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Rigidity for surfaces of non-positive curvature

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Introduction

In this paper we consider the question of boundary rigidity for surfaces of nonpositive curvature. Given a compact manifold, M, with smooth boundary N, a riemannian metric g_0 on M induces a nonnegative real valued function, d_0, on $N \times N$ where $d_0(p, q)$ is the distance in (M, g_0) between p and q. A riemannian manifold (M, g_0) is called boundary rigid if for any riemannian manifold (M_1, g_1) with the same boundary, N, if $d_1 = d_0$ then g_1 is isometric to g_0. This question was recently considered by the author in [C] where one was led to the quesiton: "Are all SGM manifolds boundary rigid?". The condition SGM is a condition on the boundary distance function d_0 which roughly speaking is equivalent to the condition that all geodesic segments in M are the unique minimizing paths between the endpoints (see [C] for a precise definition.) By geodesic segments we mean geodesics that intersect the boundary at most at the boundary points (i.e. they do not "graze" the boundary at interior points of the segment.) Any compact subdomain in the interior of a convex manifold with possibly empty boundary (i.e. between any two points there is a unique geodesic) will be SGM. Hence, in particular, any subdomain of a complete simply connected manifold of nonpositive curvature will be SGM. Also any disk of nonpositive curvature will be SGM. In this paper we show:

THEOREM A. *If (M_0^2, g_0) is a compact, nonpositively curved, SGM, surface with boundary then it is boundary rigid.*

It should be emphasized that no assumptions are made a-priori about the curvature (or even the topology) of the possible (M_1, g_1). Other manifolds are known to be boundary rigid. It has been shown by Gromov and Michel (see [G] sec. 5.5B and [M]) that (M_0, g_0) is boundary rigid in any of the following three cases: (1) M_0^n admits an isometric immersion into $\mathbb{R}^n$, (2) M_0^n admits a 1-1 immersion into a convex subset of the round n-sphere, (3) M_0^2 admits a 1-1 immersion into the hyperbolic plane. All of the above three cases are for manifolds of constant curvature.

The reader is referred to [C] for a more extensive history of this problem as well as its relationship to other problems such as the uniqueness of "geodesic lenses". Also in [C] the reader will find a case made for the condition SGM through examples that are not boundary rigid.

A problem related to the above boundary rigidity problem is the question of compact manifolds without boundary whose geodesic flows are conjugate. We will say that M_0 and M_1 have conjugate geodesic flows via F if F is a C^1 diffeomorphism, $F: UM_1 \to UM_0$, between the unit tangent bundles which commutes with the geodesic flows i.e. $\zeta_0^t \circ F = F \circ \zeta_1^t$ for all t where ζ_i^t is the goedesic flow (for time t) on UM_i.

THEOREM B. *If M_0 is a compact surface (without boundary) of genus ≥ 2 with non-positive sectional curvature and M_1 is a compact surface whose geodesic flow is conjugate via F to M_0 then $F = \zeta_0^t \circ dI$ (or $dI \circ \zeta_1^t$), where I is an isometry from M_1 to M_0 and t is a fixed number.*

We emphasize that in this theorem as well there are no a-priori assumptions about the compact surface M_1.

The question of geodesic conjugacy has come up in many contexts recently. In particular the recent work of Feres and Katok [F–K] extending the results of Kanai [K] shows that if M is a compact manifold of negative quarter pinched curvature such that at least one of the horospheric foliations is smooth then the geodesic flow on M is smoothly conjugate to the geodesic flow on a manifold of constant negative curvature. Hence a higher dimensional version of theorem B would answer part of a long standing conjecture.

In the case that both M_0 and M_1 are surfaces of negative curvature they will have conjugate geodesic flows if and only if they have the same marked length spectrum (see [B–K] sec. 10 and [F–O]) and hence by the above they will be isometric if and only if they have the same marked length spectrum. The marked length spectrum for a surface of negative curvature is the function that takes elements of π_1 (or conjugacy classes) to the length of the shortest closed geodesic in the free homotopy class. The length spectrum (the image of the above function) is not enough to determine a surface of negative curvature up to isometry as was shown by Vignéras [V] (also see [Su]) who gave examples of two nonisometric surfaces of constant negative curvature -1 with the same eigenvalue spectrum and hence (by our curvature conditions) the same length spectrum (see [D–G] or [CV]).

