i.e., a homologically non trivial oriented simple closed curve. Denote the set of sutures by $s(\gamma)$. Finally, every component of $R(\gamma) = \partial M - \mathring{\gamma}$ is oriented. Define $R_+(\gamma)$ (or $R_-(\gamma)$) to be those components of $\partial M - \mathring{\gamma}$ whose normal vectors point out of (into) M. The orientations on $R(\gamma)$ must be coherent with respect to $s(\gamma)$ i.e., if λ is a component of $\partial R(\gamma)$ and is given the boundary orientation, then λ must represent the same homology class in $H_1(\gamma)$ as some suture.

Remark 1.3. The rest of this paper involves the study of sutured manifolds embedded in S^3. Furthermore, all sutured manifolds subsequently considered satisfy $T(\gamma) = \varnothing$ and every component of ∂M intersects γ non trivially. Under these circumstances the sutured manifold is determined by M and $s(\gamma)$. Therefore, one can view γ as a set of thick oriented curves in ∂M where such curves induce the orientations on $\partial M - \mathring{\gamma}$.

One can think of a sutured manifold as a manifold with corners (equal to $\partial\gamma$) together with a vector field which points in along $R_-(\gamma)$ and out along $R_+(\gamma)$.

CONVENTION 1.4. Fix once and for all an orientation on S^3. A surface R is oriented if and only if R has a well defined normal vector field i.e., transverse orientation. The + side (− side) of R is that "side" of R where the normals point out (in). A transverse orientation on R induces an orientation on ∂R using the rule that if an observer walking along ∂R on the + side (− side) of R, sees R to the left (right), then the observer is (is not) following the orientation of ∂R.

DEFINITION 1.5. Let $R \subset S^3$ be a compact oriented surface with no closed components, then $(R \times I, \partial R \times I) = (N, \delta)$ is the sutured manifold *obtained* from R. Use Convention 1.4 to orient $R(\gamma) = R \times \{0, 1\}$.

If (M, γ) is a sutured manifold in S^3, then $(N, \delta) = (S^3 - \mathring{M}, \gamma)$ is the *complementary* sutured manifold.

If S is a "reasonable" [i.e., S is transverse to γ, each arc component of $S \cap \gamma$ is an essential arc in γ, and if λ is a circle component of $S \cap \gamma$, then λ (oriented as a component of ∂S) is homologous in γ to a component of $s(\gamma)$] properly embedded oriented surface in the sutured manifold (M_1, γ_1), then by applying the sutured manifold decomposition operation of Definition 3.1 [G5] to S and

(M_1, γ_1) we obtain the new sutured manifold (M_2, γ_2). Topologically $M_2 = M_1 - \mathring{N}(S)$. The notation for this operation is as follows.

$$(M_1, \gamma_1) \overset{S}{\rightsquigarrow} (M_2, \gamma_2).$$

If (M_1, γ_1) is a sutured manifold in S^3, then one may think of this as the operation

$$(N_1, \gamma_1) \overset{S}{\rightsquigarrow} (N_2, \gamma_2).$$

where (N_i, γ_i) is the complementary sutured manifold to (M_i, γ_i). Note that [G4] views sutured manifold decomposition from the latter point of view.

This paper focuses on a very special type of sutured manifold decomposition.

DEFINITION 1.6. A *product decomposition* is a sutured manifold decomposition

$$(M_1, \gamma_1) \overset{D}{\rightsquigarrow} (M_2, \gamma_2).$$

where D is a disc properly embedded in M_1 and $D \cap s(\gamma) = 2$ points. Therefore, $M_2 = M_1 - \mathring{N}(D)$ and $s(\gamma_2)$ is obtained by extending $s(\gamma_1) - \mathring{N}(D)$ in the natural way (figure 1.1a).

Dually, if (N_i, γ_i) is the complementary sutured manifold to (M_i, γ_i), then a *C-product* (C for complementary) decomposition is the operation

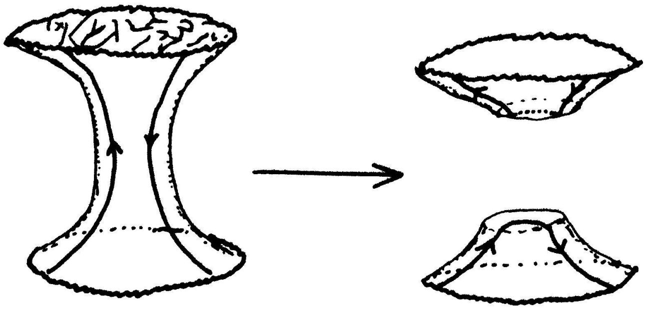
$$(N_1, \gamma_1) \overset{D}{\rightsquigarrow} (N_2, \gamma_2).$$

where D is a properly embedded disc in $S^3 - \mathring{N}_1$ with $\partial D \cap s(\gamma_1) = 2$ points. N_2 is obtained from N_1 by attaching the 2-handle D and $s(\gamma_2)$ is obtained by extending $s(\gamma_1) - \mathring{N}(D)$ in the natural way (figure 1.1b).

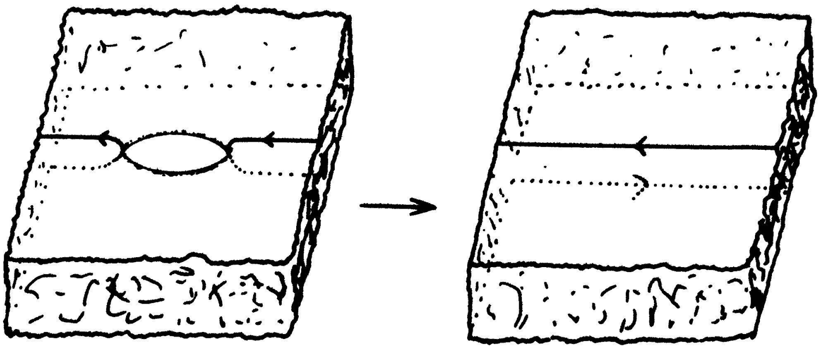
DEFINITION 1.7. Let (J_0, γ_0) be a sutured manifold in S^3. A *complete* (C)*product* decomposition of (J_0, γ_0) is a sequence of (C)product decompositions

$$(J_0, \gamma_0) \overset{D_1}{\rightsquigarrow} (J_1, \gamma_1) \rightsquigarrow \cdots \overset{D_p}{\rightsquigarrow} (J_p, \gamma_p)$$

where ∂J_p is a union of 2-spheres $S_1, \ldots, S_k$ with $s(\gamma_p) \cap S_r =$ a unique simple closed curve for $r = 1, \ldots, k$.



a) A Product Decomposition



b) A C-Product Decomposition

Figure 1.1

A *product decomposition* of a smooth surface R in S^3 is a complete C-product decomposition of the sutured manifold $(N_0, \gamma_0) = (R \times I, \partial R \times I)$, i.e., the sutured manifold obtained from R.

DEFINITION 1.8. An oriented link L is a *fibred link* in S^3 with fibre R if $\partial R = L$ (oriented boundary) and $S^3 - \mathring{N}(L)$ fibres over S^1 with fibre R.

THEOREM 1.9. *Let R be an oriented surface in S^3, L the oriented link ∂R, then L is a fibred link with fibre R if and only if R has a product decomposition.*

Proof. L is a fibred link with fibre R if and only if $(S^3 - \mathring{N}(L)) - \mathring{N}(R) = R \times I$.

$\Rightarrow$ Let $D_i = \gamma_i \times I$ where $\gamma_1, \ldots, \gamma_n$ is a set of pairwise disjoint properly embedded arcs in R such that

$$R - \bigcup_{i=1}^{n} \mathring{N}(\gamma_i) = D^2.$$

If follows that

$$(R \times I, \partial R \times I) = (N_0, \gamma_0) \overset{D_1}{\rightsquigarrow} (N_1, \gamma_1) \overset{D_2}{\rightsquigarrow} (N_2, \gamma_2) \rightsquigarrow \cdots \overset{D_n}{\rightsquigarrow} (N_n, \gamma_n) = (D^2 \times I, \partial D^2 \times I)$$

is a product decomposition for R.

$\Leftarrow$ This is 4) of Theorem 2.1 of [G4]. We give an alternative proof. A product decomposition gives a prescription to show that $(S^3 - \mathring{N}(L)) - \mathring{N}(R)$ is a product. Let

$$(N_0, \gamma_0) \overset{D_1}{\rightsquigarrow} \cdots \overset{D_n}{\rightsquigarrow} (N_n, \gamma_n)$$

be a product decomposition of R. Let (M_k, γ_k) be the complementary sutured manifold to (N_k, γ_k). Starting with $(M_n, \gamma_n) = (E \times I, \partial E \times I)$ where E is a union of 2-discs, one inductively observes that each (M_k, γ_k), hence (M_0, γ_0), is a product sutured manifold, i.e., of the form $(R_k \times I, \partial R_k \times I)$. $\square$

EXAMPLE 1.10. a) Figure 1.4b) of [G4] shows a product decomposition of an oriented surface.

b) Figure 1.2 shows a product decomposition of an oriented surface.

DEFINITION 1.11. If L is a fibred link with fibre R, then the *monodromy* of L is represented by $f: R \to R$ if there exists an orientation preserving homeomorphism

$$g: S^3 - \mathring{N}(L) \to \frac{R \times I}{(x, 0) \sim (f(x), 1)} \quad \text{where}$$

a) $g|R$ is an orientation preserving homeomorphism onto R

b) $g(m_i) = x_i \times [0, 1]$ for every meridian m_i of L

c) $f|\partial R = id$.

Here $R \times I$ is oriented so that the identity map $id: R \to R \times 0$ is orientation preserving and one standing at $R \times \frac{1}{4}$ sees the $+$ side of $R \times 0$.

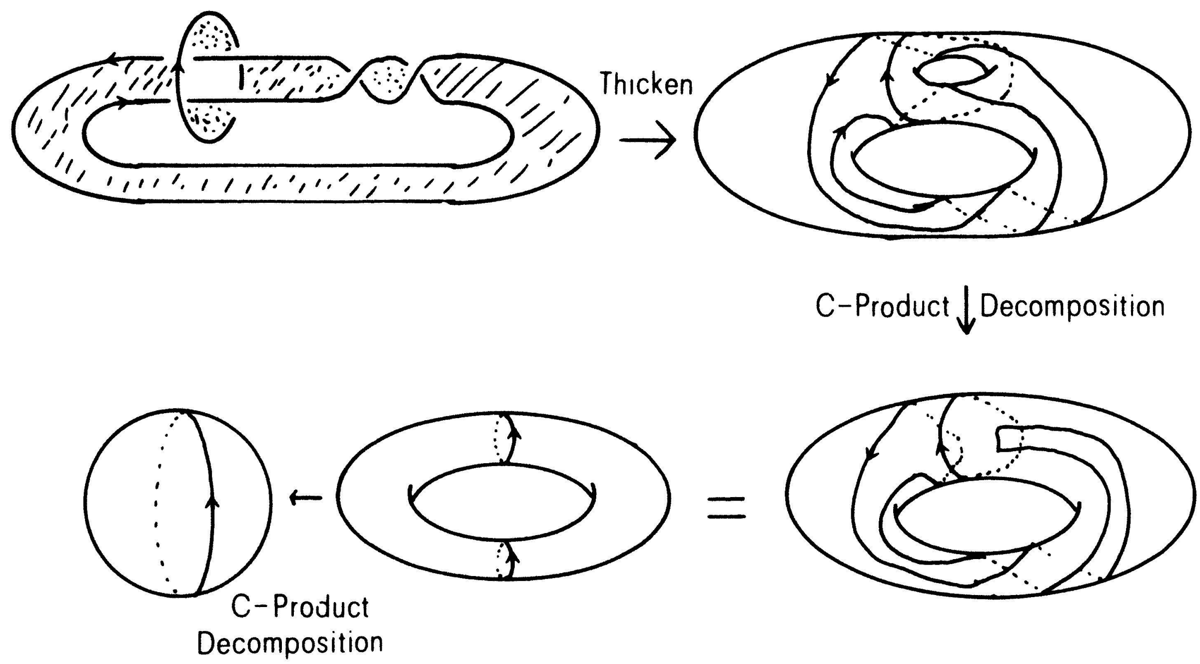


Figure 1.2

Note that $f_1, f_2 : R \to R$ represent the monodromy of the same fibred link if and only if f_1 is isotopic (rel ∂) to hf_2h^{-1} for some orientation preserving homeomorphism h of R.

Remark 1.12. One can view the monodromy of f as the automorphism of $\pi_1(R)$ obtained by pushing loops (with basepoint on ∂R) off the $+$ side of R through $S^3 - R$ and onto the $-$ side of R where the basepoint travels along a meridian.

§2. Technical lemmas

DEFINITION 2.1. A *product annulus* in the sutured manifold (M, γ) is an annulus A properly embedded in M such that $\partial A \subset R(\gamma)$, $\partial A \cap R_+(\gamma) \neq \varnothing$, and $\partial A \cap R_-(\gamma) \neq \varnothing$. A *product disc* is a disc D properly embedded in M such that $\partial D \cap \gamma$ equals two essential arcs in γ. Product discs and annuli detect where a sutured manifold is locally a product. (M, γ) is a *product sutured manifold* if $M = R \times I$, $\gamma = \partial R \times I$, $R_+(\gamma) = R \times 1$, and $R_-(\gamma) = R \times 0$.

