Reduction theory using semistability, II.

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Reduction theory using semistability, II

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In this paper we extend the result of [G] to include the case of an arithmetic subgroup Γ of a semisimple algebraic group G. We represent the symmetric space $X = G/K$ (where K = a maximal compact subgroup of G) as a certain space of inner products H on the Lie algebra $\mathfrak{g}$ of G, namely, those inner products that come from the Killing form via a Cartan involution.

If we assume that Γ is the stabilizer of a lattice $\mathfrak{g}_{\mathbb{Z}} \subset \mathfrak{g}$, then we may use the notions of semistability and of Harder–Narasimhan canonical filtration to study the action of Γ on X. Our main result is the explicit construction of a closed submanifold (with boundary) in X (of codimension zero) which is contractible, compact modulo Γ, and has boundary Γ-homotopy equivalent to the Tits building of G. Thus this subspace of X provides an alternate route toward the theorems of Borel–Serre [BS] about the homological properties of Γ.

The proof in [G] of contractibility for the intersections of the neighborhoods of the cusps [G, 7.18(c), 7.7] had the side effect of proving something already known, namely contractibility of the space X. In section 4 we use a simpler technique that makes use of the contractiblity of X.

We make essential use of the assumption that G is semisimple; it would have been nice to avoid it altogether, the way Borel–Serre do.

The idea of formulating the results of reduction theory in terms of functions which measure the distance to the cusps, here and in [G], is due to Harder [Ha1, Ha2]. Indeed, his function $\pi(P, \mathcal{R})$ of [Ha1, 2.4, p. 54] is presumably closely related to our $\operatorname{vol}(L \cap W^1)$, where W^1 is the nilpotent radical of a parabolic subalgebra of $\mathfrak{g}$. In a sense, we answer the question in the last sentence of [Ha1], hopefully by a method as direct and transparent as the one anticipated there by Harder, thereby completing the program he laid out.

Motivation for doing reduction theory via semistability of Lie algebra bundles comes from the paper of Atiyah and Bott [AB], where the canonical filtration for a principal G-bundle on a Riemann surface is introduced.

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1. Inner products on a Lie algebra

Let G be a connected semisimple algebraic group defined over $\mathbb{Q}$. Since we will be dealing with the adjoint representation of G, we will replace G by $G' = G/Z(G)$, where $Z =$ "center of." Since G is semisimple, $Z(G)$ is finite, and G' is also a semisimple linear algebraic group; its adjoint representation is now faithful, so $Z(G') = 1$ [W, 3.50]. Any arithmetic subgroup $\Gamma \subset G(\mathbb{R})$ has a congruence subgroup $\Gamma' \subset \Gamma$ of finite index which maps injectively to $G'(\mathbb{R})$; its image in G' is an arithmetic subgroup of G' [B, 7.13(2)]. Since all theorems we intend to prove about Γ are (equivalent to a theorem) of the form "Γ has a subgroup Γ' of finite index such that Γ' satisfies . . . ," we may as well replace Γ by Γ'.

Let $\mathfrak{g}$ be the Lie algebra of the group of real points $G(\mathbb{R})$ of G; it is a semisimple Lie algebra, so the Cartan–Killing form B (induced from the canonical bilinear form $(X, Y) \mapsto \operatorname{tr}(XY)$ on $\operatorname{End}_{\mathbb{R}}(\mathfrak{g})$ via $ad : \mathfrak{g} \to \operatorname{End}_{\mathbb{R}}(\mathfrak{g})$) is nondegenerate. The form B is characteristic (i.e. for all $\theta \in \operatorname{Aut}_{\text{Lie alg}}(\mathfrak{g})$ we have $B(X, Y) = B(\theta X, \theta Y)$) and is invariant (i.e. $B([X, Y], Z) = B(X, [Y, Z])$).

Let H be an inner product on $\mathfrak{g}$, i.e. a positive definite real symmetric bilinear form. If we think of a bilinear form also as a map $\mathfrak{g} \to \mathfrak{g}^*$ (or as a matrix formed with respect to a basis of $\mathfrak{g}$ and its dual basis of $\mathfrak{g}^*$), then an expression like $H^{-1}B$ makes sense as an $\mathbb{R}$-linear endomorphism of $\mathfrak{g}$ (and is independent of the choice of basis). We say that H is *compatible* (with the Lie bracket) if $-H^{-1}B$ is a Lie algebra automorphism of $\mathfrak{g}$. One consequence of compatibility is that $\theta := -H^{-1}B$ is a Cartan involution of $\mathfrak{g}$, as we see now. The fact that $\theta^2 = 1$ follows from ${}^t\theta B \theta = B$ (B is characteristic) and ${}^t\theta B = -BH^{-1}B = B\theta$. Letting $\mathfrak{k} = (+1)$-eigenspace of θ and $\mathfrak{p} = (-1)$-eigenspace, we see from $B\theta = -H < 0$ that B is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$. It follows from ${}^t\theta B \theta = B$ that $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal with respect to B, and thus that θ is a Cartan involution.

Another consequence of compatibility is that H is compatible with B in the sense of [G, Definition 7.3]. For the equation $\theta^2 = 1$ may be rewritten as ${}^tBH^{-1}B = H$, which says that B is an isometry $\mathfrak{g} \to \mathfrak{g}^*$ with respect to the inner products H on $\mathfrak{g}$ and H^{-1} on $\mathfrak{g}^*$.

Now let X be the space of all compatible inner products on $\mathfrak{g}$; it is a closed subspace of the space of all inner products on $\mathfrak{g}$. Suppose F is any Lie algebra automorphism of $\mathfrak{g}$: then ${}^tFBF = B$ because B is characteristic. For any $H \in X$ we find that $H' := {}^tFHF \in X$, too. See that by computing $-H'^{-1}B = F^{-1}\theta F$. Thus we may let G act on X via the adjoint representation, i.e. $(H, g) \mapsto {}^t(\mathrm{Ad}\, g)H(\mathrm{Ad}\, g)$, $g \in \mathfrak{g}$, $H \in X$. In terms of θ, the action of G on X is expressed by $\theta \mapsto (\mathrm{Ad}\, g)^{-1}\theta(\mathrm{Ad}\, g)$.