The fact that two surfaces of negative curvature are isometric if they have the same marked length spectrum was proved independently (and apparently some months earlier than the author) by Otal [O1] and had been conjectured in [B–K]. A result similar to Theorem A was also proved independently by Otal [O2]. The methods used in Otal's papers are different from the ones used here and the results stated here are more general. In particular Otal makes additional assumptions about the metric of M_1 in both his theorems (some of these assumptions are not hard to drop) and he needs to assume negative rather than nonpositive curvature. An earlier version of theorem A (with additional assumptions) can be found in [G–N].

In the final section of this paper we discuss the case where M_0 has genus 1 (i.e. is a flat torus.) We show that M_1 must be isometric to M_0 but F need not be of the form $\zeta_0^{\prime} \circ dI$.

It should be pointed out that for general surfaces there is no theorem like Theorem B. In particular Zoll surfaces have geodesic flows that are conjugate to the geodesic flow on the round sphere (see [W]).

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I. Preliminaries

We begin with an analytic lemma that will be used in the proof of both Theorem A and Theorem B.

LEMMA 1.1. *Let j and $\bar{j}$ be positive real valued continuous functions defined on intervals of $\mathbb{R}^1$. For constants C_1 and C_2 with $C_2 > 0$ define $f: [a, b] \to [\bar{a}, \bar{b}]$ by:*

$$C_2 \cdot \int_a^{f(t)} \frac{ds}{\bar{j}^2(s)} + C_1 = \int_a^t \frac{ds}{j^2(s)} \tag{i}$$

where j is assumed to be defined at least on $[a, b]$ and $\bar{j}$ on $[a, \bar{b}] \cup [\bar{a}, \bar{b}]$. Then we have:

$$\int_a^b \frac{C_2 \cdot j(t)}{\bar{j}(f(t))} dt \geq \left[\frac{(b-a)^3 \cdot C_2}{(\bar{b}-\bar{a})}\right]^{1/2}$$

with equality if and only if

$$f(t) = \frac{\bar{b}-\bar{a}}{b-a}(t-a) + \bar{a} \quad \text{and} \quad \frac{j(t)}{\bar{j}(f(t))} = \left[\frac{(b-a)}{C_2(\bar{b}-\bar{a})}\right]^{1/2}.$$

Proof. Differentiating (i) with respect to t we see that

$$f'(t) = \frac{\bar{j}^2(f(t))}{C_2 \cdot j^2(t)}.$$

Hence using the substitution $u = f(t)$ gives:

$$\int_a^b \frac{C_2 \cdot j(t)}{\bar{j}(f(t))}\, dt = \int_{\bar{a}}^{\bar{b}} \frac{C_2^2 \cdot j^3(f^{-1}(u))}{\bar{j}^3(u)}\, du$$

(Note that $C_2 > 0$ implies $f'(t) > 0$ and hence that $f^{-1}(u)$ is well defined.) A Hölder inequality applied to the right hand side, RHS, of the above yields:

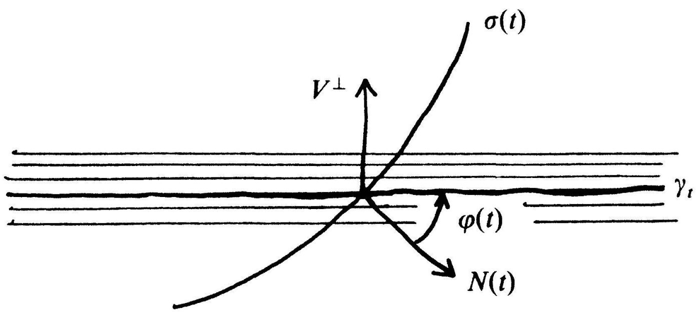
$$[\text{RHS}]^{2/3} \cdot [\bar{b} - \bar{a}]^{1/3} \geq \int_{\bar{a}}^{\bar{b}} \frac{C_2^{4/3} \cdot j^2(f^{-1}(u))}{\bar{j}^2(u)}\, du = C_2^{1/3} \cdot (b - a). \qquad \text{(ii)}$$

The equality above comes from the substitution $t = f^{-1}(u)$. The inequality in (ii) will be equality if and only if $j(f^{-1}(u))/(\bar{j}(u))$ is a constant, say F. Rearranging (ii) yields the inequality in the lemma. If equality holds then we see that $C_2 \cdot F \cdot (b - a) = [C_2(b - a)^3/(\bar{b} - \bar{a})]^{1/2}$ and hence $F = [(b - a)/\{C_2(\bar{b} - \bar{a})\}]^{1/2}$. Further our computation of $f'(t)$ yields in the equality case $f'(t) = 1/(C_2 \cdot F^2) = (\bar{b} - \bar{a})/(b - a)$. These results plus the fact that $f(a) = \bar{a}$ yield the equality case in the lemma. ■

In both applications of the lemma C_2 will be 1.

