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# Commutative ring-spectra of characteristic 2

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## 1. Introduction and statement of the main result

Let $E^*(-)$ be a ring-theory, i.e. a multiplicative cohomology theory represented by a ring-spectrum $E$. Experience shows that if $E$ is of characteristic 2, i.e. if $2 \cdot \pi_*(E) = 0$, then often some care is needed because the product on $E$ might not be commutative. For example this happens for the spectra $MU\mathbb{Z}/_2$ or $K\mathbb{Z}/_2$ which represent complex cobordism theory resp. complex $K$-theory with coefficients $\mathbb{Z}/_2$, and many cobordism theories of manifolds with singularities. On the other hand one knows many ring-theories of characteristic 2 which are commutative. For example, if $\Lambda$ denotes a graded commutative $\mathbb{Z}/_2$-algebra,

$$H^*(X; \Lambda) = \prod_i H^{*+i}(X; \Lambda^{-i})$$

is such a ring-theory which is represented by the graded Eilenberg–MacLane spectrum

$$H(\Lambda) = \bigvee_i \Sigma^i H(\Lambda^{-i}) = \prod_i \Sigma^i H(\Lambda^{-i}).$$

Other examples are given by $MO^*(-)$, the unoriented cobordism theory, or $MPL^*(-)$, the unoriented $PL$-cobordism theory. Now it is well-known that both spectra $MO$ and $MPL$ are equivalent to graded Eilenberg–MacLane spectra whereas $MU\mathbb{Z}/_2$ and $K\mathbb{Z}/_2$ are not. The purpose of this paper is to show that these examples represent a general phenomenon. More precisely, we will prove the

(1.1) THEOREM. *Let $E$ be a commutative ring-spectrum with coefficient ring $\pi_*(E)$ of characteristic 2. Then there is an equivalence of ring-theories*

$$E^*(-) \cong H^*(-; E^*)$$

*on the stable homotopy category* **S**.

(1.2) Remarks and Examples. (1) Theorem (1.1) immediately produces a conceptual proof of the fact that $MO$, $MPL$ and a lot of other spectra considered in [5] are equivalent as ring-spectra to graded Eilenberg–MacLane spectra.

(2) Following Rourke [13] we call a spectrum $E$ a $\mathbb{Z}/_p$-spectrum if it is a $(-1)$-connected ring-spectrum and $\pi_0(E) \cong \mathbb{Z}/_p$. $E$ is said to represent ordinary $\mathbb{Z}/_p$-cohomology if the Thom map $T: E^*(X) \to H^*(X; \mathbb{Z}/_p)$ is epic for all $CW$-complexes $X$, i.e. if the Atiyah–Hirzebruch spectral sequence collapses for all $X$. From (1.1) it is obvious that a $\mathbb{Z}/_2$-spectrum with commutative coefficient ring represents ordinary $\mathbb{Z}/_2$-cohomology iff it is commutative. This corrects Corollary 1 of [13] (see also [14]).

(3) Let $E$ be a commutative ring-spectrum. From [3] one knows that if $\eta^*: E^*(S^2) \to E^*(S^3)$ is zero ($\eta: S^3 \to S^2$ denotes the Hopf map) then $E\mathbb{Z}/_2$ admits an admissible product. Moreover, in this case there is a commutative admissible product on $E\mathbb{Z}/_2$ if and only if $0 = \bar{\eta}^*(1) \in E^*(M_2)$ where $\bar{\eta} \in \{\Sigma^2 M_2, S^2\}$ is a generator and $M_2$ denotes the mod 2 Moore spectrum (see [3], (7.7)). So in this situation (1.1) implies that $E\mathbb{Z}/_2$ is equivalent to a graded Eilenberg–MacLane spectrum iff $\bar{\eta}^*(1) = 0$. This means that the class $\bar{\eta}^*(1)$ may be viewed as a sort of "total $k$-invariant" of the spectrum $E\mathbb{Z}/_2$.

(4) Let $\eta: S^1 \to BO$ be the non-trivial map. Since $BO$ is an infinite loop space, corresponding to $\eta$ there is a map $\sigma^2(\eta): S^3 \to B^2O$ where $\Omega^2 B^2 O = BO$. Applying $\Omega^2$ one gets a map

$$h = \Omega^2\sigma^2(\eta): \Omega^2 S^2 \to BO.$$

Let $\gamma$ be the universal bundle over $BO$ and set $\bar{\gamma} = h^*(\gamma)$. A theorem of Mahowald [11] asserts that the Thom spectrum of $\bar{\gamma}$, $M(\bar{\gamma})$, is equivalent to the Eilenberg–MacLane spectrum $K(\mathbb{Z}/_2, 0)$. Theorem (1.1) may be used to produce a very simple and conceptual proof of this fact. For this we remark that $M(\bar{\gamma})$ is $(-1)$-connected and $\pi_0(M(\bar{\gamma}))$ is cyclic since $M(\bar{\gamma})$ can be constructed to have a single zero cell. Since $h^*(w_1) \neq 0$, the zero cell extends over the Moore spectrum and $\pi_0(M(\bar{\gamma})) \cong \mathbb{Z}/_2$. Because $h$ is an $H$-map, $M(\bar{\gamma})$ is a commutative ring-spectrum ([10], Theorem (1.1)). From Theorem (1.1) it now follows that $M(\bar{\gamma})$ is a wedge of Eilenberg–MacLane spectra. Using the homology Thom isomorphism $H_*(\Omega^2 S^3; \mathbb{Z}/_2) \xrightarrow{\cong} H_*(M(\bar{\gamma}); \mathbb{Z}/_2)$ and the well-known fact that