Now choose $\Theta\colon G(\mathbb{R}) \to G(\mathbb{R})$ a Cartan involution with $\theta = d\Theta$. We require Θ to extend to an algebraic map $G \to G$ as in [BS, 1.6]; this makes Θ unique even if $G(\mathbb{R})$ is not connected, because G is. It is known that $K := \{g \in G(\mathbb{R}) \mid \Theta(g) = g\}$ is a maximal compact subgroup of $G(\mathbb{R})$. We now prove that K is the stabilizer of θ (and thus is also the stabilizer of H). Let $C_g\colon G \to G$ denote conjugation by g, and let $(\ )^0$ denote "connected component of the identity in." then $\theta = (\mathrm{Ad}\, g)^{-1}\theta(\mathrm{Ad}\, g)$ iff $d(C_g^{-1}\Theta C_g) = d\Theta$ iff for all $h \in G(\mathbb{R})^0$ $\Theta(h) = g^{-1}\Theta(g)\Theta(h)\Theta(g^{-1})g$ iff $g^{-1}\Theta(g) \in Z(G(\mathbb{R})^0) = 1$ iff $g \in K$. (We know $Z(G(\mathbb{R})^0) = 1$ because $Z(G(\mathbb{C})) = 1$ and $G(\mathbb{R})^0$ is Zariski dense in G). Since any two Cartan involutions are conjugate under $G(\mathbb{R})$ [V, Propositions 2 & 9, p. 193–195, Part II, Section 1], we have a diffeomorphism $X \approx G(\mathbb{R})/K$. In particular, we now know that X is contractible [BHC, Lemma 1.7], [B, 9.10].

Since G is defined over $\mathbb{Q}$, its Lie algebra has a rational structure $\mathfrak{g}_{\mathbb{Q}} \subset \mathfrak{g}$. Moreover, the adjoint representation $G \to \mathrm{Aut}(\mathfrak{g}_{\mathbb{Q}})$ is rational (regular map with rational number coefficients). Let $\Gamma \subset G$ be any arithmetic subgroup. Then by [B, 7.13(1)] we may find an integral structure $\mathfrak{g}_{\mathbb{Z}} \subset \mathfrak{g}_{\mathbb{Q}}$ such that $(\mathrm{Ad}\, \Gamma)\mathfrak{g}_{\mathbb{Z}} \subset \mathfrak{g}_{\mathbb{Z}}$. Since $\mathfrak{g}_{\mathbb{Q}}$ is a Lie algebra, by clearing denominators (i.e. replacing $\mathfrak{g}_{\mathbb{Z}}$ by $N\mathfrak{g}_{\mathbb{Z}}$ for some large integer N) we can achieve $[\mathfrak{g}_{\mathbb{Z}}, \mathfrak{g}_{\mathbb{Z}}] \subset \mathfrak{g}_{\mathbb{Z}}$. Having done this, we see that the form B is integral (meaning $B(\mathfrak{g}_{\mathbb{Z}}, \mathfrak{g}_{\mathbb{Z}}) \subset \mathbb{Z}$). For any $H \in X$, the pair $L = (\mathfrak{g}_{\mathbb{Z}}, H)$ is an example of what we call a *lattice* in [G]. Henceforth $\mathfrak{g}_{\mathbb{Z}}$ remains fixed.

By a $\mathbb{Q}$-subspace W of $\mathfrak{g}$ we mean an $\mathbb{R}$-subspace of the form $W = W_{\mathbb{Q}} \otimes \mathbb{R}$, with $W_{\mathbb{Q}} \subset \mathfrak{g}_{\mathbb{Q}}$. For any such W and any inner product H on $\mathfrak{g}$ we defined a number $d_W(H) = d(W, H) > 0$ in [G, 2.1] with the property that $d_W(H) > 1$ iff W is in the canonical filtration of L.

For $t \geq 1$ we define $X_W(t) := X(W, t) := \{x \in X \mid d_W(H) > t\}$; this is an open subset of X because d_W is a continuous function of H, see [G, 3.1]. We will be interested mainly in the case where W is a parabolic $\mathbb{Q}$-subalgebra of $\mathfrak{g}$, because the other W's will correspond to cusps for the larger group $\mathrm{Gl}(\mathfrak{g})$ which are "uninhabited" by X (i.e. X does not approach the ends of these other cusps without also approaching the cusp corresponding to a $\mathbb{Q}$-parabolic). We make this precise as follows. Define $X_{ss}(t) = X - \bigcup_W X(W, t)$, where W runs over all parabolic $\mathbb{Q}$-subalgebras of $\mathfrak{g}$; this is a closed subspace of X.

If $g \in \Gamma$, then $\mathrm{Ad}\, g$ is a Lie algebra automorphism of $\mathfrak{g}_{\mathbb{Z}}$, and so we have $g \cdot X(W, t) = X((\mathrm{Ad}\, g) \cdot W, t)$ as in [G, 2.2(b)]. Thus $X_{ss}(t)$ is stable under Γ.

THEOREM 1.1 *$X_{ss}(t)$ is compact modulo Γ.*

The other properties of $X_{ss}(t)$ present fewer difficulties than compactness.

THEOREM 1.2. *There is a constant $c>0$ such that for all $H \in X$ if some $v \in \mathfrak{g}_{\mathbb{Z}} - \{0\}$ has ${}^t vHv < c$ then there is a parabloic $\mathbb{Q}$-subalgebra W of $\mathfrak{g}$ such that $d_W(H) > t$.*

Proof of 1.1 *from* 1.2. The set $X_{ss}(t)$ is closed in the space of all inner products on $\mathfrak{g}$, so we may apply Mahler's compactness criterion [M], [B, Prop. 8.2], [MT, 1.1], or [G, 5.2]. From the equation ${}^t BH^{-1}B = H$ we find $\det H = |\det B|$, so $\det H$ is bounded as H ranges over X: this is one of the hypotheses required for Mahler's criterion. The other is a positive lower bound on the numbers ${}^t vHv$ with $v \in \mathfrak{g}_{\mathbb{Z}} - \{0\}$ and $H \in X_{ss}(t)$, which comes from 1.2. QED

For $H \in X$, let $0 = L_0 \subset L_1 \subset \cdots \subset L_s = L = (\mathfrak{g}_{\mathbb{Z}}, H)$ be the canonical filtration of L, and let $\sigma_i = \text{slope}\,(L_i/L_{i-1})$. Notice that $\text{vol}\, L = (\det H)^{1/2} = |\det B|^{1/2}$ is independent of H. As in [G, 1.23] we let $\min L = \sigma_1$ and $\max L = \sigma_s$. We will come back to the proof of 1.2 after some preliminaries.

2. Operations on lattices

Recall that a lattice L is a finitely generated free abelian group equipped with an inner product H on $V = L \otimes_{\mathbb{Z}} \mathbb{R}$. In [G, 7.1] we made the dual L^* into a lattice with inner product H^{-1}. We can also make $L_1 \otimes_{\mathbb{Z}} L_2$ into a lattice by using $H_1 \otimes H_2$ as inner product; alternatively, an orthonormal basis for $V_1 \otimes V_2$ will be the tensor product $\{e_i \otimes f_j\}$ of orthonormal bases $\{e_i\}$ for V_1 and $\{f_j\}$ for V_2.