Similarly, a *C-product annulus* in the sutured manifold (N, δ) is an annulus A properly embedded in $S^3 - \mathring{N}$ such that $\partial A \subset R(\delta)$, $\partial A \cap R_+(\delta) \neq \varnothing$, and $\partial A \cap R_-(\delta) \neq \varnothing$. A *C-product disc* is a disc D properly embedded in $S^3 - \mathring{N}$ such that $\partial D \cap \delta$ equals two essential arcs in δ.

The following lemmas are helpful in deciding whether or not a sutured manifold (M, γ) is a product sutured manifold. To understand §3, §4, §5, and most of §6 one needs only the very elementary Lemma 2.2.

LEMMA 2.2. *Let* $(M, \gamma) \overset{D}{\rightsquigarrow} (M_1, \gamma_1)$ *be a product decomposition, then* (M_1, γ_1) *is a product sutured manifold if and only if* (M, γ) *is a product sutured manifold.*

Proof. $\Rightarrow$ clear.

$\Leftarrow$ Certainly the homeomorphism type of (M_1, γ_1) is unchanged if one replaces D by an isotopic disc E where the isotopy is done $\operatorname{rel} s(\gamma)$. If $(M, \gamma) = (R \times I, \partial R \times I)$, then view D as $I \times I$ where $I \times 0$, $I \times 1$ are properly embedded arcs in $R \times 0$, $R \times 1$ respectively and $0 \times I$, $1 \times I$ are properly embedded arcs in $\gamma = \partial R \times I$. Now isotope $(\operatorname{rel} s(\gamma) = \partial R \times 1/2)D$ to be of the form $\alpha \times I$ where α is a properly embedded arc in R. $\square$

In Lemma 2.4 we give a more general version of $\Leftarrow$ Lemma 2.2 which is only needed in §6.

DEFNITION 2.3. (M, γ) is *taut* if M is irreducible and $R(\gamma)$ is Thurston norm minimizing. I.e., if T is a properly embedded incompressible surface in M having the properties that $\partial T \subset \gamma$ and $[T, \partial T] = [R(\gamma), \partial R(\gamma)] \in H_2(M, \gamma)$, then $\chi(T) \leq \chi(R(\gamma))$.

LEMMA 2.4. *If* $(M, \gamma) \overset{S}{\rightsquigarrow} (M_1, \gamma_1)$ *is a sutured manifold decomposition such that* (M_1, γ_1) *is taut, then* (M_1, γ_1) *is a product sutured manifold if* (M, γ) *is a product sutured manifold.*

Proof. Since the product sutured manifolds are exactly those with minimal sutured manifold complexity (§4 of [G5]) this result follows from Lemma 4.11 of [G5]. $\square$

The following lemma describes a sutured manifold decomposition (not used in this paper) which is very helpful in practice. The proof follows as in Lemma 2.2.

LEMMA 2.5. *Let* $(M, \gamma) \overset{A}{\rightsquigarrow} (M_1, \gamma_1)$ *be a sutured manifold decomposition where A is a product annulus, then* (M_1, γ_1) *is a product sutured manifold if and only if* (M, γ) *is a product sutured manifold.* $\square$

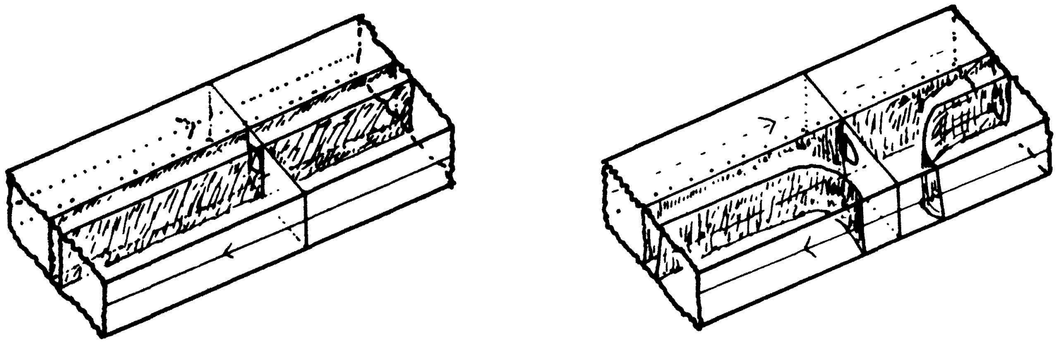
The following lemma and corollary are used only in §6.

LEMMA 2.6. *Let* $(M, \gamma) \overset{S}{\rightsquigarrow} (M_1, \gamma_1)$, $(M, \gamma) \overset{-S}{\rightsquigarrow} (M_2, \gamma_2)$ *be sutured manifold decompositions such that* (M_1, γ_1) *and* (M_2, γ_2) *are taut,* (M, γ) *is a product sutured manifold and* $-S$ *denotes* S *oppositely oriented, then* $S = \lambda \times I$ *for some properly embedded curve* λ *in* R.

Proof.
CASE 1. $\partial R \neq \varnothing$. Let $D_1, D_2, \ldots, D_n$ be a set of pairwise disjoint product discs in M such that

$$(M, \gamma) \overset{D}{\rightsquigarrow} (M', \gamma') = (D^2 \times I, \partial D^2 \times I) \quad \text{where} \quad D = \bigcup_{i=1}^{n} D_i.$$

Isotope S so that $|S \cap D|$ is minimal and $S \cap D = S \cap D - A(\gamma)$. [If a component of ∂S lay completely in $A(\gamma)$, then one of the decompositions of the lemma would not be defined.] It follows as in p. 462–464 of [G5] that $S \cap D$ appears as in figure 2.1a and if S' is the surface obtained by doing the boundary compression of figure 2.1b then S' satisfies the hypothesis of the theorem. Since the result is evidently true if $D \cap S = \varnothing$ the lemma follows by induction.



a) b)

Figure 2.1

CASE 2. $\partial R = \varnothing$. Note that $\partial S \neq \varnothing$ else S is isotopic to $R \times 1/2$, but then one of (M_1, γ_1) or (M_2, γ_2) would not be taut. Now isotope S and find an annulus $A = \delta \times I \subset R \times I$ such that $A \cap S$ is a union of arcs of the form $P \times I$.

We now have the commutative diagram where $S' = S - \mathring{N}(A)$. Since (M_1, γ_1) is taut, Lemma 3.12 [G5] implies (M_1', γ_1') is taut. Since a similar diagram holds for $-S$, $-S'$ the hypothesis of the lemma holds for (M', γ') and $\pm S'$. The result now follows from Case 1. □

$$\begin{tikzcd}
(M, \gamma) \arrow[r, squiggly, "S"] \arrow[d, squiggly, "A"] & (M_1, \gamma_1) \arrow[d, squiggly, "\text{product discs}"] \\
(M', \gamma') \arrow[r, squiggly, "S'"] & (M_1', \gamma_1')
\end{tikzcd}$$

COROLLARY 2.7. *If the sutured manifold decompositions* $(M, \gamma) \overset{S}{\rightsquigarrow} (M_1, \gamma_1)$, $(M, \gamma) \overset{-S}{\rightsquigarrow} (M_2, \gamma_2)$ *yield taut sutured manifolds and some component of S is neither a product annulus nor a product disc, then* (M, γ) *is not a product sutured manifold.* □

§3. Applications

THEOREM 3.1. *Let* $R \subset S^3$ *be a Murasugi sum* ([*M*] *or* [*G*2]) *of oriented surfaces* R_1, R_2 *in* S^3, *then* $L = \partial R$ *is a fibred link with fibre R if and only if for* $i = 1, 2$ $L_i = \partial R_i$ *is a fibred link with fibre* R_i.

Remark. $\Leftarrow$ was proven algebraically by Stallings in 1976. A geometric proof can be found in [G2].

$\Rightarrow$ was proved algebraically in [G2] and geometrically in [G3].

What follows is a completely elementary geometric proof.

Proof. R being a Murasugi sum implies that there exists a 2-sphere $Q \subset S^3$ separating S^3 into two 3-balls B_1, B_2 where $R \cap B_i = R_i$ and $R \cap Q$ is a $2-n$ gon D. Let (N, δ), (N_1, δ_1), (N_2, δ_2) be the sutured manifolds obtained from R, R_1, R_2 respectively and let (M, γ), (M_1, γ_1), (M_2, γ_2) respectively be the complementary sutured manifolds. Note that (figure 3.1)

$$\left.\begin{array}{l} B_1 \cap R_-(\delta) = R_-(\delta_1),\ B_1 \cap R_+(\delta) = R_+(\delta_1) - \mathring{N}(D) \\ B_2 \cap R_-(\delta) = R_-(\delta_2) - \mathring{N}(D),\ B_2 \cap R_+(\delta) = R_+(\delta_2). \end{array}\right\} \qquad (*)$$

Here we assume that the normal to D (D is a square in Figure 3.1) points into B_2.

Proof of $\Leftarrow$. Each (M_i, γ_i) is a product sutured manifold so there exist product discs

$D_1, \ldots, D_n$ in (M_1, γ_1)

$D_{n+1}, \ldots, D_r$ in (M_2, γ_2)

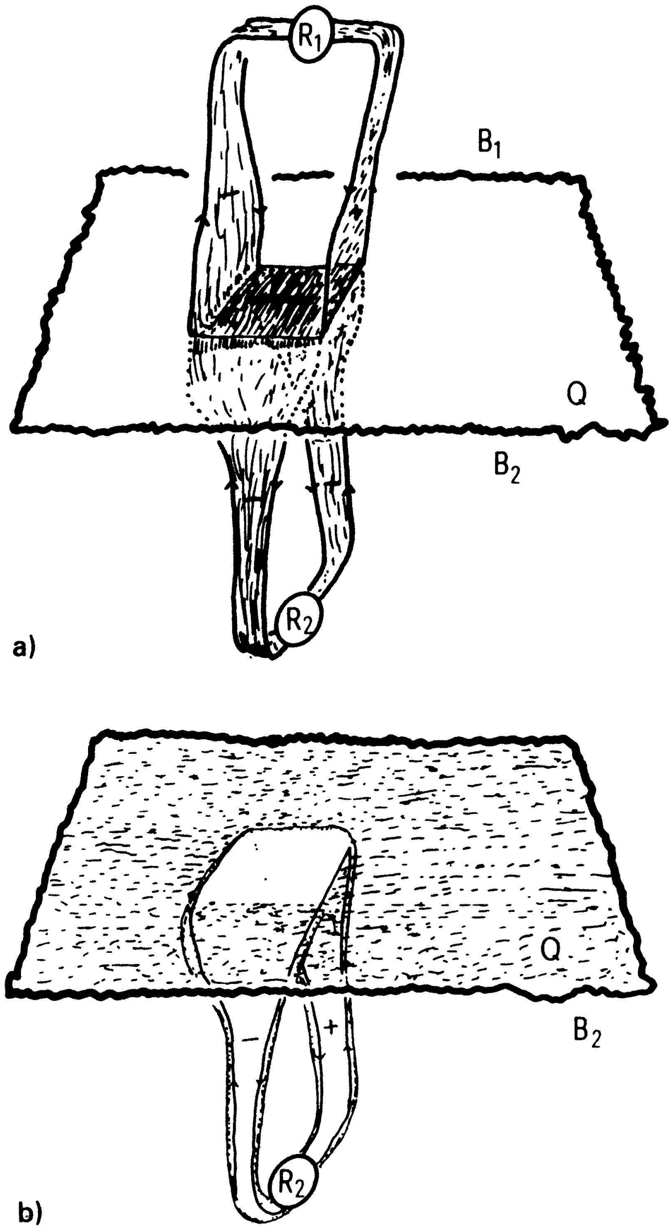


Figure 3.1

such that

$$D_i \cap N(Q) \cap R_+(\gamma_1) = \varnothing \qquad i \leq n$$
$$D_i \cap N(Q) \cap R_-(\gamma_2) = \varnothing \qquad i \geq n+1$$

and

$$R_\pm(\gamma_1) - \bigcup_{i=1}^{n} \mathring{N}(D_i), \qquad R_\pm(\gamma_2) - \bigcup_{i=n+1}^{r} \mathring{N}(D_i) \quad \text{are discs.}$$

Using (*) (recall $R_\pm(\gamma) = R_\pm(\delta)$) we can view each D_i as a product disc in (M, γ). We conclude that the sequence

$$(M, \gamma) \overset{D_1}{\rightsquigarrow} (M_1', \gamma_1') \rightsquigarrow \cdots \overset{D_r}{\rightsquigarrow} (M_r', \gamma_r')$$

is a product decomposition for (M, γ). $\square$

Proof of $\Rightarrow$. (This sharp proof was inspired by a question from Michel Boileau.) Assume that R (figure 3.2a) was summed along a square for the other cases follow similarly. By summing R_1 and R_2 along the square $S^2 - \mathring{D} = E$ we obtain the surface T (figure 3.2b) which is disjoint from R in $S^3 - \mathring{N}(L)$ (figure 3.2c) and homologous to R in $H_2(S^3 - \mathring{N}(L), \partial N(L))$. Since L fibres with fibre R, R and T are isotopic by [N] hence separate $S^3 - \mathring{N}(L)$ into 2 product sutured manifolds (H_1, δ_1) and (H_2, δ_2) where (H_1, δ_1) is the sutured manifold whose