The next lemma will help in interpreting the boundary term in the Gauss-Bonnet theorem in two dimensions.

Let $H : (-\varepsilon, \varepsilon) \times (a, b) \to M^2$ be a C^2-differentiable variation of unit speed geodesics in M. That is, for each fixed t, $\gamma_t(s) = H(t, s)$ is a unit speed geodesic in M. Let $h : (-\varepsilon, \varepsilon) \to (a, b)$ be a function such that $\sigma(t) = H(t, h(t))$ is an embedded C^2-differentiable curve transverse to γ_t for all t near 0. Let $N(t)$ be the continuous unit normal to σ at t near 0 with $\langle \gamma_t'(h(t)), N(t) \rangle > 0$. We now let $Kg(t)$ be the geodesic curvature of $\sigma(t)$ with respect to $N(t)$ and $\varphi(t) \in (-\pi/2, \pi/2)$ be the angle between $N(t)$ and $\gamma_t'(h(t))$. Let $J_t(s)$ be the variation field (Jacobi field) along γ_t of this variation. Let $V^\perp = -\sin(\varphi(t)) \cdot N(t) + \cos(\varphi(t)) \cdot \sigma'(t)/|\sigma'(t)|$ be a unit vector perpendicular to $\gamma_t'(h(t))$ (see figure.)



LEMMA 1.2. *In the situation described above we have*:

$$\nabla_{\gamma_t'(h(t))} J_t = \left(\frac{d\varphi}{dt} - Kg(t) \cdot |\sigma'(t)|\right) \cdot V^{\perp}$$

Proof. We first claim that we may assume that $h(t) = 0$ for all t near 0. Consider the new variation $H^N(t, s) = H(t, s + h(t))$, then $\sigma(t) = H^N(t, 0)$ and the new variation field J_t^N satisfies $J_t^N(s) = J_t(s - h(t)) + h'(t)\gamma_t'(s - h(t))$ and hence its covariant derivative in γ_t' direction agrees with that of J_t. Hence we may assume that $h(t) = 0$.

We let $T(t) = \sigma'(t)/|\sigma'(t)|$ then $\gamma_t'(0) = \cos(\varphi(t)) \cdot N(t) + \sin(\varphi(t)) \cdot T(t)$. Thus

$$\begin{aligned} \nabla_{\gamma_t'(0)} J_t(s)|_{s=0} = \nabla_{J_t(0)}\gamma_t'(0) = \nabla_{\sigma'(t)}\gamma_t'(0) &= -\sin(\varphi(t)) \cdot \frac{d\varphi}{dt} \cdot N(t) \\ &+ |\sigma'(t)| \cdot \cos(\varphi(t)) \cdot \nabla_{T(t)} N(t) + \cos(\varphi(t)) \cdot \frac{d\varphi}{dt} \cdot T(t) \\ &+ |\sigma'(t)| \cdot \sin(\varphi(t)) \cdot \nabla_{T(t)} T(t). \end{aligned}$$

Now from $\nabla_T T = Kg \cdot N$ and $\nabla_T N = -Kg \cdot T$ we find that $\nabla_{\gamma_t'(0)} J_t(s)|_{s=0}$ is

$$\begin{aligned} &\sin(\varphi(t)) \cdot \left\{\frac{-d\varphi}{dt} + Kg(t) \cdot |\sigma'(t)|\right\} \cdot N(t) + \cos(\varphi(t)) \cdot \left\{\frac{d\varphi}{dt} - Kg(t)|\sigma'(t)|\right\} \cdot T(t) \\ &= \left\{\frac{d\varphi}{dt} - Kg(t)|\sigma'(t)|\right\} \cdot V^{\perp} \end{aligned}$$

and the lemma follows. ■

For $\theta \in S^1$ let $a(\theta) < b(\theta)$ be bounded functions such that a is continuous and b is C^1-differentiable for all but a finite set $\{\theta_1, \theta_2, \ldots, \theta_k\}$ where the derivatives have left and right limits (b need not be continuous at θ_i.) Let

$$Q = \text{closure}\{(\theta, s) | a(\theta) \le s \le b(\theta)\} \subset S^1 \times \mathbb{R}^1.$$

Q has two boundary components

$\partial_0 = \{(\theta, a(\theta))\}$ and $\partial_1 = \{(\theta, b(\theta))\} \cup_{i=1}^k \{(\theta_i, s) | s$ is in the interval between the two half limits of b at $\theta_i\}$.