$$H_*(\Omega^2 S^2; \mathbb{Z}/_2) \cong \mathbb{Z}/_2[x_1, x_2, \ldots]; \; |x_i| = 2^i - 1$$

one sees that there is only one summand $K(\mathbb{Z}/_2, 0)$ in the wedge decomposition of

$M(\bar{\gamma})$, which finishes the proof. There are certainly other applications of (1.1) in this context. We hope to come back to this somewhere else.

The proof of (1.1) will appear in section 3. It proceeds by an induction on the Postnikov factors of the spectrum $E$ and uses in a crucial way a property of products in the spectra $P(n)$ which is of independent interest and which we will discuss in section 2.

## 2. Products on the spectra $P(n)$ for $p=2$

Let BP be the Brown–Peterson spectrum at the prime $p$. Then $\mathrm{BP}_* = \pi_*(\mathrm{BP}) \cong \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$ where $|v_i| = 2(p^i - 1)$. For all $n \geq 0$ there are BP-module spectra $P(n)$ with $P(n)_* \cong \mathrm{BP}_*/I_n$ where $I_n = (p, v_1, \ldots, v_{n-1})$ denotes the $n$-th invariant prime ideal of $\mathrm{BP}_*$ (see [8] for details). For different $n$, the spectra $P(n)$ are related by exact triangles in the stable homotopy category **S**:

$$\begin{array}{ccccc} P(n) & \xrightarrow{v_n} & P(n) & \xrightarrow{\eta_n} & P(n+1) \\ \uparrow & & & & \big\downarrow \\ \multicolumn{5}{c}{\underleftarrow{\qquad\qquad \partial_n \qquad\qquad}} \end{array} \tag{2.1}$$

All maps displayed above are morphisms of BP-module spectra. Note that $P(0) = \mathrm{BP}$ by definition and that $P(1) \cong \mathrm{BP}\mathbb{Z}/_p$. Let $\mu_n : \mathrm{BP} \to P(n)$ denote the composition $\eta_{n-1} \circ \cdots \circ \eta_0$ and $\nu_n : \mathrm{BP} \wedge P(n) \to P(n)$ the module map.

Recall [19] that $\mathrm{BP}^*(\mathrm{BP})$ is isomorphic as left $\mathrm{BP}^*$-module to $\mathrm{BP}^* \hat{\otimes} R$ where $R$ denotes the free $\mathbb{Z}_{(p)}$-module with basis elements the Quillen operations $r_E$ indexed over exponent sequences $E = (e_1, e_2, \ldots, e_N, 0, \ldots)$. The degree of $r_E$ is $|E| = \sum_{i \geq 1} 2(p^i - 1)e_i$. Let $\Lambda(Q_0, \ldots, Q_{n-1})$ be the $\mathbb{Z}/_p$-exterior algebra with generators $Q_i$ of degree $2p^i - 1$. There is an isomorphism of left $P(n)^*$-modules (see [8], 2.12.)

$$\Phi_n : P(n)^* \hat{\otimes} R \otimes \Lambda(Q_0, \ldots, Q_{n-1}) \xrightarrow{\sim} P(n)^*(P(n)) \tag{2.2}$$

defined by the rule

$$\Phi_n(v^E \otimes r_F \otimes Q^{(\varepsilon_0, \ldots, \varepsilon_{n-1})}) = v^E (r_F)_n \circ (Q^{(\varepsilon_0, \ldots, \varepsilon_{n-2})})_n \circ Q_{n-1}^{\varepsilon_{n-1}}$$

where $Q_{n-1} = \eta_n \circ \partial_n$ and the operations $(r_F)_n$, $(Q^{(\varepsilon_0, \ldots, \varepsilon_{n-2})})_n$ are recursively defined as extensions of $(r_F)_{n-1}$, $(Q^{(\varepsilon_0, \ldots, \varepsilon_{n-3})})_{n-1} \circ Q_{n-2}$ to $P(n)^*(P(n))$ using [8, (2.9)]. In particular one has $Q_{n-1}^2 = 0$.

By an admissible product on $P(n)$ we mean a product $m_n : P(n) \wedge P(n) \to$

$P(n)$ which satisfies the following two conditions:

(2.3) (i) The diagram

$$\begin{array}{ccc} \mathrm{BP} \wedge \mathrm{BP} & \xrightarrow{\mu_n \wedge \mu_n} & P(n) \wedge P(n) \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m_n} \\ \mathrm{BP} & \xrightarrow{\mu_n} & P(n) \end{array}$$

commutes in **S**.