We make $\text{Hom}_{\mathbb{Z}}(L_1, L_2) = L_1^* \otimes L_2$ into a lattice by combining the previous two definitions. Alternatively, if $g \in \text{Hom}_{\mathbb{Z}}(L_1, L_2)$ has matrix g_{ij} with respect to orthonormal bases of V_1 and V_2, then $\|g\| = (\sum_i \sum_j g_{ij}^2)^{1/2}$.

Given $f \in \text{Hom}\,(L_1, L_2)$, we define lattices $\text{im} f$ and $\text{coim} f$; both have $f(L_1) \subset L_2$ as underlying abelian group, but the inner product on $\text{im} f$ comes from L_2 by restriction, whereas the inner product on $\text{coim} f$ comes from L_1 by orthogonal projection [cf. G, Section 1].

Minkowski's theorem about finding lattice vectors in bounded convex symmetric closed subsets of Euclidean space may be phrased as follows.

THEOREM 2.1 [Minkowski]. *Given $n>0$, there is a constant c so that for any lattice L of dimension n some vector $v \in L - \{0\}$ has* $\log \|v\| < \text{slope}\, L + c$.

Proof. Let $B(r)$ be the closed ball of radius r in V. Then Minkowski proved that if $\text{vol}\, B(r) \geq 2^n \cdot \text{vol}\, L$, then $B(r) \cap L \neq \{0\}$. Now take $r = 2((\text{vol}\, L)/$

$(\operatorname{vol} B(1))^{1/n}$ to fulfill the inequality, producing v. Then $\log \|v\| \le \log r = \log 2 + \operatorname{slope} L - 1/n \log \operatorname{vol} B(1)$. QED.

THEOREM 2.2. *Suppose L and M are lattices. There is a number c, depending only on the ranks of L and M, so that* $\min \operatorname{Hom}(L, M) \ge \min M - \max L - c$.

Proof. By 2.1 applied to the first member of the canonical filtration of $\operatorname{Hom}(L, M)$, it is enough to find c so for all nonzero $f \in \operatorname{Hom}(L, M)$ we have $\log \|f\| \ge \min M - \max L - c$. Consider such an f, and let $L'' = \operatorname{coim} f$, $M' = \operatorname{im} f$, and $f' \in \operatorname{Hom}(L'', M')$ be the bijection induced by f. Now M' is a sublattice of M and L'' is a quotient lattice of L, so

$$\operatorname{slope} M' \ge \min M,$$

and

$$\operatorname{slope} L'' \le \max L,$$

and thus

$$\operatorname{slope} M' - \operatorname{slope} L'' \ge \min M - \max L.$$

If we choose orthonormal bases for L and M compatibly with L'' and M', then the matrix of f has the form $f = \begin{pmatrix} 0 & f' \\ 0 & 0 \end{pmatrix}$, so apparently $\|f\| = \|f'\|$. The matrix of f' appearing here has been formed with respect to orthonormal bases of L'' and M', so $\operatorname{vol} M' = |\det f'| \cdot (\operatorname{vol} L'')$. Each matrix entry satisfies $|f'_{ij}| \le \|f'\|$, so letting $n = \dim M' = \dim L''$ we see that $|\det f'| = |\sum_\sigma \pm f'_{\sigma 1,1} \cdots f'_{\sigma n,n}| \le n! \cdot \|f'\|^n$. Combining everything, we get

$$\begin{aligned}
\log \|f\| &= \log \|f'\| \\
&\ge \frac{1}{n} \log |\det f'| - \frac{1}{n} \log (n!) \\
&= \frac{1}{n} \log \operatorname{vol} M' - \frac{1}{n} \log \operatorname{vol} L'' - \frac{1}{n} \log (n!) \\
&= \operatorname{slope} M' - \operatorname{slope} L'' - \frac{1}{n} \log (n!) \\
&\ge \min M - \max L - \frac{1}{n} \log (n!) \\
&\ge \min M - \max L - c,
\end{aligned}$$

where $c = \sup \{1/n \log (n!) : n \le \dim L \text{ and } n \le \dim M\}$. QED.

COROLLARY 2.3. *If L and M are lattices, then there is a constant c depending only on the ranks of L and M so that*

$$\min L \otimes M \geq \min M + \min L - c,$$

and

$$\max L \otimes M \leq \max M + \max L + c.$$

Proof. Apply 2.2 to $\operatorname{Hom}(L^*, M) = L \otimes M$ and use $\min L = -\max L^*$. QED.

COROLLARY 2.4. *Suppose L, M, and N are lattices, and $f: L \otimes M \to N$ is a nonzero map. There is a constant c (depending only on $\|f\|$ and the ranks of L, M, and N) such that any sublattices $L' \subset L$, $M' \subset M$, and $N' \subset N$ with $\min N/N' > \max L' + \max M' + c$ also satisfy $f(L' \otimes M') \subset N'$.*

Proof. Let $N'' = N^*/N'$, and consider the map $f': L' \otimes M' \to N''$ induced by f. For suitable orthonormal bases, we see that the matrix of f' occurs as part of the matrix of f, so $\|f\| \geq \|f'\|$. Applying 2.2 and 2.3 we find constants with

$$\begin{aligned} \min \operatorname{Hom}(L' \otimes M', N'') &\geq \min N'' - \max L' \otimes M' - c_1 \\ &\geq \min N'' - \max L' - \max M' - c_2. \end{aligned}$$

If $f' \neq 0$, then we know that

$$\log \|f'\| \geq \min \operatorname{Hom}(L' \otimes M', N'').$$

Letting $c = \log \|f\| + c_2$ we find

$$\min N/N' \leq \max L' + \max M' + c. \quad \text{QED.}$$

3. The search for parabolic subalgebras

We need the following convenient characterization of parabolic subalgebras, which was used in [AB].

PROPOSITION 3.1 [AB]. *Suppose $\mathfrak{g}$ is semisimple, $W \subset \mathfrak{g}$ is a subalgebra, and the orthogonal $W^\vee$ (formed with respect to the Killing form B) is contained in W and is $\operatorname{ad}_{\mathfrak{g}}$-nilpotent. Then W is parabolic. The converse is also true.*

Proof [Ranga Rao]. We tensor everything with $\mathbb{C}$, so we are dealing with complex Lie algebras, and are in the case where Borel subalgebras exist. Since $W^\vee$ is solvable, it is contained in a Borel subalgebra $\mathfrak{b}$, Let $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$, and write $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$, where $\mathfrak{h}$ is a Cartan subalgebra.