Let $H : Q \to M^2$ be a map into a two dimensional riemannian manifold with the following properties:

(i) Each curve $\gamma_\theta(s) = H(\theta, s)$ is a unit speed geodesic in M.

(ii) On the interior of Q, H is a C^1 immersion.

(iii) The image $H(Q)$ is a manifold whose boundary is the 1-1 image of ∂_1.

(iv) The image of ∂_0 lies in the interior of $H(Q)$.

We will let $J(\theta, s)$ be the variation field $H_*(d/d\theta)$. Hence for fixed θ, $J(\theta, s)$ is a Jacobi field along γ_θ. We also choose a unit normal field $\gamma_\theta^\perp$ along each geodesic γ_θ which we assume has $\langle J(\theta, s), \gamma_\theta^\perp \rangle > 0$ for $a(\theta) < \theta < b(\theta)$ (we can do this since H is an immersion on the interior of Q).

LEMMA 1.3. *If in the above M has nonpositive curvature then we have*:

$$2\pi \geq \int_{S^1} \langle \nabla_{\dot\gamma_\theta(a(\theta))} J(\theta, s), \gamma_\theta^\perp \rangle \, d\theta$$

If M has negative curvature then equality will hold if and only if H is one to one on the interior of Q and $H(Q)$ is a disk.

Proof. For m in M let $k(m)$ represent the curvature of M at m. Since H may be more than 1 to 1 and since $k(m) < 0$ we have

$$\int_{H(Q)} k(m)\, dm \geq \int_0^{2\pi} \int_{a(\theta)}^{b(\theta)} k(H(\theta, s)) \langle J(\theta, s), \gamma_\theta^\perp \rangle \, ds\, d\theta$$

with equality holding when M has negative curvature if and only if H is one to one on the interior of Q. Using the jacobi equation along $\gamma_\theta^\perp$ we find that the integrand of the right hand side is

$$-\langle \nabla_{\dot\gamma_\theta(s)} \nabla_{\dot\gamma_\theta(s)} J(\theta, s), \gamma_\theta^\perp \rangle = -\frac{d}{ds} \langle \nabla_{\dot\gamma_\theta(s)} J(\theta, s), \gamma_\theta^\perp \rangle .$$

Hence the right hand side becomes

$$\int_0^{2\pi} \langle \nabla_{\dot\gamma_\theta(a(\theta))} J(\theta, s), \gamma_\theta^\perp \rangle \, d\theta - \int_0^{2\pi} \langle \nabla_{\dot\gamma_\theta(b(\theta))} J(\theta, s), \gamma_\theta^\perp \rangle \, d\theta.$$

Since the boundary component of $H(Q)$ is a single circle the euler characteristic is ≤ 1 (in our applicaitons $H(Q)$ will in fact always be a disk) and hence the left hand side is less than or equal to 2π—boundary term, $B\partial$, of Gauss–Bonnet. Hence the lemma follows when we see that

$$B\partial = \int_0^{2\pi} \langle \nabla_{\dot\gamma_\theta(b(\theta))} J(\theta, s), \gamma_\theta^\perp \rangle \, d\theta.$$

In the case that the boundary $H(\partial_1)$ and the variation is piecewise C^2 integrating the result of Lemma 1.2 will show that this is true. In general we can approximate by such and see the above result in each of the approximating cases and hence in the limit. ■

II. The proof of Theorem A

We now consider the case of Theorem A, that is (M_0, g_0) is an SGM riemannian manifold of nonpositive curvature with boundary N and (M_1, g_1) is another riemannian manifold with the same boundary N and the same boundary distance function (and hence is also SGM.) Since all geodesics, γ, hit the boundary we will always parameterize them by arclength with parameter ≥ 0 and $\gamma(0) \in N$.

M_0 and M_1 are equivalent as lenses (see [C]). This means that for every geodesic segment γ in M_1 the geodesic segment $\bar{\gamma}$ in M_0 which begins at the same boundary point with the same angle as γ intersects N again at the same point with the same angle and at the same parameter value as γ. This allows us to define a natural map F from UM_1 (the unit tangent bundle of M_1) to UM_0 as follows. Given u in UM_1 let v be the unique unit vector of M_1 at the boundary N such that $u = \gamma_v'(t)$ where γ_v is the geodesic of M_1 with $\gamma_v'(0) = v$ and $0 \leq t \leq 1(\gamma)$ where $1(\gamma)$ is the first parameter value greater than zero with $\gamma(1(\gamma)) \in N$. (In particular if γ "grazes" N then it is considered to stop there.) We then let $F(u) = \bar{\gamma}_v(t)$ where $\bar{\gamma}_v$ is the geodesic in M_0 with the corresponding initial condition as above. It is not hard to see that the map F above is continuous, it commutes with the geodesic flow, and $F(-u) = -F(u)$. In particular F is measure preserving and hence Vol $(UM_1) =$ Vol (UM_0) so M_1 and M_2 have the same volume. All of the above holds in all dimensions. For further details see [C].