(ii) Set $\psi_n = (v_n \wedge v_n) \circ (id \wedge T \wedge id) : \mathrm{BP} \wedge \mathrm{BP} \wedge P(n) \wedge P(n) \to P(n) \wedge P(n)$. Then the diagram

$$\begin{array}{ccc} \mathrm{BP} \wedge \mathrm{BP} \wedge P(n) \wedge P(n) & \xrightarrow{\psi_n} & P(n) \wedge P(n) \\ {\scriptstyle id \wedge id \wedge m_n}\downarrow & & \downarrow{\scriptstyle m_n} \\ \mathrm{BP} \wedge \mathrm{BP} \wedge P(n) & \xrightarrow{v_n \circ (m \wedge id)} & P(n) \end{array}$$

commutes in **S**.

Written in terms of elements the second condition looks as follows: For all $u \in \mathrm{BP}^*(X)$, $v \in \mathrm{BP}^*(Y)$, $x \in P(n)^*(X)$, $y \in P(n)^*(Y)$ one has

$$(u \cdot x) \wedge_n (v \cdot y) = (-1)^{|v||x|}(u \wedge v) \cdot (x \wedge_n y).$$

(2.3) just means that an admissible product makes $P(n)$ a BP-algebra-spectrum compatible with the given BP-module structure $v_n$.

It is known that for $p$ odd, there is exactly one admissible product on $P(n)$ and this product is associative and commutative (see [18], (2.12)). If $p = 2$ the situation is as follows:

(2.4) PROPOSITION. *Suppose $p = 2$ and $n \geq 1$. Then there are exactly two admissible products $m_n, \bar{m}_n : P(n) \wedge P(n) \to P(n)$. Both are associative and have a two-sided unit. $m_n$ and $\bar{m}_n$ are related by the formula*

$$\bar{m}_n = m_n \circ T = m_n + v_n \cdot m_n(Q_{n-1} \wedge Q_{n-1}). \tag{2.5}$$

*Moreover, $\eta_{n-1} : P(n-1) \to P(n)$ is a map of ring spectra with respect to any admissible product chosen on $P(n-1)$ resp. $P(n)$.*

(2.6) REMARKS. (a) Proposition (2.4) is implicitly contained in [18] (see (3.7) of [18]), and was known to the author since 1977. Using completely different methods the formula (2.5) has been independently proved by Mironov [12] for a geometrically defined product on $P(n)$.

(b) Using (2.4) and the Conner–Floyd theorem mod $I_n$ one easily sees that an analogous statement is also true for $K(n)$, the $n$-th Morava $K$-theory. For $K^*(X; \mathbb{Z}/_2)$, complex $K$-theory mod 2, a formula like (2.5) has been proved by Araki–Toda in [3].

(c) Besides the application we will discuss in this paper, (2.4) has other uses. For example, it may be used to calculate the Morava $K$-theories for Eilenberg–MacLane spaces for the case $p = 2$ (see the appendix of [9]) and it should be possible to determine the algebra structure of $P(n)_*(P(n))$ for $p = 2$ with its help.

Before we give the proof of (2.4) let us recall some facts which will be used repeatedly.

(2.7) Lemma. *Let $E$ be a ring-spectrum. If $X$ is a connective spectrum such that $H_*(X; E_*)$ is free over $E_*$ and the spectral sequence $H_*(X; E_*) \Rightarrow E_*(X)$ collapses, then the evaluation*

$$\varepsilon : E^*(X) \to \mathrm{Hom}_{E^*}(E_*(X), E_*)$$

*is an isomorphism. If $X$ is a ring-spectrum, $\varepsilon$ induces a* 1-1*-correspondence between maps of ring-spectra $X \to E$ and homomorphisms of $E_*$-algebras $E_*(X) \to E_*$.*

*Proof.* See [1], (4.2) and the proof of (4.6).

Let $E$ be a complex-orientable ring spectrum (i.e. the canonical complex line bundle $\eta_\infty$ is $E$-orientable). Let $u \in E^2(\mathbb{C}P_\infty)$ be a $\mathbb{C}$-orientation and let $F$ denote the formal group associated to $u$. If $E^*$ is a $\mathbb{Z}_{(p)}$-algebra we may assume $F$ is $p$-typical. Then there is a unique map of ring-spectra $\mu_E : \mathrm{BP} \to E$ such that $\mu_E(u^{\mathrm{BP}}) = u$.