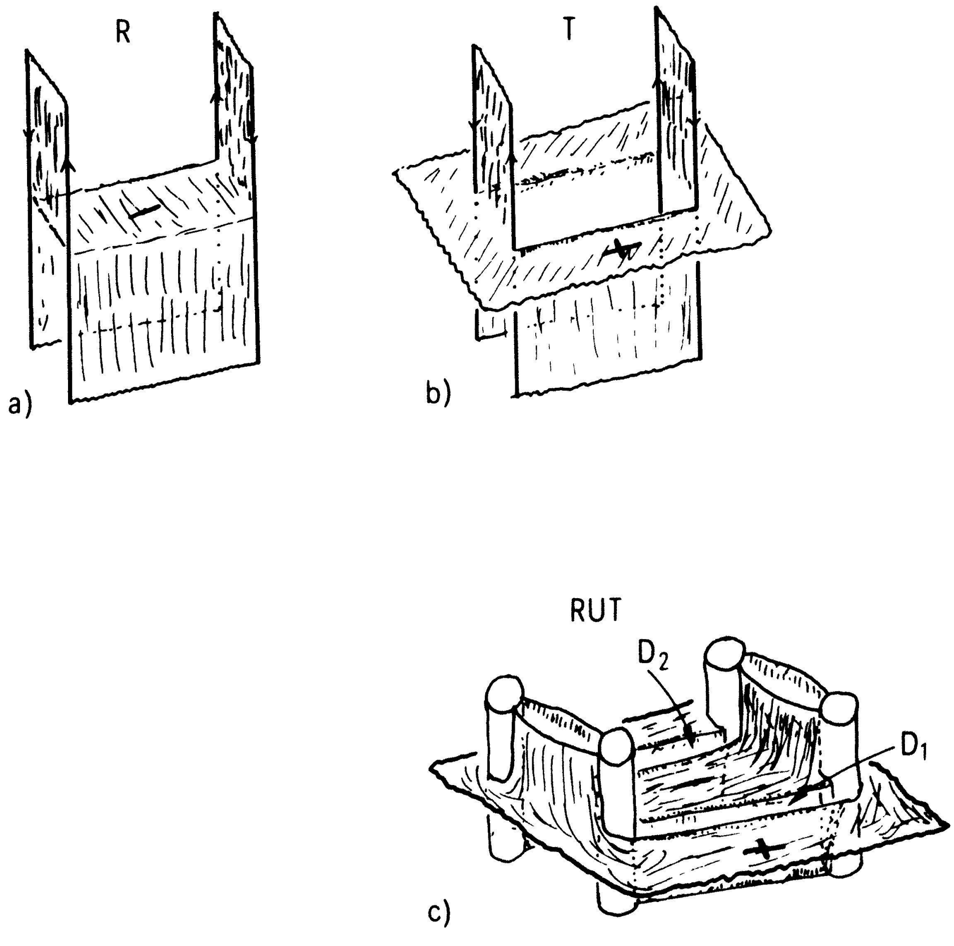


Figure 3.2

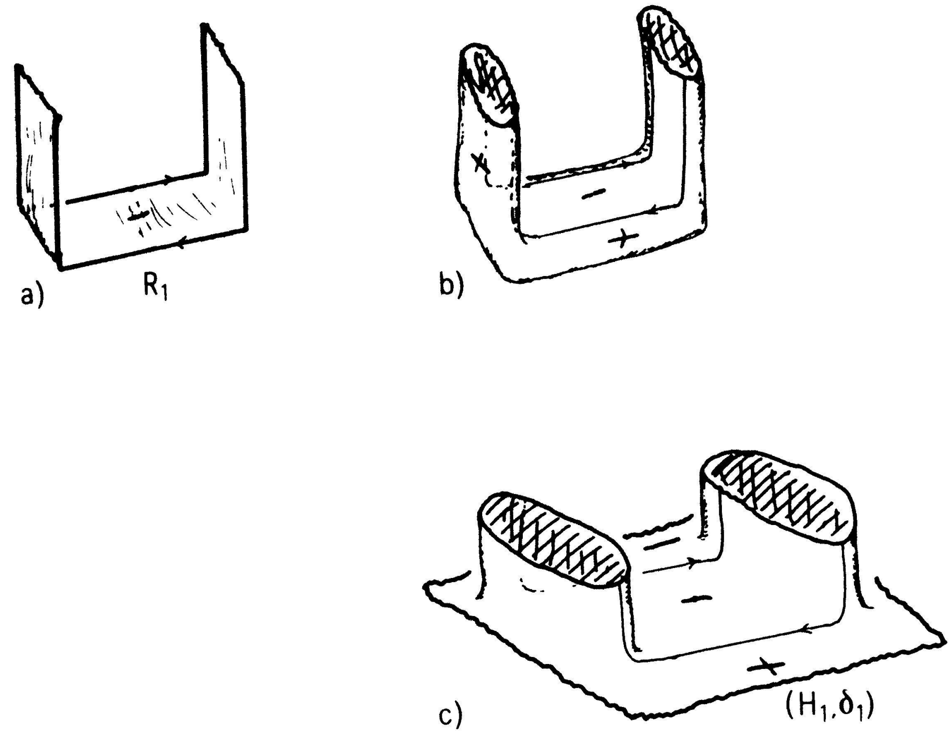


Figure 3.3

"thick part" lies above Q. The sutured manifold decomposition

$$(H_1, \delta_1) \overset{D_1}{\rightsquigarrow} (H_1', \delta_1') \overset{D_2}{\rightsquigarrow} (H_1'', \delta_1'')$$

along the product discs D_1, D_2 (figure 3.2c) yields by Lemma 2.2 (H_1'', δ_1'') a product sutured manifold. The sutured manifold obtained by splitting $S^3 - \mathring{N}(L)$ open along R_1 (figure 3.3) is the component (H, δ) of (H_1'', δ_1'') which is contained in B_1, which is a product, so L_1 is a fibred link with fibre R_1. Similarly, L_2 fibres with fibre R_2. □

COROLLARY 3.2 (with Michel Boileau). If R is a Murasugi sum of R_1 and R_2 and L is an oriented link with R its unique incompressible Seifert surface, then for some i $L_i = \partial R_i$ is a fibred link with fibre R_i.

Proof. Any minimal genus surface for L is incompressible, hence R is minimal genus. The surface T (of the $\Rightarrow$ proof of Theorem 3.1) is a Seifert surface for L. Since $\chi(T) = \chi(R)$, T is minimal genus hence incompressible. Therefore T is isotopic to S so some component of $(S^3 - \mathring{N}(L)) - N(T \cup S)$ is a product. The proof follows as in the $\Rightarrow$ proof of Theorem 2.1. □

Remark 3.3. This result was known to Eisner [E] for connected sums.

DEFINITION 3.4. A compact oriented surface $R \subset S^3$ can be *desummed* in $R_1, \ldots, R_n$ if one can obtain R by performing a finite number of Murasugi sum operations starting with $R_1, \ldots, R_n$.

QUESTION 3.5. How to decide whether or not an oriented link in S^3 is a fibred link?

PRACTICAL ANSWER. First find a surface S of minimal genus for L. If L fibres with fibre F, then S is isotopic to F hence is a fibre. Next try to desum S. By Theorem 3.1 if any of the desummed pieces $S_1, \ldots, S_n$ is not a fibre, then L does not fibre. Next analyze each of the S_i's. Let (N_i, δ_i) be the sutured manifold obtained from S_i and perform as many C-product decompositions as possible to (N_i, γ_i). We either might obtain a complete C-product decomposition or still be left with a non trivial sutured manifold (N, δ) with complementary sutured manifold (M, γ). If we can show that (M, γ) is not a product sutured manifold then by Lemma 2.2, S_i is not a fibre.

Algebraically by [S1] if M is irreducible and connected then (M, γ) is a product sutured manifold if and only if $\pi_1(R_+(\gamma)) \to \pi_1(M)$ is an isomorphism and both $R_+(\gamma)$, $R_-(\gamma)$ are connected. Geometrically if $M \neq B^3$ and there exists no product disc in (M, γ), then (M, γ) is not a product. One can use other methods (see §2) to help decide if (M, γ) is a product. We demonstrate those techniques in §5 and §6.

§4. Fibred knots of ≤10 crossings, links of ≤9 crossings

If L is an alternating knot or link see §5.

Table 4.1

NonFibred NonAlternating Knots of ≤10 Crossings			
9_{46}	10_{130}	10_{142}	10_{163}
9_{49}	10_{131}	10_{144}	10_{165}
10_{128}	10_{134}	10_{146}	10_{166}
10_{129}	10_{135}	10_{147}	
NonFibred NonAlternating Oriented Links of ≤9 Crossings			
9^2_{45} 2, 4	9^2_{51} 4	8^3_8 1, 3	9^3_{20} 3, 3
9^2_{46} 2, 2	9^2_{52} 2	9^3_{15} 3	9^3_{21} 1, 1, 3, 3
9^2_{48} 2, 4	9^2_{54} 2	9^3_{16} 3, 3, 3	8^4_2 2, 2, 2, 2
		9^3_{17} 1, 3, 3	8^4_3 0, 0, 2, 2, 2, 2

FACT 4.1. These tables summarize work found in chapter 8 of [G1]. There I explicitly exhibited a minimal genus surface for each non alternating knot of ≤ 10 crossings and oriented link of ≤ 9 crossings. Using 3.5 one easily decides whether such an oriented link fibres or not. For a given link (resp. knot) of n components there exists 2^{n-1} orientation classes to analyze. In §2 of [G4] I tabulated the absolute value (resp. genus) of the Euler characteristic of a minimal genus surface for a given orientation class. Table 4.1 lists all such non alternating *nonfibred* knots and links. We use Rolfsen's notation to describe a given unoriented link. The numbers associated to a given unoriented link indicate the absolute values of the Euler characteristics of minimal genus surfaces spanning orientation classes which do not fibre.

Kanenabu also computed the fibred knots of ≤ 10 crossings. Pictures of the fibres of such non alternating knots can be found in [K].

Remark 4.2. It is well known (see [R]) that if k is a fibred knot, then the genus of k is equal to one half of the degree of the Alexander polynomial of k. Also the leading term of the polynomial is 1. Conversely Murasugi [M2] showed that the fibred alternating knots are exactly those alternating knots whose Alexander polynomial has leading coefficient 1. From that fact and this list of nonfibred nonalternating knots of ≤ 10 crossings it follows that the fibred knots of ≤ 10 crossings are exactly those knots whose Alexander polynomial has leading coefficient 1. Since there exist knots of 11 crossing with trivial Alexander polynomial this condition on the leading term is not sufficient to distinguish fibred knots.

§5. Fibred alternating links

Recall that a Hopf band is an annulus spanning a $(2,2)$ torus link.

THEOREM 5.1. *Let L be an oriented link with an alternating projection. L is a fibred link if and only if the surface R obtained by applying Seifert's algorithm to the alternating projection is connected and* (*obviously*) *desums* (3.4) *into a union of Hopf bands.*

Remarks. Murasugi [M2] showed that alternating links whose Seifert surfaces desum into Hopf bands are exactly those with finitely generated commutator subgroup; hence, this result follows from [M2] and [S1].

See [R] or [G4] for the definition of Seifert's algorithm.

Proof of $\Leftarrow$. Since these bands clearly have product decompositions the result follows by Theorem 3.1.

Proof of $\Rightarrow$. By [M1], [C], [G4], or [G6] (for an easy proof), the Seifert surface R obtained by applying Seifert's algorithm to an oriented alternating projection is minimal. R canonically desums into surfaces $R_1, \ldots, R_n$ where each R_i is the surface obtained by applying Seifert's algorithm to the oriented alternating link L_i, where the oriented projection to L_i has no nested Seifert circles (i.e., the Seifert circles bound disjoint discs in S^2). Therefore it suffices to assume, by Theorem 3.1 that R has no nested Seifert circles, hence we can view R as lying in a $S^2 \subset S^3$ except in small neighborhoods of the crossings.

We prove the theorem by induction on the number n of crossings of L. If $n \leq 2$ then R is a Hopf band or disc. Assuming that the result is true for links of $\leq n$ crossings we prove it for links of $n+1$ crossings. Let (N, δ) be the sutured manifold obtained from R. Let E be a properly embedded non boundary parallel disc in $S^3 - \mathring{N}$ with $s(\delta) \cap E = 2$ points (i.e., E is a non trivial C-product disc for (N, δ)) and $E \cap S^2$ has the fewest number of components of intersection of all such discs. Assume that $E \cap s(\delta) \cap S^2 = \varnothing$.

CASE 1. $E \cap S^2 = \varnothing$.

If the points of $E \cap s(\delta)$ "occurred" at the same crossing (figure 5.1a), then E was boundary parallel, hence the points of $E \cap s(\delta)$ "occur" at different crossings (figure 5.1b). Therefore R is a Murasugi sum of R_1 and R_2, each of which is the surface obtained by applying Seifert's algorithm to an oriented alternating link L_i of at most n crossings. The result follows by induction and Theorem 3.1.

CASE 2. $E \cap S^2 \neq \varnothing$.

$E \cap S^2$ is a union of arcs, else $E \cap S^2$ does not intersect S^2 minimally. There exists an innermost disc $F \subset E$ (i.e., F is the closure of a component of $E - S^2$ and $F \cap S^2$ is connected) with $F \cap s(\delta)$ being at most 1 point.

CASE 2A. $\partial F \cap s(\delta) = \varnothing$.

The intersections of $\partial(F \cap S^2)$ occur at distinct crossings else we could have isotoped E to reduce the number of intersections with S^2, contradicting the minimality of E. We conclude that R is a Murasugi sum (in fact a connected sum) (figure 5.2) and the result follows as in Case 1.

CASE 2B. $\partial F \cap s(\delta) = 1$ point.

In this case, since E is of minimal complexity, F must appear as in figure 5.3a), b), or c). In either case R is a Murasugi sum and the result follows as in Case 1.