We claim that $W^\vee \subset \mathfrak{n}$. Choose $y \in W^\vee$ and write $y = h + n$, $h \in \mathfrak{h}$, $n \in \mathfrak{n}$. We have an ordering on the roots α so that $\mathfrak{n} = \sum_{\alpha>0} \mathfrak{g}_\alpha$, and $\mathfrak{b} = \sum_{\alpha \geq 0} \mathfrak{g}_\alpha$. If we choose a basis of $\mathfrak{b}$ which contains bases for the root spaces, and is totally ordered in a way that extends the partial ordering of the roots, then the matrix of $\mathrm{ad}_{\mathfrak{b}}\, n$ is strictly lower triangular (because $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$), and the matrix of $\mathrm{ad}_{\mathfrak{b}}\, h$ is diagonal. But $\mathrm{ad}_{\mathfrak{b}}(h+n) = \mathrm{ad}_{\mathfrak{g}}\, y$ is nilpotent by hypothesis, so $\mathrm{ad}_{\mathfrak{b}}\, h = 0$. Thus $\alpha(h) = \mathrm{ad}_{\mathfrak{g}_\alpha}(h) = 0$ for $\alpha > 0$. Now the simple roots are a basis of $\mathfrak{h}^*$, and so $h = 0$, and thus $W^\vee \subset \mathfrak{n}$.

Next we claim that $\mathfrak{n}^\vee = \mathfrak{b}$. This follows from $B(H, \mathfrak{g}_\alpha) = 0$ $(\alpha \neq 0)$, $B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ $(\alpha + \beta \neq 0)$, and $\dim \mathfrak{n} = \operatorname{codim} \mathfrak{b}$.

Finally, we have $\mathfrak{b} = \mathfrak{n}^\vee \subset W^{\vee\vee} = W$, so W contains a Borel subalgebra, and is parabolic (by definition). QED.

Proposition 3.1 describes parabolic subalgebras completely in terms of trilinear and bilinear data, namely $[\cdot,\cdot] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ and $B : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$. We now see how to apply the facts from section 2. Recall the notation $L = (\mathfrak{g}_{\mathbb{Z}}, H)$ for inner products $H \in X$.

LEMMA 3.2. *Let $b = [\cdot,\cdot] \in \mathrm{Hom}(L \otimes L, L)$. The norm $\|b\|$ is independent of the choice of $H \in X$.*

Proof [D. Kazhdan]. Suppose $H' \in X$ also, and $L' = (\mathfrak{g}_{\mathbb{Z}}, H')$. Then for some $g \in G(\mathbb{R})$ we have $H' = {}^t(\mathrm{Ad}\, g) H (\mathrm{Ad}\, g)$. Since $\mathrm{Ad}\, g$ is a Lie algebra automorphism of $\mathfrak{g}$ we get a commutative diagram

$$\begin{array}{ccc} V \otimes V & \xrightarrow{b} & V \\ {\scriptstyle \mathrm{Ad}\, g \otimes \mathrm{Ad}\, g}\downarrow & & \downarrow{\scriptstyle \mathrm{Ad}\, g} \\ V' \otimes V' & \xrightarrow{b} & V' \end{array}$$

where $V = L \otimes \mathbb{R}$, $V' = L' \otimes \mathbb{R}$, and the vertical maps are isometries. The vertical maps do not preserve the integral structure; nevertheless, the diagram shows that $\|b\| = \|b\|'$. QED.

For $H \in X$ and $L = (\mathfrak{g}_{\mathbb{Z}}, H)$, we let $0 = L_0 \subset L_1 \subset \cdots \subset L_{s-1} \subset L_s = L$ be the canonical filtration of L, $\sigma_i = \mathrm{slope}\,(L_i / L_{i-1})$, and $L_i^\vee =$ the dual with respect to

B. We introduce the notation err (c) to denote any (undetermined) number of absolute value $\leq c$, i.e. $x = y + \text{err}\,(c)$ will mean $|x - y| \leq c$.

PROPOSITION 3.3. *There is a constant* $c > 0$ *so that for all* $H \in X$ *we have*

$$\text{(P1)}\quad \forall i, j, k \quad \sigma_i + \sigma_j + c \leq \sigma_{k+1} \Rightarrow [L_i, L_j] \subset L_k$$

$$\text{(P2)}\quad \forall i \quad \sigma_{i+1} \geq \sigma_i + c \Rightarrow \exists ! j \;\; L_j = L_i^{\vee},\ \sigma_j = -\sigma_{i+1} + \text{err}\,(c),$$

$$\textit{and } \sigma_{j+1} = -\sigma_i + \text{err}\,(c).$$

Proof. P1 follows from (3.2) and (2.4) because $\sigma_1 = \max L_i$ and $\sigma_{k+1} = \min L/L_k$.

P2 follows from [G, 7.14] and the proof of [G, 7.13]. QED.

PROPOSITION 3.4. *With* c *as in* 3.3, *assume* i *satisfies* $1 \leq i \leq s - 1$ *and*
(A) $2c \leq \sigma_{i+1}$,
(B) $\sigma_i + c \leq \sigma_{i+1}$, *and*
(C) $2\sigma_i + c \leq \sigma_{i+1}$.
Then L_i *is a parabolic subalgebra of* $\mathfrak{g}$.

Proof. By (B) and (P2) we know $L_i^{\vee} = L_j$ for some j. In addition, $\sigma_j = -\sigma_{i+1} + \text{err}\,(c) \leq -2c + \text{err}\,(c) \leq -c$. Were it true that $j \geq i + 1$ we would have $\sigma_j \geq \sigma_{i+1} \geq 2c$ as well, so we must have $j \leq i$ and thus $L_j \subset L_i$.

By (P1) and (C) we know $[L_i, L_i] \subset L_i$, so L_i is a subalgebra of $\mathfrak{g}$.

For all k we have $\sigma_j + \sigma_k + c \leq -c + \sigma_k + c = \sigma_{(k-1)+1}$, so by (P1) we have $[L_j, L_k] \subset L_{k-1}$, which shows that $L_i^{\vee} = L_j$ is $\text{ad}_{\mathfrak{g}}$-nilpotent. Now by (3.1) we know L_i is parabolic. QED.

Now we have enough to prove Theorem 1.2.