Observe that

$$[p]_F(x) = \mu_E\Big(\sum_{i \geq 0}{}^{F_{\mathrm{BP}}} v_i x^{p^i}\Big) = \sum_{i \geq 0}{}^{F} w_i x^{p^i} \tag{2.8}$$

where $w_i = \mu_E(v_i)$ and the $v_i$ are Araki-generators [2]. If $\mu_E(I_n) = (p, w_1, \ldots, w_{n-1}) = (0) \subset E^*$ we obtain for $0 \leq k \leq n$ using (2.1) an isomorphism of $E_*$-modules

$$E_*(P(k)) \cong E_*(\mathrm{BP}) \otimes \Lambda(a_0, \ldots, a_{k-1}). \tag{2.9}$$

If $p$ is odd, (2.9) is an isomorphism of $E_*$-algebras [18].

*Proof of* (2.4). We assume the reader is familiar with [18]. Let $\mathrm{Prod}_n$ denote the set of admissible products on $P(n)$. In [18] it was shown that $P(n)^*(\mathrm{BP} \wedge \mathrm{BP})$ is a coalgebra with structure map $\psi_0^*$ and that $P(n)^*(P(n) \wedge P(n))$ is a $P(n)^*(\mathrm{BP} \wedge \mathrm{BP})$-comodule with structure map $\psi_n^*$. Moreover, there is a 1-1-correspondence between the set $\mathrm{Prod}_n$ and the set of primitive elements $a \in P(n)^0(P(n) \wedge P(n))$ which satisfy $(\mu_n \wedge \mu_n)^*(a) = \mu_n m_0 \in P(n)^0(\mathrm{BP} \wedge \mathrm{BP})$. But ([18], (4.12)) there is an isomorphism of $P(n)^*$-modules

$$\Pr P(n)^*(P(n) \wedge P(n)) \cong \Lambda_{P(n)^*}(Q_0', \ldots, Q_{n-1}', Q_0'', \ldots, Q_{n-1}'')$$

and the same proof as for [18], (4.13) shows that in the case $p = 2$ the sparseness of the ring $P(n)^*$ implies that there are two elements $m_n, \bar{m}_n$ in $\mathrm{Prod}_n$ corresponding to 1 and $1 + v_n Q_{n-1}' Q_{n-1}''$. $m_n$ induces an isomorphism ([18], (3.8))

$$P(n)^*(P(n)) \hat{\otimes} P(n)^*(P(n)) \xrightarrow{\sim} P(n)^* P(n) \wedge P(n))$$

from which the relation

$$\bar{m}_n = m_n + v_n m_n (Q_{n-1} \wedge Q_{n-1})$$

follows. Associativity and existence of a two-sided unit for both products $m_n$ and $\bar{m}_n$ are proved as in [18], (5.4), as is the multiplicativity of $\mu_n$. It thus rests to show that $\bar{m}_n = m_n \circ T$. It is obvious that $\alpha \in \mathrm{Prod}_n$ implies $\alpha \circ T \in \mathrm{Prod}_n$. So either both admissible products are commutative or none of them. In the latter case we obtain $\bar{m}_n = m_n \circ T$ and the proposition is proved. Suppose $m_n$ and $\bar{m}_n$ are both commutative and write $P(n)$ (resp. $\bar{P}(n)$) for the spectrum $P(n)$ endowed with the multiplication $m_n$ (resp. $\bar{m}_n$). Adapting the arguments of ([18], section 6) to the case $p = 2$ one sees using the commutativity of $m_n$ that for any complex-oriented commutative ring spectrum $E$ of characteristic 2 and 2-typical formal group of height $\geqslant n$ (i.e. $\mu_E(I_n) = 0$) there is an isomorphism of $E_*$-algebras

$$E_*(P(n)) \cong E_*[a_1, \ldots, a_n, b_{n+1}, \ldots]$$

where $b_j = (\mu_E \wedge \mu_E)_*(t_j)$, $a_i^2 = (\mu_E \wedge \mu_E)_*(t_i)$, $i = 1, \ldots, n$. From (2.7) it follows that there is a map of ring-spectra $\theta : P(n) \to E$ given by $\varepsilon(\theta)(a_i) = 0$, $\varepsilon(\theta)(b_j) = 0$, $\varepsilon(1) = 1$. Clearly, $\theta$ is an extension of the map $\mu_E$. This applies in particular to the case $E = \bar{P}(n)$. Because $m_n$ and $\bar{m}_n$ are both admissible products, $\theta$ is a map of BP-module-spectra $P(n) \to P(n)$. From the obvious BP-version of [16], (4.17) it follows that there is one and only one such map. Hence $\theta = id_{P(n)}$ and so $m_n = \bar{m}_n$, a contradiction. This means there can be no commutative product in $\mathrm{Prod}_n$ and the proposition follows.

We close this section by stating two lemmas which will be used for the proof of Theorem (1.1) in section 3.