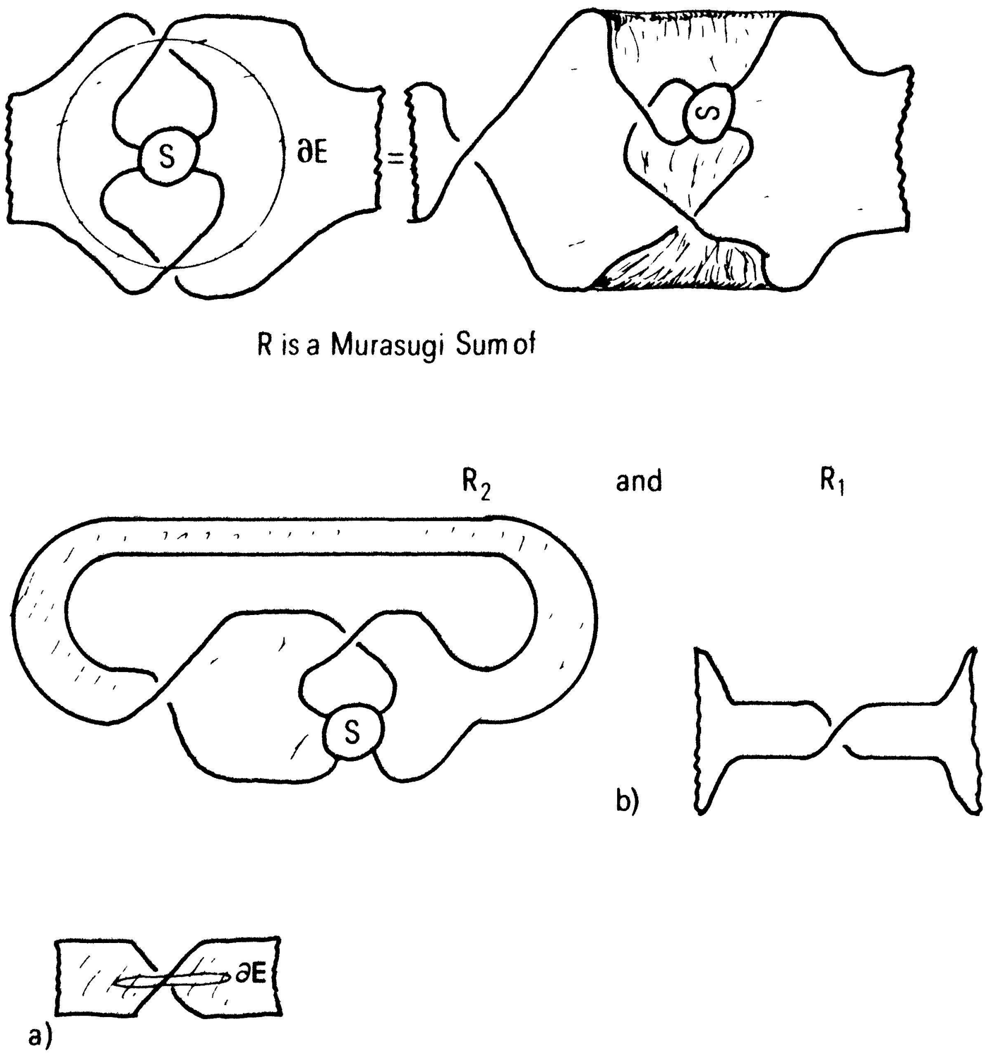


Figure 5.1

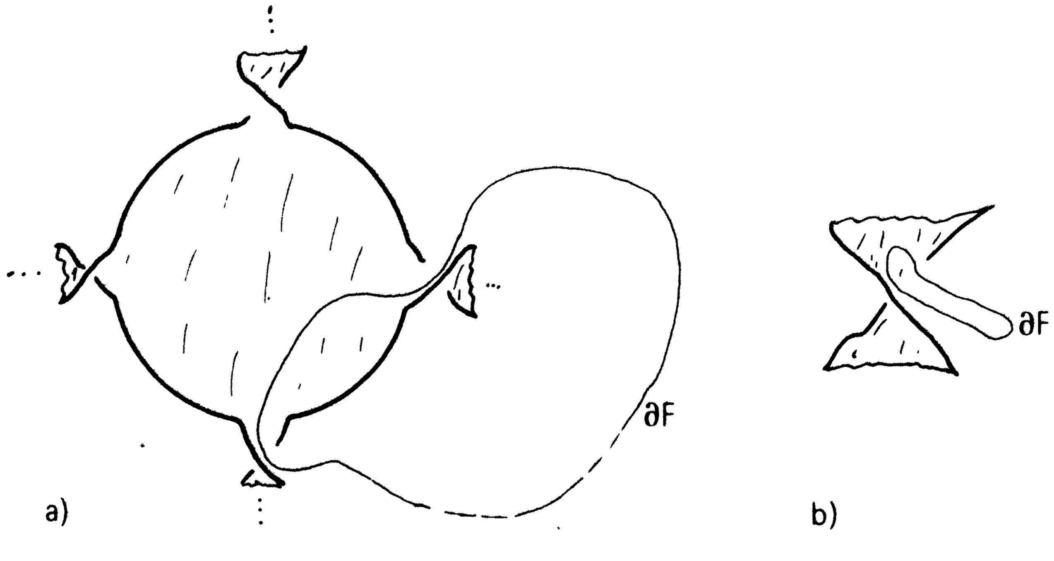


Figure 5.2

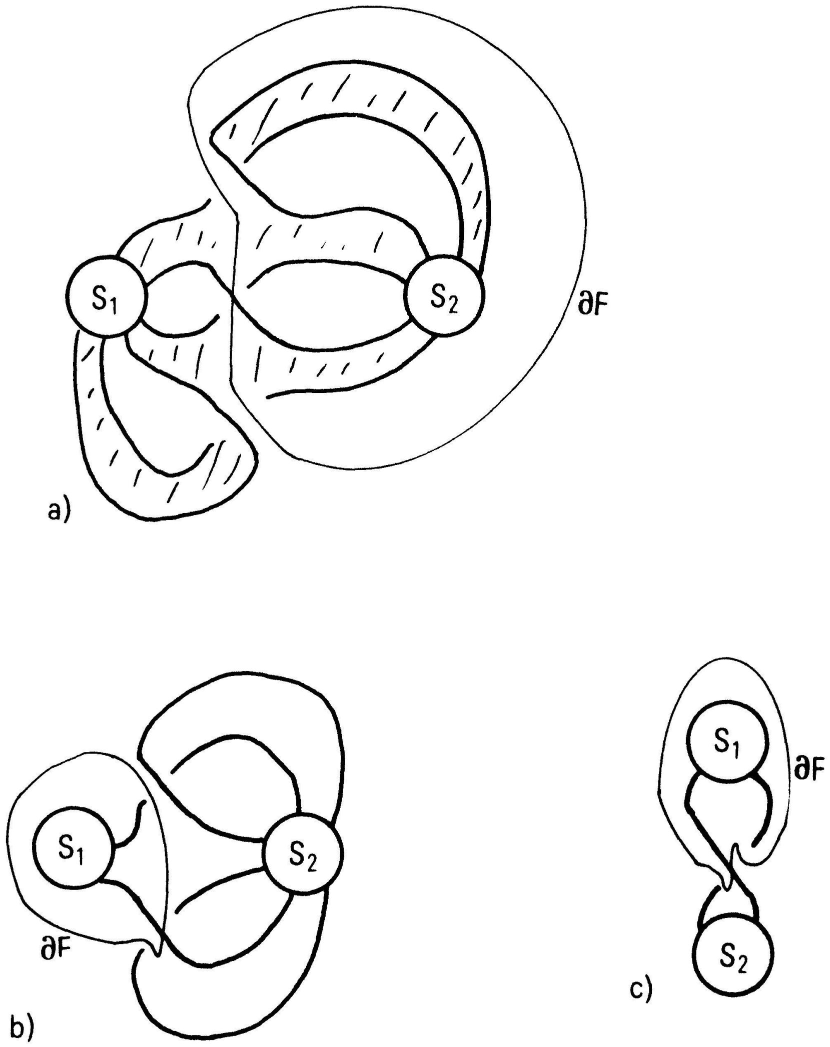


Figure 5.3

We have shown that if $R \neq D^2$ has a product decomposition, then R desums in a recognizable way as a union of Hopf bands. □

§6. Fibred pretzel links

In this section we give a complete list of all the oriented prime fibred pretzel links together with their fibres. We follow the program of 3.5. For a given oriented pretzel link L we utilize [G7] to find a surface S of minimal genus for L. We then apply the geometric methods to decide whether or not L fibres.

DEFINITION 6.1. An unoriented pretzel link $(n_1, \ldots, n_k)$ is a link of the following form (figure 6.1).

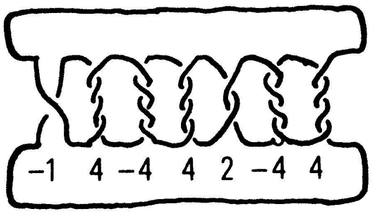
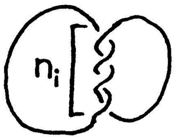
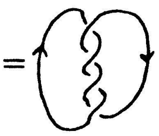
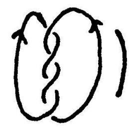


Figure 6.1

Remarks 6.2. If $\{0,0\} \subset \{n_1, \ldots, n_k\}$, then L is a split link hence does not fibre since $\pi_2(S^3 - L) \neq 0$.

If exactly one n_i (say n_1) equals 0, then l is a connected sum of $L_2, \ldots, L_n$ where the unoriented L_i equals 

. If $|n_i| \geq 4$, n_i even and $L_i =$  (rather than ), then L_i does not fibre, otherwise L_i fibres.

By Theorem 3.1 L fibres if and only if each L_i $i \geq 2$ fibres (since any incompressible Seifert surface for L can be expressed as a connected sum of incompressible Seifert surfaces, one for each L_i).

If no $n_i = 0$, then by [O] L is prime.

NORMALIZATION 6.3. If $\{1, -1\} \subset \{n_1, \ldots, n_k\}$, then L is the pretzel link $(n_1, \ldots, n_{k-2})$ obtained by deleting a 1 and a -1 from $(n_1, \ldots, n_k)$. We now assume that $n_i \neq 0$ and $\{1, -1\} \subset \{n_1, \ldots, n_k\}$.

Let R be the surface obtained by applying Seifert's algorithm to an oriented pretzel presentation of L. R must appear as one of the following 3 types.

TYPE I.6.4. (e.g., figure 6.2). L is oriented so that R is the pretzel spanning surface.

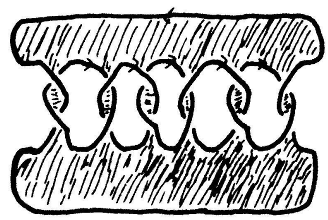


Figure 6.2

TYPE II.6.5. Figure 6.3. Here we *require* some m_{ij} exists. Note that r is even and each $|m_{ij}| \geq 2$ and even. Associated to L is the oriented pretzel link

$$L' = \left(\frac{-2m_1}{|m_1|}, m_{11}, m_{12}, \ldots, m_{1l_1}, \frac{-2m_2}{|m_2|}, \right.$$
$$\left. m_{21}, \ldots, m_{2l_2}, \ldots, \frac{-2m_r}{|m_r|}, m_{r1}, \ldots, m_{rl_r} \right)$$

where the term

$$\frac{-2m_i}{|m_i|}$$

is deleted if $|m_i| = 1$. L' is oriented so that the surface obtained by applying Seifert's algorithm is of type I.

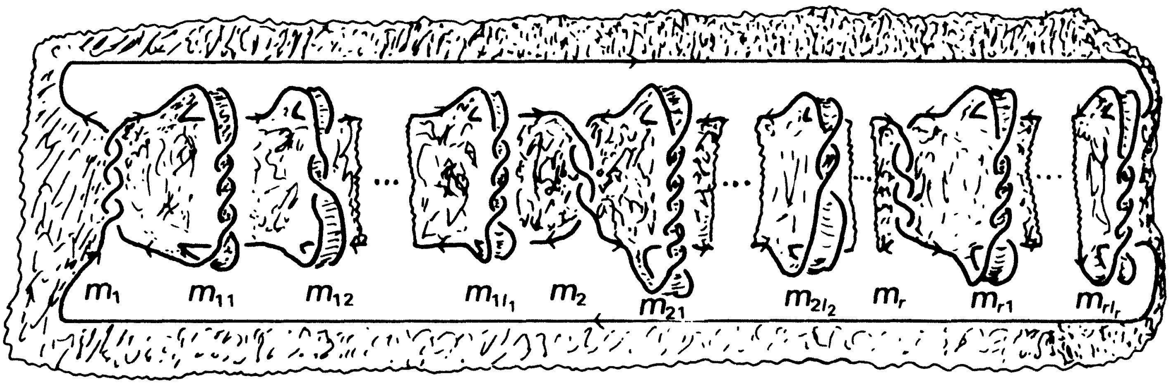


Figure 6.3

TYPE III.6.6. Figure 6.4. A type III surface is a type II surface where no m_{ij} exists.

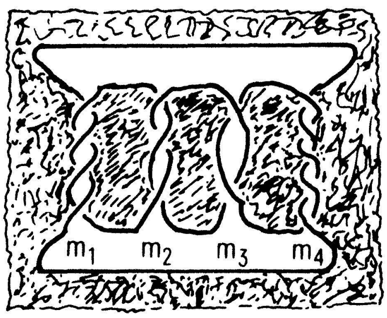


Figure 6.4

THEOREM 6.7. *The following is an effective algorithm to decide whether or not the prime oriented pretzel link L fibres. If L fibres we describe the fibre.*

CASE 1. The surface R obtained by applying Seifert's algorithm is of type I.