Proof. of Theorem 1.2. Suppose we have an $H \in X_{ss}(t)$. Then for all j $(1 \leq j \leq s)$ we have either $j = 1$ (in which case $\sigma_j \leq \text{slope}\, L$); or L_{j-1} is parabolic, in which case $\sigma_j \leq \sigma_{j-1} + \log t$; or, finally, L_{j-1} is not parabolic, in which case 3.4 shows that either $\sigma_j < 2c$, or $\sigma_j < \sigma_{j-1} + c$, or $\sigma_j < 2\sigma_{j-1} + c$. We apply descending induction to prove that for each $i < s$ there are a finite number of polynomials $P_\alpha^i(T)$ with positive coefficients such that $\sigma_s < \max_\alpha P_\alpha^i(\sigma_i)$. We find that $\max L = \sigma_s < \max_\alpha P_\alpha^1(\sigma_1) \leq \max_\alpha P_\alpha^1(\text{slope}\, L)$; the latter number is independent of L because slope L is. The dependence of that number on s can be removed because $1 \leq s \leq \dim \mathfrak{g}$, yielding an upper bound for $\max L$ which is independent of H. Clearly an upper bound on $\max L$ provides also a lower bound on $\min L$, again using that slope L is independent of H. But $\min L \leq \log \|v\|$ for any nonzero $v \in L$. Thus we have proved the contrapositive of 1.2. QED.

4. Topology

To prove results about X_{ss} we introduce flows on X for each parabolic subalgebra $W \subset \mathfrak{g}$, namely, the geodesic action of [BS].

LEMMA 4.1. *Suppose $W \subset \mathfrak{g}$ is a parabolic subalgebra. Then for some s, $\mathfrak{g}$ has a filtration by subspaces $0 = W^{s+1} \subset \cdots \subset W^1 \subset W^0 \subset W^{-1} \subset \cdots \subset W^{-s} = \mathfrak{g}$ such that $W^0 = W$, $W^i = (W^{-i+1})^\vee$ all i, and $[W^i, W^j] \subset W^{i+j}$ all i, j (using the obvious convention when $i + j$ falls outside the range $[s+1, -s]$). Moreover, if W is a $\mathbb{Q}$-subspace, so is each W^i.*

Proof. Define $W^0 = W$, $W^1 = W^\vee$ (=orthogonal with respect to Killing form B); for $i > 1$ define W^i inductively by $W^i = [W^{i-1}, W^1]$; for $i < 0$ define $W^i = (W^{-i+1})^\vee$. This is clearly a filtration, and $W^{s+1} = 0$ for some s because W^1 is nilpotent (3.1). To prove that $[W^i, W^j] \subset W^{i+j}$ we may first tensor with $\mathbb{C}$ and make use of roots relative to a Cartan subalgebra $\mathfrak{h} \subset W^0$. [This approach was explained to me by Ranga Rao; I was also able to make a complicated combinatorial proof, using only the Jacobi identity and the invariance of B.] Let S be a set of simple roots chosen so $\alpha \in S \Rightarrow \mathfrak{g}_\alpha \subset W$. Then by [H, Exercise 6, p. 87] find a subset $S' \subset S$ so W is generated by the subspaces $\mathfrak{g}_\alpha$ for $\alpha \in S$, $\mathfrak{g}_{-\alpha}$ for $\alpha \in S'$, and $\mathfrak{h}$. The root spaces $\mathfrak{g}_\alpha$ are 1-dimensional, and if α, β, $\alpha + \beta$ are all roots, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ [H, Prop. 8.4(d)]; if α is a root, then $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}$ [H, Prop. 8.3(d)]; and $[\mathfrak{h}, \mathfrak{g}_\alpha] \subset \mathfrak{g}_\alpha$. We also know that $B(\mathfrak{h}, \mathfrak{g}_\alpha) = 0$; and $B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ if $\alpha + \beta \neq 0$. Introduce the notation $\mathfrak{g}(R) = \bigoplus_{\alpha \in R \cap \Delta} \mathfrak{g}_\alpha$, where $\Delta =$ the roots, and R is any subset of $\mathbb{Z}\Delta$ (we include $\alpha = 0$, and let $\mathfrak{g}_0 = \mathfrak{h}$). Then we see that $W^0 = \mathfrak{g}(\mathbb{N}S + \mathbb{Z}S')$, $W^1 = \mathfrak{g}(\mathbb{Z}\Delta \setminus -(\mathbb{N}S + \mathbb{Z}S')) = \mathfrak{g}(T_1)$, where $T_i = \{\sum_{\alpha \in S} n_\alpha \alpha \in \mathbb{Z}\Delta : \sum_{\alpha \in S-S'} n_\alpha \geq i\}$ and "\" denotes "complement of." Using the fact that if $\sum n_\alpha \alpha$ is a root, then all n_α have the same sign, we deduce that $W^i = \mathfrak{g}(T_i)$ for $i \geq 1$, and then that $W^{-i} = (W^{i+1})^\vee = \mathfrak{g}(\mathbb{Z}\Delta \setminus -T_{i+1}) = \mathfrak{g}(T_{-i})$. The conclusion then follows easily. QED.

CONSTRUCTION 4.2. Fix W and W^i as in 4.1. Fix an inner product $H \in X$ and its corresponding Cartan involution θ. Use $(\ )^\perp$ to denote the orthogonal complement with respect to H. Define $V^i = W^i \cap (W^{i+1})^\perp$, so $\mathfrak{g} = V^s \oplus \cdots \oplus V^1 \oplus V^0 \oplus V^{-1} \oplus \cdots \oplus V^s$ is a decomposition orthogonal with respect to H.

By compatibility of H and B, and using [G, proof of 7.5] deduce that

$(W^{i+1})^{\perp} = (W^{-i})^{\vee\perp} = \theta W^{-i}$ and $V^i = W^i \cap \theta W^{-i}$. Since θ preserves the bracket, we see that $[V^i, V^j] \subset V^{i+j}$ (all i, j) and thus we have a Lie algebra grading of $\mathfrak{g}$. [In terms of roots (after tensoring with $\mathbb{C}$) we see that $V^i = \mathfrak{g}(U_i)$ where $U_i = \{\sum_{\alpha\in S} n_\alpha \alpha : \sum_{\alpha\in S\setminus S'} n_\alpha = i\}$, provided $\mathfrak{h}$ is chosen to be $\mathfrak{b} \cap \theta\mathfrak{b}$, where $\mathfrak{b}$ is a Borel subalgebra contained in W.] Given $r \in \mathbb{R}$, $r > 0$, the graded map $F_r : \mathfrak{g} \to \mathfrak{g}$, defined as multiplication by r^{-i} on V^i, is a Lie algebra automorphism of $\mathfrak{g}$. Define $H' = {}^t F_r \cdot H \cdot F_r \in X$. The corresponding lattice $L' = (\mathfrak{g}_{\mathbb{Z}}, H')$ will be denoted by $L\{W; r\}$; the same notation was used in [G, 7.16] for a related concept in a different context.