(2.10) LEMMA. *Let $E$ be a commutative $\mathbb{C}$-oriented ring-spectrum with coefficient ring of characteristic $p>0$ and $p$-typical formal group. If $\mu_E(I_n)=(0)$, then for all $0\leqslant k\leqslant n$ there is a map $\theta_k: P(k)\to E$ such that $\theta_0=\mu_E$ and the diagrams*

$$\begin{array}{ccc} P(k) & \xrightarrow{\theta_k} & E \\ {\scriptstyle \eta_{k-1}}\uparrow & \nearrow{\scriptstyle \theta_{k-1}} & \\ P(k-1) & & \end{array} \tag{2.11}$$

*commute. If $p$ is odd or if $p=2$ and $k<n$, the $\theta_k$ may be chosen to be maps of ring-spectra.*

*Proof.* From (2.9) we have an isomorphism of $E_*$-modules

$$E_*(P(k))\cong E_*[t_1, t_2, \ldots]\otimes\Lambda(a_0, \ldots, a_{k-1}). \tag{2.12}$$

For $p$ odd this is an isomorphism of $E_*$-algebras and by (2.7) the lemma follows for this case if we define $\theta_k$ by $\varepsilon(\theta_k)(t_i)=0$ $(i>0)$, $\varepsilon(\theta_k)(a_j)=0$ $(j=0, \ldots, k-1)$ and $\varepsilon(\theta_k)(1)=1$. Using the same method one also obtains maps $\theta_k$ such that (2.11) commutes for $p=2$. To show that the $\theta_k$ may be chosen to be maps of ring-spectra for $k<n$ one observes that because $w_k=\mu_E(v_k)=0$ for $k<n$, Proposition (2.4) implies that $E_*(P(k))$ is a commutative $E_*$-algebra if $k<n$. Now adapting the argument of [18], section 6, to this situation one gets an isomorphism of $E_*$-algebras

$$E_*(P(k))\cong E_*[a_0, \ldots, a_{k-1}, t_{k+1}, t_{k+2}, \ldots]$$

with $|a_i|=2^{i+1}-1$. As above for $p$ odd one now sees that for $k<n$ the $\theta_k$ may be chosen to be maps of ring spectra.

(2.13) LEMMA. *Let $E$ be a complex-oriented ring-spectrum with coefficient ring of characteristic $p>0$ and formal group $F$. If $[p]_F(X)=0$ there is an isomorphism of ring-theories*

$$\Phi: H^*(X; E^*)\xrightarrow{\sim} E^*(X)$$

*on the category* $\mathbf{CW}_*$.

*Proof.* Because $E^*$ is a $\mathbb{Z}_{(p)}$-algebra we may assume that $F$ is $p$-typical. Because $[p]_F(X) = 0$, $\mu_E(I_\infty) = 0$ and it follows from (2.10) that for all $k > 0$ there are maps of ring-spectra $\theta_k : P(k) \to E$ such that $\theta_k \circ \eta_{k-1} = \theta_{k-1}$. The $\theta_k$ may be seen to produce a map of ring spectra $\theta_\infty : H\mathbb{Z}/_p \cong \lim_{\to} P(k) \to E$. Then $\Phi = \theta_\infty \otimes 1 : H^*(X; \mathbb{Z}/_p) \otimes E^* \to E^*(X)$ is an equivalence of ring-theories over the category of finite complexes which is easily seen to extend to an equivalence over $\mathbf{CW}_*$.

(2.14) *Remark.* With the additional hypothesis that $E^\circ$ is a finite field (2.13) has been proved in [17] by different methods. Notice also that by a theorem of J. M. Boardman [4], if $E$ is a ring spectrum with $\pi_*(E)$ a free $\mathbb{Z}/_p$-module which is equivalent to a graded Eilenberg–MacLane spectrum $H(\pi_*(E))$, then there is automatically an equivalence of ring-spectra $E \cong H(\pi_*(E))$.

## 3. Proof of Theorem (1.1)

First we observe that we may assume the spectrum $E$ is $(-1)$-connected:

(3.1) Lemma. *Let $E$ be a ring-spectrum such that $p \cdot 1 = 0 \in E^0$ for some prime $p$ and let $\tilde{E}$ denote the $(-1)$-connected cover of $E$. Then any isomorphism of ring theories $\tilde{\Phi} : H^*(X; \tilde{E}^*) \to \tilde{E}^*(X)$ induces an isomorphism of ring theories $\Phi : H^*(X; E^*) \to E^*(X)$.*

*Proof.* The $(-1)$-connected cover $\tilde{E}$ of a ring-spectrum $E$ is again a ring-spectrum and there is a canonical map of ring-spectra $\pi : \tilde{E} \to E$. Define a multiplicative transformation $\phi : H^*(X; \mathbb{Z}/_p) \to E^*(X)$ by the composition

$$\phi : H^*(X; \mathbb{Z}/_p) \xrightarrow{i} H^*(X; \tilde{E}^*) \xrightarrow{\tilde{\Phi}} \tilde{E}^*(X) \xrightarrow{\pi} E^*(X).$$

For $X = S^0$, $\phi$ is just the inclusion $\mathbb{Z}/_p \subset E^0 \subset E^*$. $\phi$ induces a multiplicative transformation of cohomology theories on the category $\mathbf{CW}^f_*$ of finite complexes $\Phi : H^*(X; E^*) \cong H^*(X; \mathbb{Z}/_p) \otimes E^* \to E^*(X)$. Because $\Phi|_{S^0}$ is an isomorphism, $\Phi$ is an equivalence. It is easy to see that $\Phi$ extends to an equivalence on the category $\mathbf{CW}_*$.