L fibres if and only if L fibres with fibre R. Moreover this happens if and only if one of the following holds:

(A) each $n_i = \pm 1$ or ∓ 3 and some $n_i = \pm 1$.

(B) $(n_1, \ldots, n_k) = \pm(2, -2, 2, -2, \ldots, 2, -2, n)$ $n \in \mathbb{Z}$ (here k is odd).

(C) $(n_1, \ldots, n_k) = \pm(2, -2, 2, -2, \ldots, -2, +2, -4)$ (here k is even).

CASE 2. The surface R obtained by applying Seifert's algorithm is of type II.

CASE 2A. $\sum_{j=1}^{r} \frac{m_j}{|m_j|} \neq 0.$

L fibres if and only if L fibres with fibre R. Moreover this happens if and only if the following holds:

(1) $|m_{ij}| = 2$ for all indices ij and

(2) $\sum_{j=1}^{r} \frac{m_j}{|m_j|} = \pm 2.$

CASE 2B. $\sum_{j=1}^{r} \frac{m_j}{|m_j|} = 0$ and $L' \neq \pm(2, -2, \ldots, 2, -2)$

L fibres if and only if L fibres with fibre T (figure 6.5). Moreover this happens if and only if L' fibres.

CASE 2C. $\sum_{j=1}^{r} \frac{m_j}{|m_j|} = 0$ and $L' = \pm(2, -2, \ldots, 2, -2)$.

Isotope $L = \partial T$ as exemplified in figure 6.6 to obtain a new pretzel presentation for L which bounds a type III surface. Now consult case 3.

CASE 3. The surface R obtained by applying Seifert's algorithm is of type III. If either

$$\sum_{j=1}^{k} \frac{n_j}{|n_j|} \neq 0 \quad \text{or} \quad L' \neq \pm(2, -2, 2, -2, \ldots, 2, -2)$$

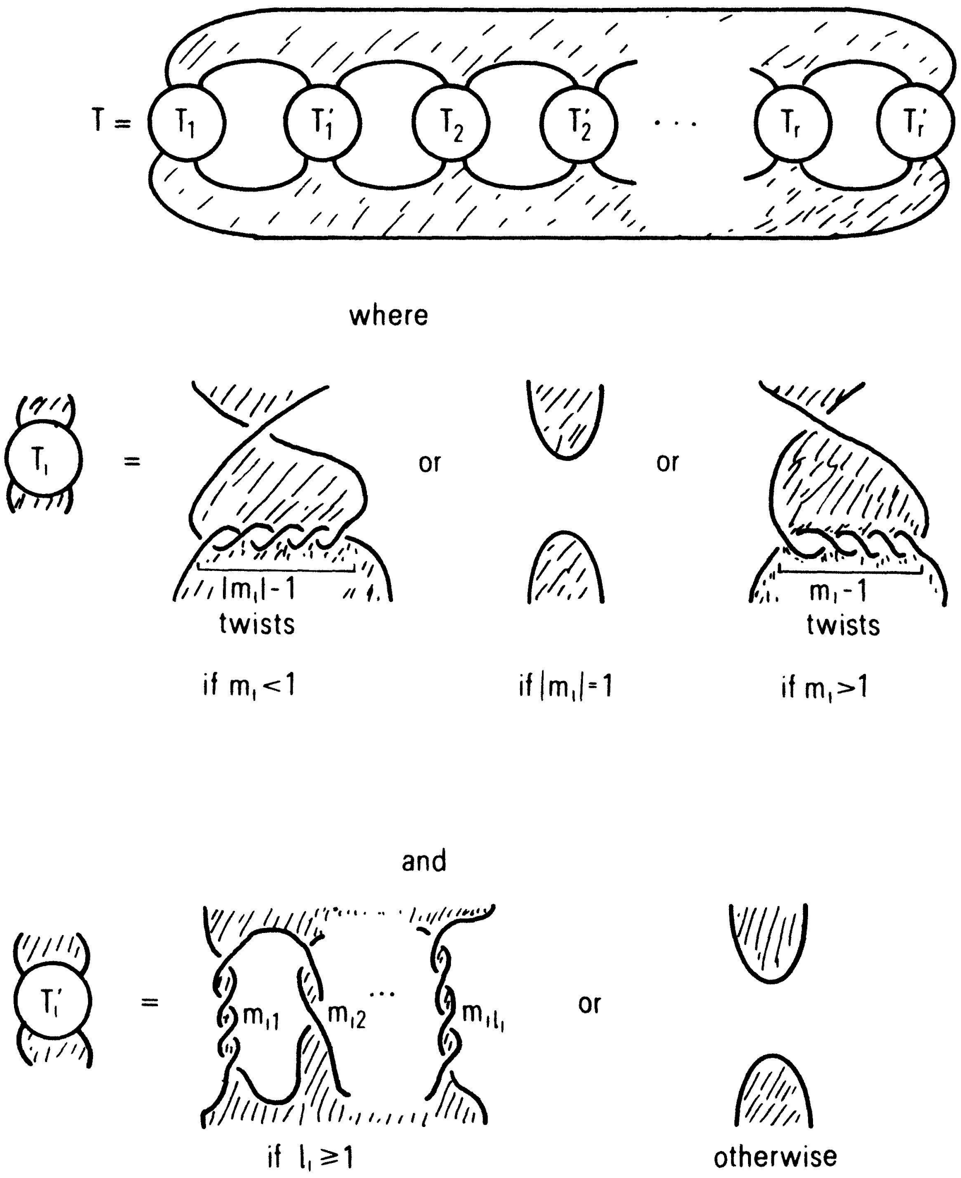


Figure 6.5

(L' defined in 6.5), then pretend R is of type II and apply case 2A or 2B. Otherwise L is fibred if and only if there exists an n_j such that $|n_j| < |n_p|$ if $j \neq p$.

To construct the fibre in this latter case first cyclically permute the n_i's and/or multiply all the n_i's by -1 so that

(1) $n_i > 0$ i odd

(2) $n_i < 0$ i even

(3) $|n_k| < |n_j|$ $j \neq k$ (k as in 6.1)

Now construct the fibre as in figure 6.7. If we multiply the n_i's by -1, then the fibre is the mirror image of the one constructed in figure 6.7.

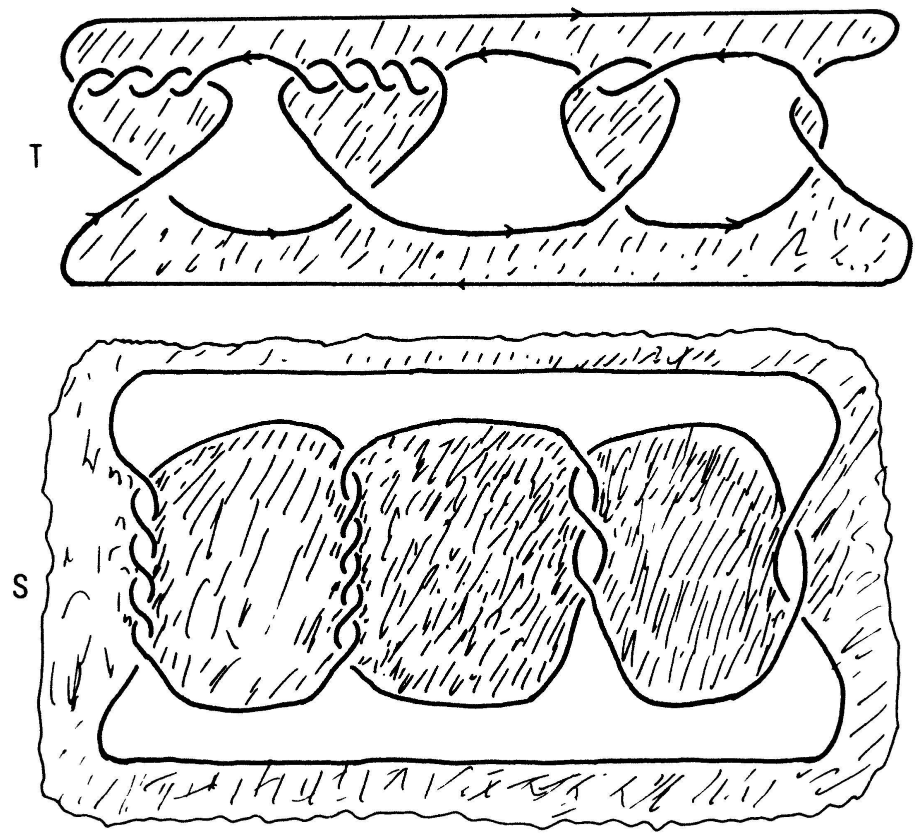
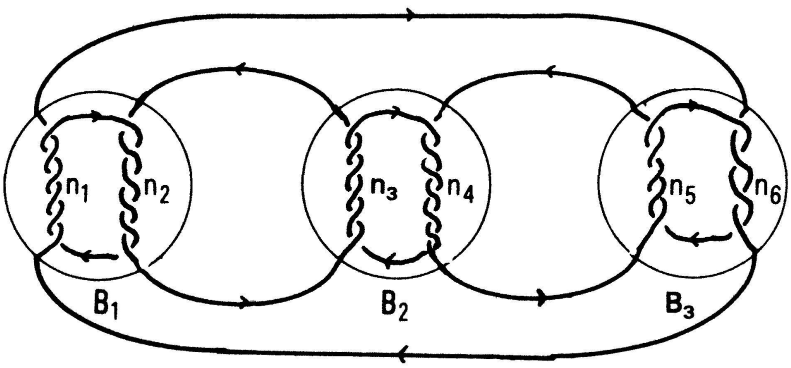


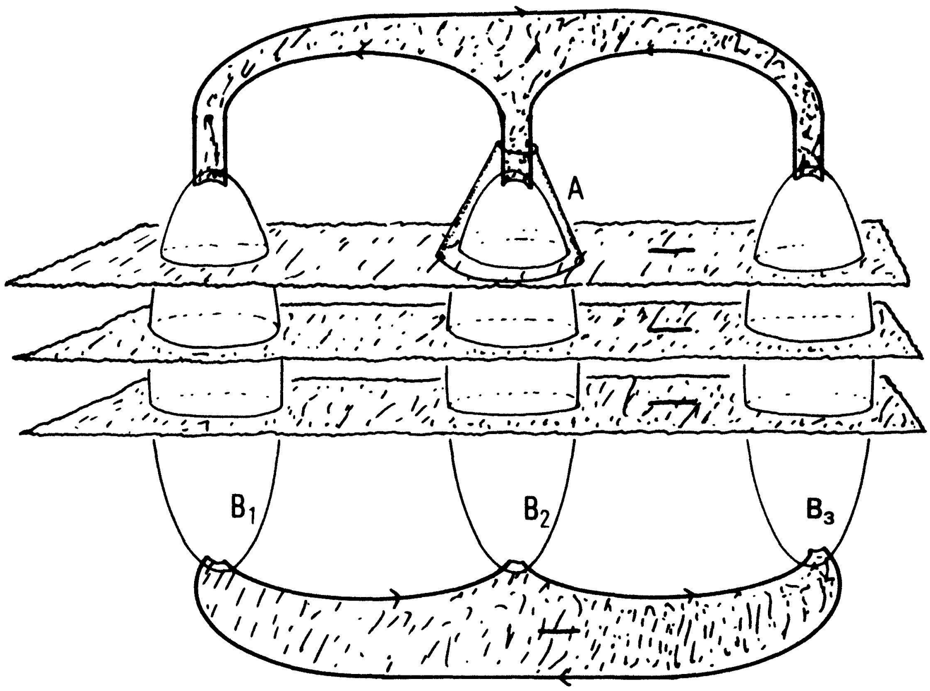
Figure 6.6



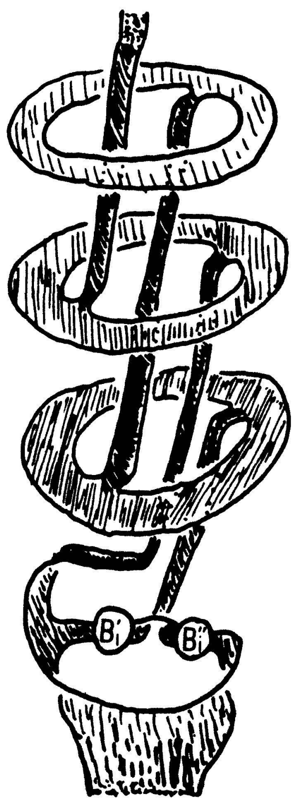
$L=(n_1,\cdots,n_6)$, here $k=6$. The B_i's are 3-cells which contain the indicated sub-tangles of L

a)

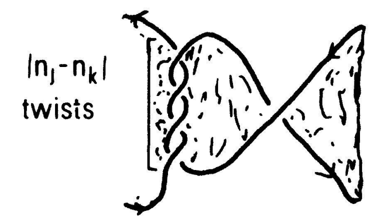
Figure 6.7



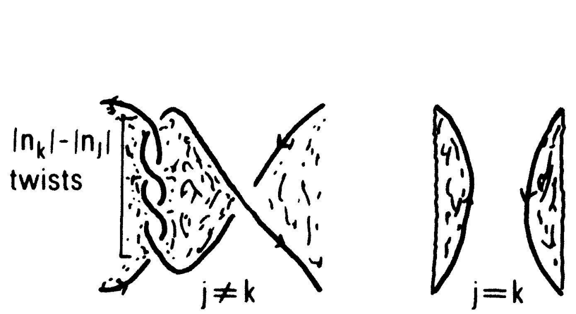
b) Construction of $S \cap (S^3 - (\mathring{B}_1 \cup \mathring{B}_2 \cup \mathring{B}_3))$



c) Construction of $S \cap B_i$



d) Construction of $S \cap B_i'$ here $j = 2i - 1$



e) Construction of $S \cap B_i''$ here $j = 2i$

Figure 6.7 continued

Remark 6.8. I was surprised that these last type III links fibred since their minimal genus Seifert surfaces look very much like the minimal genus Seifert surfaces of Kinoshita–Terasaka knots, which have trivial Alexander polynomials. See §5 of [G7] and [KT].