These operations commute to some extent. If $W' \subset W$ are $\mathbb{Q}$-parabolic subalgebras then

$$L\{W'; r'\}\{W; r\} = L\{W; r\}\{W'; r\}. \tag{4.2.1}$$

This is best seen by examining the root spaces, as in the previous paragraph, the point being that F_r is multiplication by a scalar on each root space $\mathfrak{g}_\alpha$. The hypothesis that $W' \subset W$ ensures that we may assume $\mathfrak{h} \subset W'$ and $\mathfrak{h} \subset W$, so the corresponding map F'_r is diagonal for the same root space decomposition that F_r is diagonal for, and thus F_r and F'_r commute.

LEMMA 4.3. *If $W' \subset W$ are $\mathbb{Q}$-parabolic subalgebrras of $\mathfrak{g}$, $L \in X$, and $r > 1$, then*

(a) $d(W, L\{W; r\}) \geq r \cdot d(W, L)$
(a′) $d(W, L\{w, r^{-1}\}) \leq r^{-1} \cdot d(W, L)$
(b) $d(W', L\{W; r\}) \geq d(W', L)$
(b′) $d(W', L\{W; r^{-1}\}) \leq d(W', L)$.

Remark. It would have been nice to have (b) for the case $W \subset W'$, but that is not true.

Proof. Let $V = \mathfrak{g}$. Let $0 \subset W^s \subset \cdots \subset W^1 \subset W^0 = W \subset W^{-1} \subset \cdots$ and $0 \subset W'^s \subset \cdots \subset W'^1 \subset W'^0 = W'W'^{-1} \subset \cdots$ be chosen as in 4.2. It is not always true that the union of these two filtrations is a filtration, but we can manage anyway. Notice that (a)$\Rightarrow$(a′) and (b)$\Rightarrow$(b′).

We prove (b) assuming $W' \neq W$. In the notation of [G, 2.4] we have $L\{W; r\} = L[r^{-s}][W^s; r] \cdots [W^1; r][W^0; r] \cdots [W^{1-s}; r]$ and $W^s \subset \cdots \subset W^1 \subsetneqq W' \subsetneqq W^0 \subset \cdots \subset W^{1-s}$. Since $d(W', L[r^{-s}]) = d(W', L)$ [see G, 2.1], all we need is the following lemma (from which (a) also follows). QED.

LEMMA 4.4. *If L is a lattice, $W' \subset W$ are $\mathbb{Q}$-subspaces of $V = L \otimes_{\mathbb{Z}} \mathbb{R}$ and*

$r \geq 1$, *then*

(a) $d(W, L[W; r]) = r \cdot d(W, L)$

(b) $d(W, L[W'; r]) \geq d(W, L)$

(c) $d(W', L[W; r]) \geq d(W', L)$

Proof. Part (a) was proved in [G, 2.4]. By duality [G, 7.1.] (b) will follow from (c). To prove (c) we replace L by $L/W' \cap L$; then we must show that $\min(L[W; r]) \geq \min(L)$ (notation from [G, 1.23]. So for each $\mathbb{Q}$-suspace $U \subset V$ we must show that $\operatorname{vol}(L[W; r] \cap U) \geq \operatorname{vol}(L \cap U)$. Let $m = \dim U$, choose an orthonormal basis for V which contains an orthonormal basis for W, and write $\Lambda^m(U \cap L) = \mathbb{Z}u$, with $u = \sum a_I e_I$, where $I = (i_1, \ldots, i_m) \in \mathbb{N}^m$ and $e_I = e_{i_1} \wedge \cdots \wedge e_{i_m}$. Then $\operatorname{vol}(L \cap U) = \|u\| = (\sum a_I^2)^{1/2}$, but $\operatorname{vol}(L[W; r] \cap U) = (\sum (a_I r^{\langle I \rangle})^2)^{1/2}$, where $\langle I \rangle = \operatorname{card}\{j : e_{i_j} \notin W\}$. Since $r \geq 1$, the desired inequality is clear. QED.

Our next goal is to identify the homotopy types of X_{ss}, its boundary, and its complement. In order to do these three tasks simultaneously, we abstract the information available to us. We let $\mathcal{W}$ be the partially ordered set of parabolic $\mathbb{Q}$-subalgebras of $\mathfrak{g}$; for $W \in \mathcal{W}$ let $h_W : X \to \mathbb{R}$ be the continuous map defined by $h_W(x) = \log(d(W, x)/t)$, where t is the fixed number referred to in 1.1. Let $\varphi_W : X \times \mathbb{R} \to X$ be defined by $\varphi_W(L, r) = L\{W, e^r\}$. Let $\mathcal{W}_i = \{W \in \mathcal{W} : \dim W = i\}$. Let $T = \dim \mathfrak{g}$. These data satisfy the following axioms.

(A1) $\mathcal{W}$ is a partially ordered subset which is the disjoint union of subsets $\mathcal{W}_i$, $(i = 1, \ldots, T)$, such that $W \in \mathcal{W}_i$, $U \in \mathcal{W}_j$, $W < U \Rightarrow i < j$.

(A2) X is nonempty topological space.

(A3) $\forall W \in \mathcal{W}\ h_W : X \to \mathbb{R}$ is a continuous map.

(A4) $\forall W \in \mathcal{W}\ \varphi_W : X \times \mathbb{R} \to X$ is a continuous action of the topological group $\mathbb{R}$ on X, i.e. $\varphi_W(x, 0) = x$ and $\varphi_W(\varphi_W(x, r), s) = \varphi_W(x, r + s)$.

(A5) $\forall W \in \mathcal{W}\ \forall x \in X$ the map $\mathbb{R} \to \mathbb{R}$ defined by $r \mapsto h_W(\varphi_W(x, r))$ is an increasing bijection.

(A6) $\forall x \in X\ \forall W < U \in \mathcal{W}\ [r \geq 0 \Rightarrow h_W(\varphi_U(x, r)) \geq h_W(x)]$ and $[r \leq 0 \Rightarrow h_W(x) \geq h_W(\varphi_U(x, r))]$.

(A7) The sets $\bar{X}_W := \{x \in X : h_W(x) \geq 0\}$ form a locally finite family, and moreover, any $x \in X$ has a neighborhood $\mathcal{U}$ such that $\{W \in \mathcal{W} : \mathcal{U} \cap \bar{X}_W \neq \phi\}$ is a chain in $\mathcal{W}$.

We also define $X_{ss} := \{x \in X : \forall W\ h_W(x) \leq 0\}$, and $X^+ = \bigcup_{W \in \mathcal{W}} \bar{X}_W$. The following proofs will be based solely upon the axioms.