For $E$ a $(-1)$-connected spectrum, the proof of (1.1) will proceed by an induction on the Postnikov factors of $E$ using Lemma (2.13) and Proposition (2.4). Recall from [7] that for any spectrum $E$ and any $k \in \mathbb{Z}$ the $k$-th Postnikov factor $E[k]$ of $E$ may be defined as representing spectrum of the cohomology theory

$$\mathbf{CW}_* \ni X \mapsto E[k]^q(X) := \operatorname{im}\{E^q(X^{k+q+1}) \to E^q(X^{k+q})\}. \tag{3.2}$$

The coefficients of $E[k]^*(-)$ are given by

$$E[k]^q = \begin{cases} E^q, & q \geqslant -k \\ 0, & q < -k. \end{cases} \tag{3.3}$$

There are obvious morphisms $\pi_k : E \to E[k]$ which on the coefficients induce the identity for $q \geqslant -k$ and zero otherwise. For different $k$, the Postnikov factors are related by exact sequences

$$\cdots \to E[k]^{q-1}(X) \xrightarrow{\Delta} H^{q+k+1}(X; E^{-k-1}) \to E[k+1]^q(X) \xrightarrow{\xi_k}$$

$$\to E[k]^q(X) \xrightarrow{\Delta} H^{q+k+2}(X; E^{-k-1}) \to \cdots.$$

If $E$ is a ring-spectrum the Postnikov-factors $E[k]$ are in general not ring-spectra. However, if $E$ is a $(-1)$-connected ring-spectrum the $E[k]$ are also ring-spectra and the $\pi_k$, $\xi_k$ are maps of ring-spectra (see [15]).

Let $\eta_k$ denote the canonical complex line bundle over $\mathbb{C}P_k$.

(3.5) Lemma. *Let $E$ be a $(-1)$-connected ring-spectrum, $k \geqslant 0$. Then* a) *$\eta_\infty$ is $E[2k]$-orientable iff $\eta_k$ is $E$-orientable* b) *$\eta_\infty$ is $E[2k+1]$-orientable iff $\eta_{k+1}$ is $E$-orientable.*

*Proof.* Let $s = \Sigma^2(1) \in E^2(S^2) = E^2(\mathbb{C}P_1) = E[k]^2(\mathbb{C}P_1)$. $\eta_k$ is $E$-orientable if and only if there is an element $u_k \in E^2(\mathbb{C}P_{k+1})$ such that $u_k \mid \mathbb{C}P_1 = \pm s$. One has $\mathbb{C}P_\infty^{2l} = \mathbb{C}P_l = \mathbb{C}P_\infty^{2l+i}$, so from (3.2) we get

$$E[2k]^2(\mathbb{C}P_\infty) = \operatorname{im}\{E^2(\mathbb{C}P_{k+1}) \xrightarrow{id} E^2(\mathbb{C}P_{k+1})\} = E^2(\mathbb{C}P_{k+1})$$

$$E[2k+1]^2(\mathbb{C}P_\infty) = \operatorname{im}\{E^2(\mathbb{C}P_{k+2}) \to E^2(\mathbb{C}P_{k+1})\}.$$

The lemma follows.

The next lemma is the main step in the proof of Theorem (1.1). For any $n \geqslant 0$ put $\bar{n} = 2(2^n - 1)$.

(3.6) Lemma. *Let $E$ be a $(-1)$-connected commutative ring-spectrum with coefficient ring of characteristic 2. If $E[\bar{n}]$ is $\mathbb{C}$-orientable and $[2]_{F_{\bar{n}}}(x) = 0$, then $E[\overline{n+1}]$ is also $\mathbb{C}$-orientable and $[2]_{F_{\overline{n+1}}}(x) = 0$.*

*Proof.* The proof may be divided into two steps. (A): $E[\overline{n+1}]$ is $\mathbb{C}$-orientable and (B): $[2]_{F_{\overline{n+1}}}(x) = 0$.

(A): From Lemma (3.5) we see that we must prove the bundle $\eta_{(\overline{n+1})/2}$ is $E$-orientable. Let $u_k \in E^2(\mathbb{C}P_{k+1})$ be an $E$-orientation of $\eta_k$ and let $\gamma: S^{2(k+1)+1} \to \mathbb{C}P_{k+1}$ be the Hopf map. Then $\eta_{k+1}$ is $E$-orientable iff $\gamma^*(u_k) = 0 \in E^{1-2(k+1)}$. From [6, (6.8)] we learn that this obstruction is zero if either $k+2$ is not a power of a prime $p$ or if $k+2 = p^r$ for some prime $p$ and $1/p \in E^*$. In our case $2 \cdot 1 = 0$ in $E^*$, hence $1/p \in E^*$ for all odd primes $p$. So a possibly non-zero obstruction for extending an orientation of $\eta_k$ to an orientation of $\eta_{k+1}$ can only occur if $k+2 = 2^r$. The hypothesis of (3.6) and Lemma (3.5) imply that $\eta_{\bar{n}/2}$ is $E$-orientable. From the remarks above it follows that then $\eta_{(\overline{n+1})/2-1} = \eta_{2^{n+1}-1}$ is also $E$-orientable, so $E[(\overline{n+1})-2]$ is $\mathbb{C}$-orientable. By (3.5) to prove the $\mathbb{C}$-orientability of $E[\overline{n+1}]$ it will suffice to show that $E[\overline{n+1}-1]$ is $\mathbb{C}$-orientable.