HISTORY. Crowell and Trotter [CT] determined which "classical" pretzel knots (i.e., k and each n_j is odd) of 3 strings (i.e., $k=3$) fibred. Parris [P] showed exactly which classical pretzel knots fibred. Such knots bound Type I surfaces. Goodman and Tavares [GT] and Kanenobu [K] independently showed exactly which Type I surfaces were fibres. Their methods were algebraic and relied on Stallings' work [S1].

Proof. Recall program 3.5.

Proof of Case 1.

n_i odd all i. L had been normalized so that $\{1, -1\} \subset \{n_1, \dots, n_k\}$, hence by §3 of [G7] R is a minimal genus surface for L. If some $n_i = +1$, say $i = 1$, then R desums into surfaces $R_2, \dots, R_k$ where R_j is a $n_j + 1$ twisted band. By Theorem 3.1, L is a fibred link with fibre R if and only if for every j, $n_j + 1 = \pm 2$. If some $n_i = -1$, the result follows similarly. If $|n_i| \geq 3$ all i, then geometric arguments show that there do not exist any C-product discs for the sutured manifold obtained from R, hence R does not have a product decomposition and L does not fibre.

n_i even all i. By [G7] R is a minimal genus surface for L unless $L = \pm(2, -2, 2, -2, \dots, 2, -2)$ in which case its minimal genus surface is a union of two $k/2$ punctured spheres (figure 3.3 of [G7]). If L is a fibred link its fibre, hence its minimal genus surface, is connected.

Let (N, δ) be the sutured manifold obtained from R. Apply C-product decompositions to decompose maximal regions of (N, δ) corresponding to maximal sets of consecutive numbers $\pm(2, -2, \dots, 2, -2)$ or $\pm(2, -2, \dots, -2, 2)$ in the presentation $(n_1, \dots, n_k)$ of L, as in figure 6.8 to obtain the sutured manifold (N', δ').

If L satisfies (B) or (C) of Case 1, then (N', δ') will be one of the sutured manifolds of figure 6.9, so has a C-product decomposition, hence L fibres with fibre R.

If L satisfies neither (B) nor (C), then by arguing geometrically we conclude that there do not exist any nontrivial C-product decompositions for (N', δ'). By Lemma 2.2 and Theorem 1.9 (M', γ') (resp. (M, γ)) the complementary sutured manifold to (N', δ') (resp. (N, δ)) is not a product so L does not fibre. □

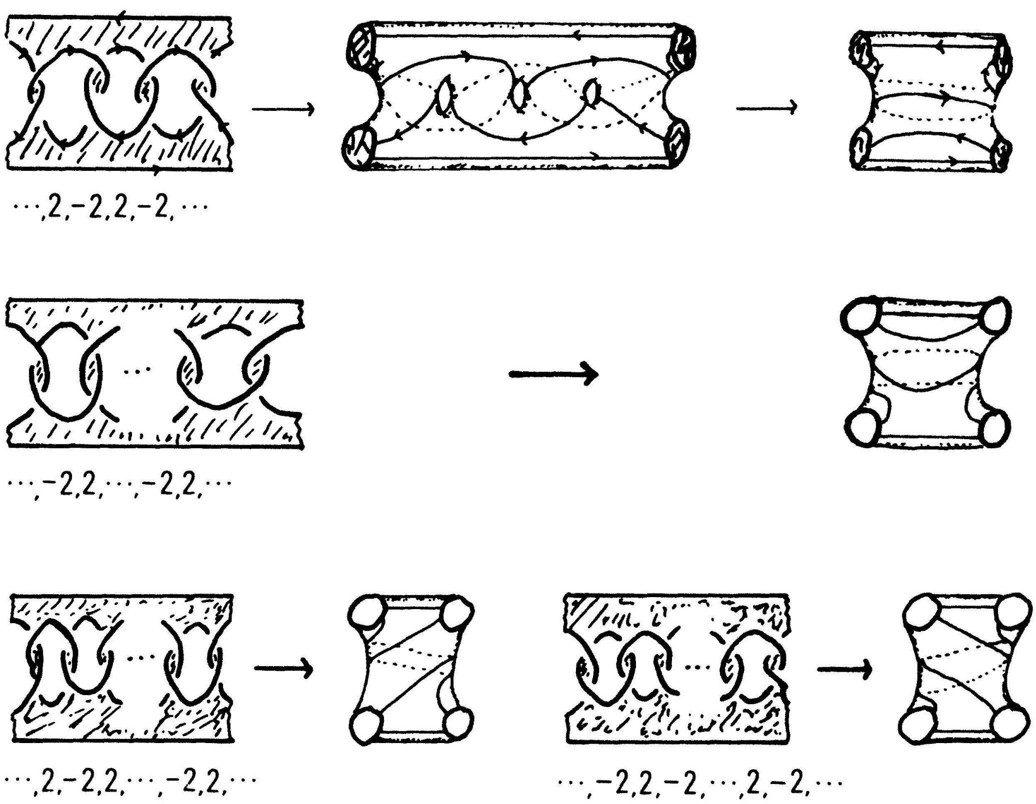


Figure 6.8

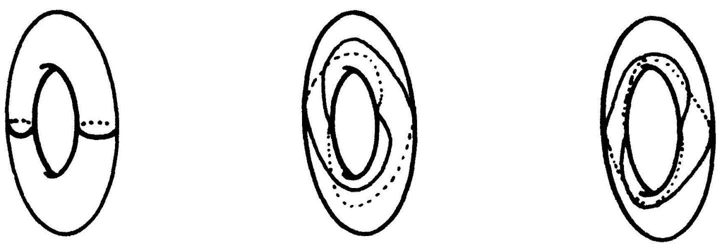


Figure 6.9

Proof of Case 2.

CASE 2A. By [G7] R is a minimal genus surface for L. R is a Murasugi sum of

(1) surfaces R_{ij} (which are m_{ij} twisted bands),

(2) a surface R' $\left(\text{which is a} -\sum_{j=1}^{r} \frac{m_j}{|m_j|} \text{ twisted band}\right)$,

(3) and $\sum_{j=1}^{r} (m_j - 1)$ Hopf bands.

The result now follows from Theorem 3.1.

CASE 2B. L bounds the Seifert surface T (figure 6.5) which is a Murasugi sum of Hopf bands (max. $(0, |m_i| - 2)$ arise from each T_i) and the type I spanning surface R' to the pretzel link L' (recall 6.5). By [G7] T is minimal genus, hence L fibres if and only if L fibres with fibre T if and only if (by Theorem 3.1 and Case 1) L' fibres with fibre R'.

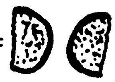
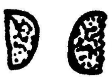
CASE 2C. There is nothing to prove here. We remark that T is not a minimal genus surface for L. □

Proof of Case 3. If both

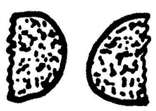
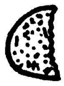
$$\sum_{j=1}^{k} \frac{n_j}{|n_j|} = 0 \quad \text{and} \quad L' = \pm(2, -2, \ldots, 2, -2),$$

then after performing a cyclic permutation of the n_i's and/or multiplying all the n_i's by -1 we can assume that

(1) $n_i > 0$ i odd
(2) $n_i < 0$ i even
(3) $|n_k| \leq |n_j|$ $j \neq k$
(4) $n_k \leq -2$.

Construct a Seifert surface S for L as follows. If $|n_k| < |n_j|$ all $j \neq k$, then proceed as in figure 6.7. Otherwise construct S as in figure 6.7 with the following modifications. If $|n_j| = |n_k|$ $j = 2i - 1$, define $S \cap B_i' =$ . If $|n_j| = |n_k|$ $j = 2i$, define $S \cap B_i'' =$ . If $|n_j| < |n_k|$ define the corresponding $S \cap B_i'$ or $S \cap B_i''$ as before.

Recall that (M, γ), (M_p, γ_p), etc. denote the complementary sutured manifolds in S^3 to the sutured manifolds (N, δ), (N_p, δ_p), etc. Using the methods of [G7] we will show that S is a surface of minimal genus for L by showing that (M, γ) has a sutured manifold hierarchy (i.e., a sequence of sutured manifold decompositions reducing (M, γ) to a union of product sutured manifolds, where (N, δ) is the sutured manifold obtained from S).

By Lemma 5.5 of [G7], if some $S \cap B_i'$ or $S \cap B_i''$ (as in figure 6.7) equals , then one can decompose (M, γ) to any one of (M_1, γ_1) or (M_2, γ_2) where $(N_j, \delta_j) \cap (S^3 - \mathring{B}_i) = (N, \delta) \cap (S^3 - \mathring{B}_i)$ $j = 1, 2$ and $(N_j, \gamma_j) \cap B_i$ appears as in figure 6.10a if $j = 1$ or figure 6.10b if $j = 2$. If for example $i = 2$, then the first step of each sequence of decompositions involves decomposing (M, γ) along the oriented annulus A (figure 6.7b), where distinct orientations on A are used in these distinct sequences. It also follows from Lemma 5.5 of [G7] that if neither $S \cap B_i'$ nor $S \cap B_i''$ equals   then we can decompose (M, γ) to (M_1, γ_1).

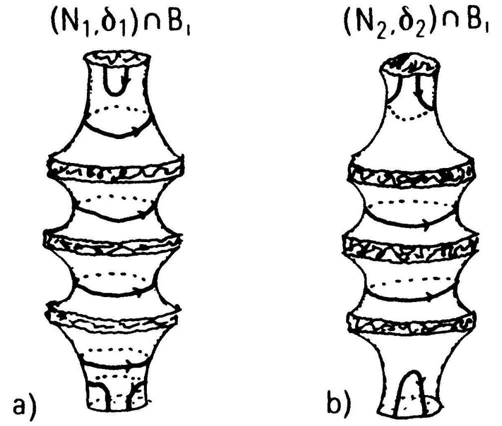


Figure 6.10

To show that (M, γ) has a sutured manifold hierarchy, decompose (M, γ) to (M', γ') where

$$(N', \delta') \cap B_i = \begin{cases} \text{fig. 6.10a } i < k/2 \\ \text{fig. 6.10b } i = k/2 \end{cases}$$

and

$$(N', \delta') \cap \left(S^3 - \bigcup_{j=1}^{k/2} \mathring{B}_j\right) = (N, \delta) \cap \left(S^3 - \bigcup_{j=1}^{k/2} \mathring{B}_j\right)$$

Now observe that $(M', \delta') = (J \times I, \partial J \times I)$ where $J = |n_k| + 1$ copies of $S^2 - k/2$(discs). Since (M', δ') is a product, (M, γ) has a sutured manifold hierarchy.

Now suppose that for some $i \neq k/2$ either $|n_{2i}| = |n_k|$ or $|n_{2i-1}| = |n_k|$. Decompose (M, γ) to (M'', δ'') where

$$(N'', \delta'') \cap B_r = \begin{cases} \text{fig. 6.10a if } r \neq i \\ \text{fig. 6.10b if } r = i \end{cases}$$

and

$$(N'', \delta'') \cap \left(S^3 - \bigcup_{j=1}^{k/2} \mathring{B}_j\right) = (N, \delta) \cap \left(S^3 - \bigcup_{j=1}^{k/2} \mathring{B}_j\right)$$

we again obtain $(M'', \delta'') = (J \times I, \partial J \times I)$. Hence we have now obtained a second sutured manifold hierarchy of (M, γ). (Compare with previous paragraph.) If for example, $i = 2$, then we could have arranged the first term of each sutured manifold hierarchy to involve decomposing along A (see figure 6.7b). We have shown that by using distinct orientations on A the decompositions (*) yield taut sutured manifolds (see 5.3 of [G5]). Since $A \cap \gamma \neq \varnothing$, A is not a product annulus so it follows by Corollary 2.7 that (M, γ) is not a product sutured manifold, hence L is not a fibred link with fibre S. Since S is a minimal genus surface for L, L cannot fibre.

$$(*)\qquad \begin{tikzcd} & (\bar{M}, \bar{\gamma}) \\ (M, \gamma) \arrow[ur, squiggly, "A"] \arrow[dr, squiggly, "-A"'] & \\ & (\hat{M}, \hat{\gamma}) \end{tikzcd}$$

We can now assume that L satisfies $|n_k| < |n_j|$ $j < k-1$ and $|n_k| \leq |n_{k-1}|$. To complete the proof of Theorem 6.7 we need to show that L fibres with fibre S if $|n_k| < |n_{k-1}|$ and does not fibre if $|n_k| = |n_{k-1}|$. By using only C-product decompositions, decompose (N, δ) to (N', δ') so that

$$(N', \delta') \cap B_i = \text{figure 6.10a if } i < k/2$$

and

$$(N', \delta') \cap \left(S^3 - \bigcup_{i=1}^{k/2-1} \mathring{B}_i\right) = (N, \delta) \cap \left(S^3 - \bigcup_{i=1}^{k/2-1} \mathring{B}_i\right)$$