THEOREM 4.5. (a) *The boundary* ∂X_{ss} *is the set* $\{x \in X_{ss} : \exists W\ h_W(x) = 0\}$.

(b) *There is a deformation retraction of* X *onto* X_{ss} *which restricts to a deformation retraction of* X^+ *onto* ∂X_{ss}.

(c) *Given* $W_1 < \cdots < W_m \in \mathcal{W}$, *there is a deformation retraction of* X *onto* $\bar{X}_{W_1} \cap \cdots \cap \bar{X}_{W_m}$.

First, we need some lemmas.

LEMMA 4.6. *Given* $W \in \mathcal{W}$, *define* $\psi_W : X \times \mathbb{R} \to \mathbb{R}$ *by decreeing that* $\psi_W(x, r) = t$ *iff* $h_W(\varphi_W(x, t)) = r$ *(such a* t *exists by* (A5)). *The map* ψ_W *is continuous.*

Proof. Let $\varphi = \varphi_W$, $h = h_W$, and $\psi = \psi_W$. Suppose $\psi(x, r) = t$ and $\varepsilon > 0$ is given. We know $h(\varphi(x, t)) = r$, so by (A5) we may choose $\delta > 0$ so $h(\varphi(x, t - \varepsilon)) < r - \delta < r + \delta < h(\varphi(x, t + \varepsilon))$. Choose $\mathcal{U}$ a neighborhood of x so $\forall y \in \mathcal{U}\ h(\varphi(y, t - \varepsilon)) < r - \delta$ and $r + \delta < h(\varphi(y, t + \varepsilon))$. If $y \in \mathcal{U}$ and $s \in (r - \delta, r + \delta)$, then $h(\varphi(y, t - \varepsilon)) < s = h(\varphi(y, \psi(y, s))) < h(\varphi(y, t + \varepsilon))$, and thus $t - \varepsilon < \psi(y, s) < t + \varepsilon$. QED.

LEMMA 4.7 [Rectification of φ_W]. *Define* $\varphi'_W : X \times \mathbb{R} \to X$ *by the formula* $\varphi'_W(x, r) = \varphi_W(x, \psi_W(x, r + h_W(x)))$. *The maps* $\{\varphi'_W\}$ *satisfy the axioms* (A1)–(A7) *as well as*

(A8) $\forall x \in X\ \forall r\ h_W(\varphi_W(x, r)) = h_W(x) + r$ [*equivariance of* h_W].

Proof. Axiom (A8) for φ'_W follows from the definition.

Check (A4). Continuity follows from the previous lemma. Check $h_W(\varphi'_W(\varphi'_W(x, r), s))) = h_W(\varphi'_W(x, r)) + s = h_W(x) + r + s = h_W(\varphi'_W(x, r + s))$; thus $\varphi'_W(\varphi'_W(x, r), s) = \varphi'_W(x, r + s)$, as both sides are in the orbit of x for the φ_W action.

It is immediate that (A8) implies (A5).

To check (A6) notice that $\psi_W(x, r + h_W(x)) \geq 0$ iff $r \geq 0$, which itself follows from (A6) for φ_W. QED.

Proof of 4.5*a*. For any $\varepsilon > 0$, $W \in \mathcal{W}$, and $x \in X_{ss}$ with $h_W(x) = 0$, we have $h_W(\varphi_W(x, \varepsilon)) > 0$, and thus $\varphi_W(x, \varepsilon) \notin X_{ss}$. since $\varphi_W(x, \varepsilon)$ is continuous in ε, we see $x \in \partial X_{ss}$.

Proof of 4.5*b*. We define the homotopy $H : X \times [0, T] \to X$ as a composite of homotopies $H_1, \ldots, H_T : X \times [0, 1] \to X$, i.e. for $s \in [i-1, i]$ and $x \in X$ we have

$$H(x, s) = H_i(H_{i-1}(H_{i-2}(\ldots H_1(x, 1) \ldots, 1), 1), s - i + 1).$$

Each H_i will satisfy $H_i(x, 0) = x$ (all $x \in X$) so H will be continuous.

Since 4.5b makes no reference to the maps φ_W, we may use 4.7 to rectify the φ_W's, and thus may assume (A8) is satisfied. Now set

$$H_i(x, s) = \begin{cases} \varphi_W(x, -sh_W(x)) & \text{if } x \in X_w \text{ and } W \in \mathscr{W}_i \\ x & \text{if } h_W(x) \leq 0 \text{ all } W \in \mathscr{W}_i \end{cases}$$

Axioms (A1, 7) ensure that the sets X_W ($W \in \mathscr{W}_i$) are disjoint. Since $\varphi_W(x, 0) = x$ we see that when the two cases in the definition of H_i overlap, they agree; moreover the closed sets involved form a locally finite family, so H_i is continuous.

Now we claim that

$$\forall j \leq i \quad \forall U \in \mathscr{W}_j \quad \forall x \in X \quad h_U(H(x, i)) \leq 0 \tag{$*$}$$

We prove this by induction on i, the case $i = 0$ being vacuous. Let $z = H(x, i)$ and $y = H(x, i-1)$, so $z = H_i(y, 1)$. By the inductive hypothesis, we know that $h_W(y) \leq 0$ for $U \in \mathscr{W}_j$, $j > i$. Look at the definition of H_i: if $h_W(y) \leq 0$ all $W \in \mathscr{W}_i$, then $z = y$ and ($*$) is clear. On the other hand, if $W \in \mathscr{W}_i$ and $x \in \bar{X}_W$, then $z = \varphi_W(y, -h_W(u))$. Pick $j \leq i$ and $U \in \mathscr{W}_j$ and establish ($*$): there are three cases. If $U = W$ then $h_U(z) = h_W(z) = h_W(y) - h_W(y) = 0$. If $U < W$, then (A6) yields $h_U(z) \leq h_U(y) \leq 0$. If $\{U, W\}$ is not a chain, then $h_W(z) = 0$ forces $h_U(z) < 0$ by (A7).

Now we check that H provides the deformation retraction we need; this means we must check the following properties:

(i) $H(x, 0) = X$
(ii) $H(x, T) \in X_{ss}$
(iii) $x \in X_{ss} \Rightarrow H(x, s) = x$, all s
(iv) $x \in X^+ \Rightarrow H(x, s) \in X^+$, all s
(v) $x \in X^+ \Rightarrow H(x, T) \in \partial X_{ss}$

Property (i) is immediate, and (v) follows from the others because $\partial X_{ss} = X^+ \cap X_{ss}$. Property (ii) is ($*$) for $i = T$. Property (iii) is immediate from the definition of H_i. Property (iv) follows from the analogous statements for each H_i, which are immediate. QED.