Let $E_r^{p,q}$ denote the Atiyah–Hirzebruch spectral sequence for $E$ and let ${}^kE_r^{p,q}$ denote the corresponding spectral sequence for the $k$-th Postnikov factor of $E$. Let $\sigma \in H^2(\mathbb{C}P_\infty; E^0) \cong E_2^{2,0}(\mathbb{C}P_\infty) \cong {}^kE_2^{2,0}(\mathbb{C}P_\infty)$ be a $\mathbb{C}$-orientation of $H^*(-; E^0)$. The obvious cohomological version of a result of Vick [15] implies that

(i) $E[k]$ is $\mathbb{C}$-orientable iff $\sigma$ is a permanent cycle in the spectral sequence ${}^kE_r^{p,q}(\mathbb{C}P_\infty)$ and

(ii) $\sigma$ is a permanent cycle in ${}^kE_r^{p,q}(\mathbb{C}P_\infty)$ iff $\sigma$ survives to the $k$-th stage in the spectral sequence $E_r^{p,q}(\mathbb{C}P_\infty)$.

Because $E[\overline{n+1}-2]$ is $\mathbb{C}$-orientable, $\sigma$ survives to $E^{2,0}_{\overline{n+1}-2}$. Put $d = d_{\overline{n+1}-2}$. To prove the $\mathbb{C}$-orientability of $E[\overline{n+1}-1]$ it is sufficient to show $d(\sigma) = 0$.

Let $\xi: E[\overline{n+1}-2] \to E[\bar{n}]$ be the obvious map. Let $u$ be a $\mathbb{C}$-orientation of $E[\overline{n+1}-2]$ with 2-typical formal group $F$ and set $G = \xi_*(F)$. For dimensional reasons we have $[2]_F(x) = \sum_{i=1}^{n}{}^F w_i \cdot x^{2^i}$, $|w_i| = -2(2^i - 1)$. Because $\xi$ induces the identity on the coefficients for dimensions $\geqslant -\bar{n} = -2(2^n - 1)$, we obtain from the assumptions of (3.6)

$$\xi_*[2]_F(x) = [2]_G(x) = \sum_{i=1}^{n}{}^G w_i x^{2^i} = 0,$$

hence $[2]_F(x) = 0$. By Lemma (2.12) this implies there is an equivalence of ring theories $E[\overline{n+1}-2]^*(-) \cong H^*(-; E[\overline{n+1}-2]^*)$. From the Gysin sequence of the bundle $\eta \otimes \eta$ one sees that the projection $p: S(\eta_\infty \otimes \eta_\infty) \simeq \mathbb{R}P_\infty \to \mathbb{C}P_\infty$ induces a monomorphism $p^*: H^*(\mathbb{C}P_\infty; E[\overline{n+1}-2]^*) \to H^*(\mathbb{R}P_\infty; E[\overline{n+1}-2]^*)$. We may assume that $\sigma$ has been chosen so that for some generator $\alpha \in H^1(\mathbb{R}P_\infty; E^0)$, $p^*(\sigma) = \alpha^2$. Now $d$ is a derivation, so $d(\alpha^2) = d(\alpha) \cdot \alpha + \alpha \cdot d(\alpha) = 0$ because $E$ is assumed to be commutative and $2 \cdot E^* = 0$. Since $p^*$ is monic we obtain $d(\sigma) = 0$.

(B): Let us write $T$ for $\overline{E[n+1]}$. From (A) we know that $T$ is $\mathbb{C}$-orientable. Choose a $\mathbb{C}$-orientation with typical formal group $F = F_{\overline{n+1}}$ and let $\mu_T : \mathrm{BP} \to T$ be the canonical map. From our assumptions we know that $[2]_{F_{\bar n}}(x) = 0$, hence $\mu_T(v_i) = w_i = 0$ for $0 \leq i \leq n$ and $[2]_F(x) = w_{n+1}x^{2^{n+1}}$. The lemma will be proved if we can show $w_{n+1} = 0$. By (2.10) there are maps $\theta_k (0 \leq k \leq n+1)$ such that the diagrams

$$\begin{array}{ccc} P(k+1) & \xrightarrow{\theta_{k+1}} & T \\ {\scriptstyle \eta_k}\uparrow & \nearrow{\scriptstyle \theta_k} & \\ P(k) & & \end{array}$$

commute and $\theta_k$ is a map of ring-spectra for $k \leq n$. There is an isomorphism

$$T^* \underset{P(k)^*}{\otimes} P(k)^*(P(k)) \xrightarrow{\sim} T^*(P(k))$$

given by $t \otimes \alpha \mapsto t \cdot \theta_k(\alpha)$ and the product $m_k$ on $P(k)$ induces a homomorphism

$$m_k^* : T^*(P(k)) \to T^*(P(k) \wedge P(k)) \cong T^*(P(k)) \underset{T^*}{\hat{\otimes}} T^*(P(k)).$$