By performing $|n_k|\,(k/2 - 2)$ C-product decompositions in $S^3 - \mathring{B}_{k/2}$ to (N', δ') we obtain the sutured manifold (N'', δ'') of figure 6.11 (here $k = 6$, $n_6 = -4$) which is isotopic to the sutured manifold of figure 6.12. If $|n_k| = |n_{k-1}|$ then $R(\gamma'')$ has 4 components, hence is not a product. Since S is a minimal genus surface, L does not fibre. If $n_k \neq n_{k-1}$, then (N'', δ'') has a C-product decomposition, hence L fibres with fibre S. $\square$

§7. Fibred links in general 3-manifolds

In this chapter we continue to analyze the structure of knots whose fibres decompose as nontrivial Murasugi sums. Our main result is Theorem 7.7. We will

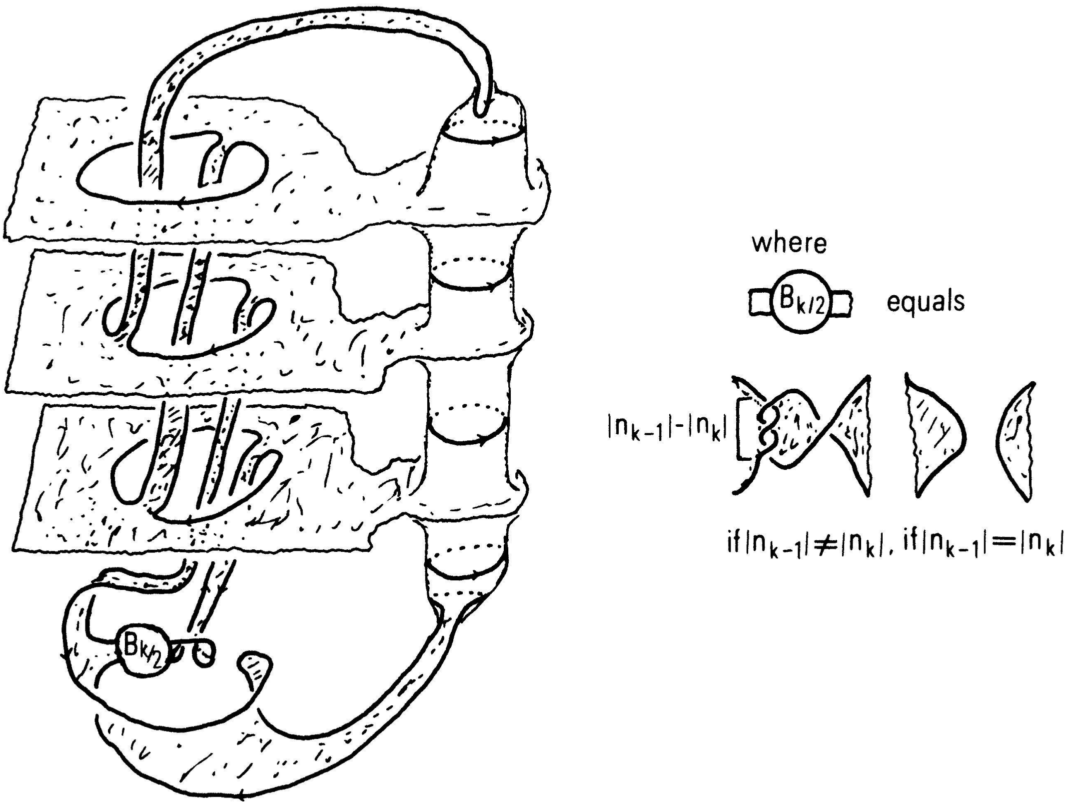


Figure 6.11

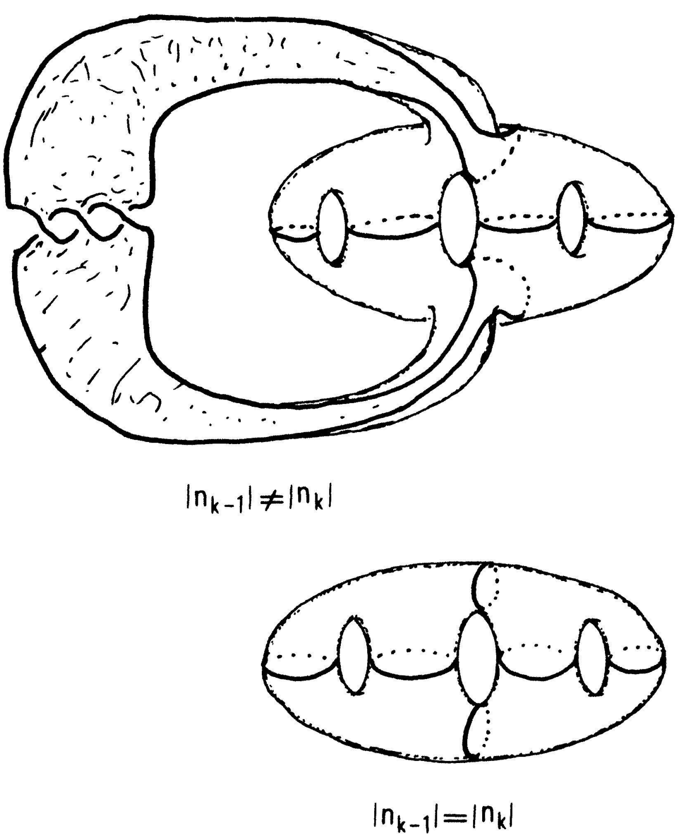


Figure 6.12

work in 3-manifolds which are not necessarily S^3. For convenience of the reader we will recall the appropriate definitions and results in this more general setting.

DEFINITION 7.1. $L \subset H$ is a *fibred link* with fibre R in the closed oriented 3-manifold H if $H - \mathring{N}(L)$ fibres over S^1 with fibre R and $\partial R \cap$ (each meridian of $L) = 1$ point.

Remark 7.2. Myers and Gonzales-Acuna (see [R]) showed that every closed oriented 3-manifold M possesses a fibred knot. Harer [H] generalized this to $\gamma \in [\pi_1(M), \pi_1(M)]$ if and only if there exists a fibred knot k homotopic to γ.

$L \subset H^3$ is a fibred link with fibre R if and only if $\partial R \cap$ (each meridian of $L) = 1$ point and $((H - \mathring{N}(L)) - \mathring{N}(R), \partial N(L) - \mathring{N}(R))$ is a product sutured manifold. It follows that an appropriately stated version of Theorem 1.9 holds for general 3-manifolds and the methods of §2 and §3 can be used for deciding whether or not a link fibres.

We show how to generalize the notion of Murasugi sum and hence Theorem 3.1 and Corollary 3.2 to general 3-manifolds.

DEFINITION 7.3. Let $R_i \subset H_i$ $i = 1, 2$ be compact oriented surfaces in the closed oriented 3-manifolds H_i. Then $R \subset H_1 \# H_2 = H$ is a Murasugi sum of R_1 and R_2 if

$$H = (H_1 - \mathring{B}_1) \bigcup_{S^2} (H_2 - \mathring{B}_2), \qquad B_i = 3 \text{ cell}, \qquad S^2 = \partial B_1 = \partial B_2$$

$$S^2 \cap R_i = 2n \text{ gon} \qquad (H_i - \mathring{B}_i) \cap R = R_i$$

PICTURE. For a view of this situation consult figure 3.1a after relabelling as follows. Replace B_1 by H_1 and B_2 by H_2.

The following result follows exactly as in the old and new proofs of Theorem 3.1.

THEOREM 7.4. *Let $R \subset H = H_1 \# H_2$ be a Murasugi sum of $R_1 \subset H_1$ and $R_2 \subset H_2$ and let $L = \partial R$, $L_i = \partial R_i$ $i = 1, 2$. $L \subset H$ is a fibred link with fibre R if and only if for $i = 1, 2$ $L_i \subset H_i$ is a fibred link with fibre R_i.* □

The following very useful result proven (e.g., see [G3] or [M]) for links in S^3 has been restated for closed oriented 3-manifolds. The proof follows exactly as before. (Recall Remark 1.12.)

LEMMA 7.5. *Let L be a fibred link in $H = H_1 \# H_2$ with fibre R which is a Murasugi sum of $R_1 \subset H_1$ and $R_2 \subset H_2$ where $\partial R_i = L_i$. If $f: R_i \to R_i$ represents the monodromy of L_i and the + side of the summing disc points into (resp. out of) the component of $H - S^2$ containing R_1, then the monodromy of L is represented by $f: R \to R$ where $f = f_2 \circ f_1$ (resp. $f_1 \circ f_2$).* $\square$

Remark 7.6. Abusing notation slightly, when we say $f: R \to R$ is periodic, reducible, or pseudo Anosov we mean the f is isotopic to such a map.

Recall that $f: R \to R$ is reducible if there exists a set $J \subset R$ of pairwise disjoint essential (i.e., neither bounds a disc nor is boundary parallel) simple closed curves and arcs such that $f(J)$ is isotopic to J.

Let H be a 3-manifold with nonempty boundary which fibres over S^1 with monodromy $f: R \to R$, $R \neq D^2$, then Thurston [T1], [T2] proves that f is pseudo Anosov if and only if every incompressible torus and annulus is boundary parallel if and only if $\mathring{H}$ has a complete hyperbolic structure of finite volume.

THEOREM 7.7. *Let L_1 be a fibred link in S^3 with fibre R_1, let L_2 be a fibred link in the compact oriented 3-manifold H with fibre R_2, let $R \subset H = S^3 \# H$ be a Murasugi sum of R_1 and R_2 summed along a square (i.e., R is a plumbing of R_1 and R_2) and let $L = \partial R$.*

One of the following must hold.

(A) *Some L_i is a 2-bridge link in S^3 (so if $i = 2$ $H = S^3$).*

(B) *The monodromy of L is pseudo Anosov.*

(C) *The monodromy of L is reducible where one invariant set of reducing curves is either contained in $R_2 - R_1$ or $R_1 - R_2$.*

Remark. By considering the trefoil knot (resp. figure 8 knot), whose fibre is a Murasugi sum of Hopf bands, one sees that possibility (A) (resp. (B)) can occur. If L is a connected sum of two hyperbolic non two bridge knots, then (C) occurs.

Proof. By Thurston [T1] there exists a $g: R \to R$ such that g is isotopic to a representative of the monodromy of L and either (B) holds, or L is the unknot in S^3 and (A) holds, or there exists an essential set of g invariant simple closed curves or arcs. (If g was periodic, then $H - \mathring{N}(L)$ is Seifert fibred and the intersection of an essential vertical annulus and R would be a set of reducing curves.) Therefore if neither (B) nor (A) holds there exists either a torus T or an annulus A which is incompressible and non boundary parallel in $H - \mathring{N}(L)$.

Let S be the 2-sphere along which R_1 and R_2 were summed.

CASE 1. There exists a non boundary parallel incompressible torus T or annulus A such that either $T \cap S = \varnothing$ or $A \cap S = \varnothing$.

Proof. Assume that we are dealing with an annulus, the other case is similar and easier.

Since $A \cap S = \emptyset$ each component of ∂A must be a meridian of $N(L)$ and $A \cap R_1 = \emptyset$ (or $A \cap R_2 = \emptyset$) hence we can isotope A so that $A \cap S = \emptyset$, ∂A is transverse to the fibres of the fibration $\mathcal{F}$, A is transverse to R, and A is transverse to $\mathcal{F}$ except possibly at a finite set of points, where at a point of tangency A looks like either a saddle or a hilltop with respect to $\mathcal{F}$. Now apply the isotopy theorem of [Ro] to isotope A to A' rel ∂A so that A' is transverse to $\mathcal{F}$. By considering Roussarie's proof we observe that the isotopy could have been performed so that $A' \cap R \subset A \cap R$, so in particular $A' \cap R_1 = \emptyset$.

Now g is the return map of a vector field transverse to $\mathcal{F}$. Since one can homotope this vector field through non singular vector fields to one keeping A invariant it follows that $g(A \cap R)$ is isotopic to $A \cap R$, hence conclusion (C) holds. □

Now suppose that each essential torus or annulus intersects S nontrivially. We now consider the case where there exists a non boundary parallel incompressible torus T. Assume that T has been chosen to minimize the number of components of intersection with S. By the usual disc swapping argument we can assume that no component of $T \cap S$ is a circle bounding a disc in $S - \mathring{N}(L)$.

CASE 2. Some component of $T - \mathring{N}(S)$ is a disc D.

No component of $T - \mathring{N}(S)$ is a disc D with ∂D boundary parallel in $S - \mathring{N}(L)$ for this would imply that there exists a sphere in H intersecting L in one point, contradicting the fact that L is homologically trivial.

Proof. We will assume that D is contained in the H factor of $S^3 \# H$ for the proof in the other case is similar. For $j = 1, 2$ $F_j = E_j \cup D$ is a 2-sphere in H which intersects L in 2-points, where E_1, E_2 are the discs in S which ∂D bounds. If some F_j does not separate H, apply Case 1 to F_j to conclude that conclusion (C) holds.

Now suppose that each F_j separates H. Let C_j be the component of $H - \mathring{N}(F_j)$ which does not contain S. First observe that $F_j - \mathring{N}(L)$ is an annulus A_j which can be extended to a torus $T_j = \partial(C_j - \mathring{N}(L))$. By the isotopy result of [Ro] we can assume that A_j is transverse to $\mathcal{F}$. Since $\partial N(L)$ is transverse to $\mathcal{F}$ we conclude that T_j is transverse to $\mathcal{F}$.