Proof of 4.5c. We replace h_W by $h'_W = -h_W$, and φ_W by φ'_W defined as $\varphi'_W(x, r) = \varphi_W(x, -r)$. The axioms (A1–6) are preserved. We replace $\mathscr{W}$ by

$\mathcal{W}' = \{\alpha_1, \ldots, \alpha_m\}$, so now (A7) is satisfied as well, because $\mathcal{W}$ is itself a chain. Now 4.5b yields what we want. QED.

In order to prove that various subspaces of X are manifolds, we introduce some new axioms, which we show later are satisfied in our situation. These are necessary because, for example, the assertion $M \times \mathbb{R}$ is a manifold does not imply that M is a manifold.

(A9) X is a C^∞-manifold

(A10) φ_W is a C^∞-map

(A11) If $W < U \in \mathcal{W}$, then φ_W and φ_U commute, i.e. $\varphi_W(\varphi_U(x, r), s)) = \varphi_W(\varphi_U(x, s), r)$.

(A12) Given $W_1 < \cdots < W_s \in \mathcal{W}$, there are numbers $n_1 \ldots, n_s > 0$ so that if we define $\varphi(x, r) = \varphi_{W_1}(\varphi_{W_2}(\ldots \varphi_{W_s}(x, n_s r) \ldots, n_2 r), n_1 r)$, then $\forall i$ $\forall r \geq 0 \, \forall x \, h_{W_i}(\varphi(x, r)) \geq r + h_{W_i}(x)$. Moreover, there is a C^∞-map $h : X \to \mathbb{R}$ so $h(\varphi(x, r)) \geq h(x) + r$, all $x \in X$, $r \geq 0$.

THEOREM 4.8. *Assume* (A1–7, 9–12) *are satisfied.*

(a) *X_{ss} is a manifold with boundary, and the boundary is ∂X_{ss}.*

(b) *Given $W_1 < \cdots < W_s \in \mathcal{W}$, the space $\bar{X}_{W_1} \cap \cdots \cap \bar{X}_{W_s}$ is a manifold with boundary.*

Proof. Using "inversion" as in the proof of 4.5c, we see that (b) follows from (a).

The idea for proving (a) comes from [G, proof of 3.4]. We see that the differential dh is nonzero everywhere (look at the composite $r \mapsto h(\varphi(x, r))$, so the level set $h^{-1}(\{0\})$ is a submanifold of X of codimension 1, and is the boundary ∂Y of the manifold $Y = h^{-1}((-\infty, 0])$.

Now suppose we choose a point $x \in X$ and try to show that X_{ss} is a manifold near x, assuming $x \in \partial X_{ss}$. We use (A7) to choose a neighborhood $\mathcal{U}$ of x and let $\{W_1 < \cdots < W_s\} = \{W \in \mathcal{W} : \mathcal{U} \cap \bar{X}_W \neq \phi\}$. Define $h'(x) = \sup_i \{h_{W_i}(x)\}$ for $x \in X$, so letting $Y' = h'^{-1}((-\infty, 0])$, we have $\mathcal{U} \cap X_{ss} = \mathcal{U} \cap Y'$, $\partial Y' = h'^{-1}(\{0\})$, and $\mathcal{U} \cap \partial X_{ss} = \mathcal{U} \cap \partial Y'$; thus it will suffice to prove that Y' is a manifold with boundary $\partial Y'$. From (A12) we see that

(A13) $h'(\varphi(x, r)) \geq h'(x) + r, \quad$ all $\quad x \in X, r \geq 0.$

Then using 4.6 for h and h', it is not hard to set up a homeomorphism $Y \simeq Y'$ which restricts to a homeomorphism $\partial Y \simeq \partial Y'$.

PROPOSITION 4.9. (A9–12) *are actually satisfied in our situation.*

Proof. (A9, 10) are clear; (A11) was proved in 4.2.1.

Prove (A12). We define $h(x) = \text{slope}\,(x/x \cap W_1) - \text{slope}\,(x \cap W_1)$; it is a C^∞ function*. We let $N = \dim \mathfrak{g}$, and define $n_i = N \cdot i$. It is easy to check the required properties in terms of root spaces. Each W_i and its orthogonal complement for the inner product H is a sum of root spaces, and each φ_{W_i} comes from a map F_r which is diagonal with respect to the root space decomposition. The numbers n_i have been chosen to make it clear that the composite of all these maps F_r has the following property: for each W_i, the scalars occurring in W_i are strictly smaller than the scalars occurring in its orthogonal complement. The result follows then from 4.4. QED.

5. Conclusion

THEOREM 5.1. *Suppose $t > 1$. The space $X_{ss}(t)$ is a manifold with boundary, is contractible, and is compact modulo Γ. The boundary is homotopy equivalent to the Tits building by a homotopy equivalence which respects the action of Γ.*

Proof. We combine 4.8b, 4.5b, and 1.1 to get the assertion about X_{ss}.

Now consider the covering $X^+ = \bigcup_{W \in \mathscr{W}} \bar{X}_W$ by closed subsets; we will use [BS, 8.1, 8.2] to identify its homotopy type.

Suppose $W_1, \ldots, W_s \in \mathscr{W}$. If they do not form a chain, then $\bar{X}_{W_1} \cap \cdots \cap \bar{X}_{W_s} = \phi$, by (A7). If they do form a chain, then $\bar{X}_{W_1} \cap \cdots \cap \bar{X}_{W_s}$ is a contractible manifold with boundary, according to 4.5c, contractibility of X, and 4.8b, and thus is an absolute retract [BS, 8.1].

Now we apply [BS, 8.2.1] to conclude that X^+ (and thus ∂X_{ss} also, by 4.5b) is Γ-homotopy equivalent to the simplicial complex associated to the poset A of $\mathbb{Q}$-parabolic subalgebras of $\mathfrak{g}$, and thus to the Tits building of G. QED.

Thus we have proved for our space X_{ss} the same qualitative properties which Borel and Serre prove for their manifold with corners $\bar{X}$, namely [BS, 8.4.2 and 9.3]. The cohomological properties for Γ deduced in [BS, Section 11, except for 11.3] can be derived using X_{ss}, except that we must appeal to [KS, p. 123] for the fact that X_{ss}/Γ is equivalent to a finite simplicial complex, and appeal to [B, 9.10] for the fact that Γ contains only finitely many conjugacy classes of torsion elements (which implies that Γ contains a torsion-free subgroup of finite index).

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* One may choose a basis so that each of these slopes is a constant times the log of the determinant of a principal minor of the matrix (or its inverse) of the inner product x.

Ultimately, I hope that both of these latter facts can be proved directly by these methods.

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