Set $\theta = \theta_{n+1}$. Using (2.2), the relation $(\eta_n^* \hat{\otimes} \eta_n^*) m_{n+1}^*(\theta) = \theta_n \hat{\otimes} \theta_n$ and the fact that $T^* = 0$ for $* < -n+1$ we obtain the expansion (we write $m$ for $m_{n+1}$ and $Q$ for $Q_n \in P(n+1)^{2^{n+1}-1}(P(n+1))$

$$m^*(\theta) = \theta \hat{\otimes} \theta + \lambda\theta(Q) \hat{\otimes} \theta(Q) + \sum_i \xi_i \theta(\alpha_i) \hat{\otimes} \theta(\beta_i Q) + \sum_j \tau_j \theta(\alpha_j' Q) \hat{\otimes} \theta(\beta_j') \tag{3.7}$$

where $\alpha_i, \alpha_i', \beta_j, \beta_j' \in \Phi_n(R \otimes \Lambda(Q_0, \ldots, Q_{n-1})) \subset P(n+1)^*(P(n+1))$ and $\lambda, \xi_i, \tau_j \in T^*$. Because $T$ is commutative we obtain ($t : P(n+1) \wedge P(n+1) \to P(n+1) \wedge P(n+1)$ denotes the switch map)

$$t^* m^*(\theta) = \theta \hat{\otimes} \theta + \lambda\theta(Q) \hat{\otimes} \theta(Q) + \sum_i \xi_i \theta(\beta_i Q) \hat{\otimes} \theta(\alpha_i) + \sum_i \tau_j \theta(\beta_j') \hat{\otimes} \theta(\alpha_j' Q). \tag{3.8}$$

From (2.4) we know that the diagram

$$\begin{array}{ccc} P(n+1)\wedge P(n+1) & \overset{\phi}{\searrow} & \\ {\scriptstyle t}\downarrow & & P(n+1) \\ P(n+1)\wedge P(n+1) & \underset{m}{\nearrow} & \end{array}$$

commutes where $\phi = m + v_{n+1}\cdot m(Q\wedge Q)$. But

$$\begin{aligned}\phi^*(\theta) &= m^*(\theta) + w_{n+1}\cdot(Q^*\hat{\otimes}Q^*)m^*(\theta)\\ &= m^*(\theta) + w_{n+1}\theta(Q)\hat{\otimes}\theta(Q)\end{aligned}$$

because $Q^2 = 0$. Since $\phi^*(\theta) = t^*m^*(\theta)$ we obtain from (3.7) and (3.8)

$$\begin{aligned} w_{n+1}\cdot\theta(Q)\hat{\otimes}\theta(Q) &= \sum_i \xi_i\theta(\alpha_i)\hat{\otimes}\theta(\beta_iQ) + \sum_j \tau_j\theta(\alpha_j'Q)\hat{\otimes}\theta(\beta_j')\\ &\quad + \sum_i \xi_i\theta(\beta_iQ)\hat{\otimes}\theta(\alpha_i) + \sum_j \tau_j\theta(\beta_j')\hat{\otimes}\theta(\alpha_j'Q)\\ &=: A + B + A' + B'. \end{aligned}\tag{3.9}$$

Applying $\eta_n^*\hat{\otimes}id$ to both sides of (3.9) we get (remember: $Q\circ\eta_n \simeq 0$)

$$0 = (\eta_n^*\hat{\otimes}id)(A) + (\eta_n^*\hat{\otimes}id)(B')$$

which implies $A + B' = 0$. Similarly, applying $id\hat{\otimes}\eta_n^*$, we get

$$0 = (id\hat{\otimes}\eta_n^*)(B) + (id\hat{\otimes}\eta_n^*)(A')$$

which implies $B + A' = 0$. Together one obtains

$$w_{n+1}\theta(Q)\hat{\otimes}\theta(Q) = 0$$

and this is only possible if $w_{n+1} = 0$. This proves the lemma.

The proof of Theorem (1.1) is now easy: First we observe that $E[0] = H(E^0)$ is clearly $\mathbb{C}$-orientable and satisfies $[2]_{F_0}(X) = 0$. So from (3.6) it follows inductively that the same is true for all Postnikov factors of $E$. This easily implies that $E$ is $\mathbb{C}$-orientable and $[2]_F(X) = 0$. So (1.1) follows from (2.13).

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