Let $\mathcal{G}$ be the foliation on $C_j - \mathring{N}(L)$ obtained by restricting $\mathcal{F}$. Each leaf of $\mathcal{G}$ is compact so by the Reeb stability Theorem [Re] $C_j - \mathring{N}(L)$ fibres over S^1 with fibre a leaf of $\mathcal{G}$. We conclude that either $C_j - \mathring{N}(L)$ is a solid torus with each leaf of $\mathcal{G}$ a disc or T_j is incompressible. The latter cannot hold else we would have contradicted the minimality hypothesis on T.

Now C_j is obtained by attaching a 2-cell (a meridianal disc of L) to a solid torus. Since each meridian intersects a fibre of $\mathcal{G}$ exactly once we conclude that C_j is a 3-cell and $L \cap C_j$ is a boundary parallel arc. It follows that if each $C_j - \mathring{N}(L)$ is a solid torus, then L_j is a rational link since, except for 2 "bridges", it can be made to lie in a plane, hence Conclusion (A) holds. □

CASE 3. Each component of $T - \mathring{N}(S)$ is an annulus. By hypothesis one side of S bounds a 3-cell C. A component E of $T \cap C$ is an annulus which can be extended to a torus $T' = E \cup E'$ where E' is the annulus which ∂J bounds in $S - L$. (If ∂J did not bound an annulus, then the minimality hypothesis on $T \cap S$ implies that T would have been boundary parallel.) T' does not bound a solid torus else one could isotope T to remove intersections with S. T' could not be incompressible for that would contradict the minimality of T. □

We now assume that there exist no essential tori. As before, let C be the closure of a 3-cell component of $H - S$.

CASE 4A. $A \cap \partial N(L)$ are meridians.

Assume that A has been chosen to have fewest number of intersections with S *and* $\partial A \subset H - C$. If a component of $A \cap S$ bounds a disc in A, then Case 4A would follow by arguing as in Case 2. If there exists an annular component E of $C \cap A$ with $\partial E \subset S$, then arguing as in Case 3 eliminates that possibility.

CASE 4B. $A \cap \partial N(L)$ contains a non meridian.

No arc of $S \cap A$ is boundary parallel in A, else either L_1 is a 2-bridge link or L_1 is a non trivial connected sum and (C) holds. If some component of $S \cap A$ was a circle α, then α bounds a disc in A hence Case 4B would follow by arguing as in Case 2.

We now assume that $S \cap A$ is a union of arcs. A component D of $C \cap A$ is a square with 2 edges on L and 2 edges in S. If D intersected a unique component of $L \cap C$, A is boundary compressible. Boundary compressing A yields a properly embedded disc $E \subset H - \mathring{N}(L)$ whose boundary is inessential in $\partial N(L)$. Since $H - \mathring{N}(L)$ is irreducible (it fibres over S^1) it follows that E, hence A, is boundary parallel.

Now assume that ∂D intersects distinct components of $L \cap C$. $\partial N(D) - \mathring{N}(S)$ is an annulus D' which can be extended to a torus $T = D' \cup D''$ where D'' is the annulus in $N(S) - L$ which $\partial D'$ bounds (figure 7.1). Since we assumed that no essential torus exists and T is contained in a 3-cell C we conclude that $C \cap L$ are

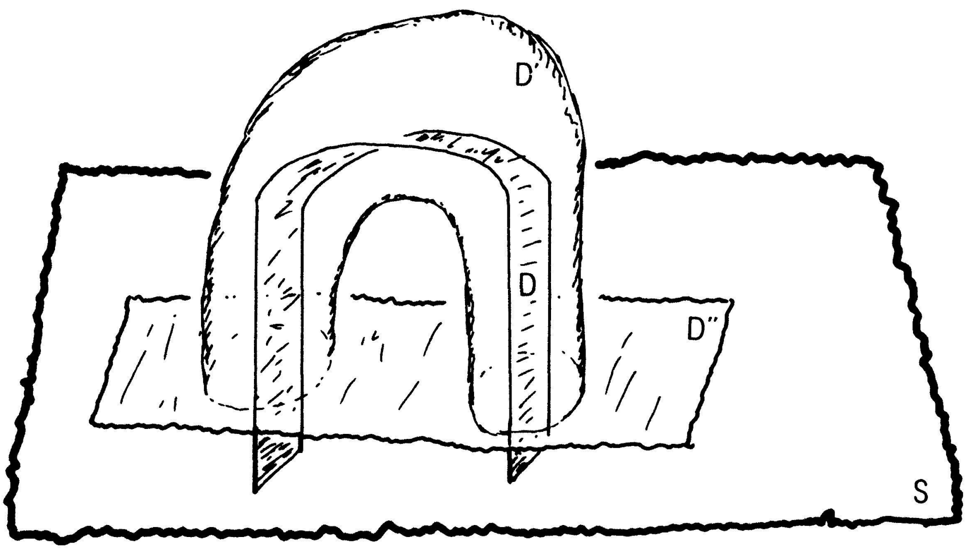


Figure 7.1

2 unknotted arcs. It follows that L_1 is a rational (or 2-bridge) link, hence conclusion (A) holds. □

The following result has been also proven (although not stated in this generality) by Soma [So].

COROLLARY 7.8. *Let H be a closed oriented 3-manifold. Then $\beta \in [\pi_1(H), \pi_1(H)]$ if and only if there exists a fibred knot $k \subset H$ in the homotopy class of β with pseudo Anosov monodromy, i.e., k is a hyperbolic fibred knot.*

Proof of $\Leftarrow$, $k = \partial R$ hence is a product of commutators. □

Idea of Proof of $\Rightarrow$. Given $\beta \in [\pi_1(H), \pi_1(H)]$ apply Harer's result to find a reasonable fibred knot k_1 in H with fibre R_1 homotopic to β. Find a sufficiently complicated (i.e., non 2-bridge) hyperbolic fibred knot k_2 with fibre $R_2 \subset S^3$. Let R be a sufficiently complicated Murasugi sum of R_1 and R_2. Finally $k = \partial R$ is the desired knot.

Proof of $\Rightarrow$. If $H = S^3$, then the figure eight knot satisfies the conclusion of the corollary. Now assume $H \neq S^3$. By Harer [H] given any $\beta \in [\pi_1(H), \pi_1(H)]$ one can find a fibred knot k_1 representing β. Let R_1 be the fibre of k with monodromy $f: R_1 \to R_1$. By [T1] f is isotopic to g where there exists a set $J \subset R$, of g invariant simple closed curves and arcs such that $R_1 - J = X \cup Y$ where $g|X$ is pseudo Anosov and $g|Y$ is periodic. By first, if necessary, doing a connected sum of k_1 and a hyperbolic fibred knot (e.g., figure eight) we can assume that $X \neq \varnothing$.

By [T1] every component of X has negative Euler characteristic and each set of pairwise disjoint arcs or simple closed curves invariant (up to isotopy) by g can be isotoped off of X.

Let λ_1 be a non boundary parallel properly embedded arc in R_1 such that $R_1 - (\mathring{X} \cup \mathring{N}(\lambda_1))$ is a union of discs and each component of $\lambda_1 \cap X$ is an essential arc in X.

Let $k_2 \subset S^3$ be a fibred knot with fibre R_2 such that k_2 is not a 2-bridge knot and the monodromy of k_2 is pseudo Anosov. For example, the pretzel knot $(5, -5, 5, -4)$ fibres by §6, is not 2-bridge by [B] and has no essential tori or annuli in its complement by [O]. Let $\lambda_2 \subset R_2$ be a non boundary parallel properly embedded arc.

Let $R \subset H = S^3 \# H$ be the surface obtained by Murasugi summing R_1 and R_2 along the squares $N(\lambda_1) \subset R_1$ and $N(\lambda_2) \subset R_2$. By Theorem 7.4 $k = \partial R$ is a fibred knot in H with fibre R. k is clearly homotopic to β. By construction and Theorem 7.7 it follows that the monodromy of k preserves no set of essential arcs or circles.

It now follows by [T1] that the monodromy f of k is pseudo Anosov. (If f was periodic, then $H - \mathring{N}(k)$ is Seifert fibred, and the intersection of an essential vertical annulus and R would be a set of reducing curves.) □

§8. A conjecture

CONJECTURE 8.1. If k is a non trivial atoroidal fibred *knot* in S^3 with fibre R, then R is a non trivial Murasugi sum.

EVIDENCE 8.2. This dubious sounding conjecture is true for alternating knots, torus knots, knots of ≤ 10 crossings and pretzel knots. (The conjecture clearly holds for fibred pretzel knots bounding type I, type II surfaces. It is a good exercise to show that a fibre constructed as in figure 6.7 is a non trivial Murasugi sum.)

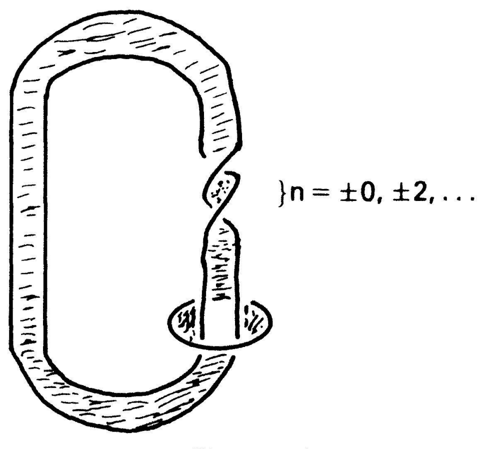


Figure 8.1

Remark 8.3. The conjecture is false for links. For example a surface T in figure 8.1 with $n \neq 0$ is not a Murasugi sum. A link L bounding a thrice punctured sphere S is not prime if S is a Murasugi sum. By [O], ∂T is prime.

The conjecture is false for knots if one drops the toroidal hypothesis.

REFERENCES

[B] F. BONAHON, *Involutions et Fibrès de Seifert dans les Variétés de Dimensions* 3, thesis de 3-ieme cycle, Orsay, France, 1979.

[C] R. CROWELL, *Genus of Alternating Link Types. Ann. of Math.* (2) *69* (1959), 258–275.

[CT] R. CROWELL and H. TROTTER, *A Class of Pretzel Knots, Duke Math. J.*, *30* (1963), 373–377.

[E] J. EISNER, *Knots with infinitely many minimal Spanning Surfaces, Trans. AMS. 229* (1977), 329–349.

[G1] D. GABAI, *Foliations and Genera of Links,* Thesis, Princeton University, 1980.

[G2] ——, *The Murasugi Sum is a Natural Geometric Operation,* Low Dimensional Topology, *Cont. Math.* AMS *20* (1983), 131–144.

[G3] ——, *The Murasugi Sum is a Natural Geometric Operation II, Cont. Math.* AMS *44* (1985), 93–100.

[G4] ——, *Foliations and Genera of Links, Topology 23* (1984), 381–394.

[G5] ——, *Foliations and the Topology of* 3*-Manifolds, J. Diff. Geom. 18* (1983), 445–503.

[G6] ——, *Genera of the Alternating Links,* to appear in the *Duke Math. J.*

[G7] ——, *Genera of the Arborescent Links, Memoirs of the AMS,* No. *339* (1986).

[GT] S. GOODMAN and G. TAVARES, *Pretzel-Fibred Links,* Preprint.

[H] J. HARER, *Representing Elements of $\pi_1(M^3)$ by Fibred Knots, Math. Proc. Cambridge Phil. Soc.*, *92* (1982), 133–138.

[K] T. KANENABU, *The Augmentation Subgroup of a Pretzel Link, Math. Seminar Notes,* Kobe University, 7 (1979), 363–384.

[KT] S. KINOSHITA and TERASAKA, *On Unions of Knots, Osaka Math. J.* 1957, 131–153.

[M] H. R. MORTON, *Fibred Knots with a given Alexander Polynomial,* Preprint.

[M1] K. MURASUGI, *On the Genus of the Alternating Knot I, II J. Math. Soc. of Japan, 10* (1958), 94–105, 235–248.

[M2] ——, *On a Certain Subgroup of an Alternating Link, Amer. J. Math.*, *85* (1963), 544–550.

[N] L. P. NEUWIRTH, *Knot Groups, Annals of Math. Studies,* 56, Princeton University Press (1965).

[O] U. OERTEL, *Aborescent Links,* Thesis, University of California, Los Angeles, 1980.

[P] R. PARRIS, *Pretzel Knots,* Thesis, Princeton University, 1978.

[R] D. ROLFSEN, *Knots and Links,* Publish or Perish 1977.

[Re] G. H. REEB, *Sur Certaines Proprietes Topologiques des Variétés Feuilletées,* Actualités Sci. Indust., No. 1183, Hermann, Paris (1952), 91–158.

[Ro] R. ROUSSARIE, *Plongements dans les Variétés Feuilletées et Classification de Feuilletages sans Holonomie,* IHES Publ. Math. No. *443* (1973) 101–142.

[So] T. SOMA, *Hyperbolic Fibred Links and Fibre Concordences, Math. Proc. Cambridge Phil. Soc.*, *96* (1984), 283–294.

[S1] J. STALLINGS, *On Fibering Certain* 3*-Manifolds, Topology of* 3*-Manifolds,* Prentice Hall (1962), 95–100.

[S2] ——, *Constructions of Fibred Knots and Links, Proc. Symp. Pure Math.* AMS *27* (1975), 315–319.

[T1] W. P. THURSTON, *On the Geometry and Dynamics of Diffeomorphisms of Surfaces I,* preprint.

[T2] ——, *Three dimensional Manifolds, Kleinian Groups, and Hyperbolic Geometry, Proc. Symp. Pure Math.* AMS *39* (1983), 87–114.

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