Although it is not clear that the map F is smooth for all u in UM_1 it is clear for those u's such that γ_v is not tangent to the boundary at 0. Further since $F(-u) = -F(u)$, F will be smooth at all u except those where γ_v is tangent to N at 0 and $1(\gamma_v)$. We will call a point x in M_1 (or M_0) "generic" if x does not lie on any geodesic that grazes N at both endpoints. Non-generic x form a set of measure zero and F is smooth at all u that are tangent to a geodesic that passes through a generic x.

We saw above that to each geodesic γ of M_1 there corresponds a geodesic $F(\gamma)$ in M_0 namely the one having the same end points (and hence the same initial conditions). Hence for every jacobi field J along γ we will associate the Jacobi field $\Phi(J)$ coming from the corresponding variation of geodesics. This means that $\Phi(J)$ is the jacobi field along $F(\gamma)$ having $J(0)$ correspond to $\Phi(J)(0)$ and $J(1(\gamma))$ correspond to $\Phi(J)(1(\gamma))$. Φ is thus linear. Note that the SGM condition guarantees that γ and $F(\gamma)$ have no conjugate points so the above correspondence can always be made. We used the fact that for $q \in N$ there is a natural isometry between T_qM_1 and T_qM_0 which is the identity on T_qN and takes inward normal to inward normal. In the above it may not be the case that $J'(0)$ corresponds to $\Phi(J)'(0)$ since the second fundamental forms of N in the two manifolds need not be the same. The relationship between $J'(0)$ and $\Phi(J)'(0)$ in all dimensions is studied in the appendix to [C]. It was shown in [M] that if M_0 (and hence M_1) is assumed to be convex then the boundaries agree up to second order hence in particular $J'(0)$ and $\Phi(j)'(0)$ do correspond.

We now restrict our attention to jacobi fields along a fixed geodesic γ and its corresponding geodesic $F(\gamma)$. We choose parallel unit vector fields X_γ and $X_{F(\gamma)}$ along γ and $F(\gamma)$ which are perpendicular to the geodesics and correspond at 0. We will show that they also agree at $1(\gamma)$. From now on we will only consider jacobi fields perpendicular to the geodesics which can and will be thought of as functions (since they are functional multiples of X_γ and $X_{F(\gamma)}$.) We can tell if a jacobi field vanishes on the interior of a geodesic simply by looking at its values at the endpoints. If it has the same sign at both endpoints then it does not vanish since it cannot vanish twice (no conjugate points) and it cannot vanish along with its derivative. On the other hand if it changes sign it clearly vanishes. We will let $J_\gamma(t)$ be the jacobi field such that $J_\gamma(0) = J_\gamma(1(\gamma)) = 1$. Since J_γ never vanishes we can define a jacobi field

$$J_\gamma^0(t) = J_\gamma(t) \cdot \int_0^t \frac{ds}{J_\gamma(s)^2}.$$

Similarly we can define $\Phi(J_\gamma)^0$. J_γ^0 is the jacobi field with $J_\gamma^0(0) = 0$ and $J_\gamma^{0\prime}(0) = 1$. Thus J_γ^0 comes from a standard variation of geodesics all starting at $\gamma(0)$. It is clear that the corresponding variation in M_0 gives rise to a jacobi field with initial value 0 and initial derivative 1, i.e. $\Phi(J_\gamma)^0 = \Phi(J_\gamma^0)$. Since $\Phi(J_\gamma^0)$ cannot vanish for $t > 0$ we see that $\Phi(J_\gamma^0)(1(\gamma)) > 0$, and hence that X_γ corresponds to $X_{F(\gamma)}$ at $1(\gamma)$. Further since J_γ^0 and $\Phi(J_\gamma^0)$ correspond at $1(\gamma)$

$$\int_0^{1(\gamma)} \frac{ds}{J_\gamma(s)^2} = \int_0^{1(\gamma)} \frac{ds}{\Phi(J_\gamma)(s)^2}.$$

(The n dimensional version of this appears in [C].)

Now fix $a \in [0, 1(\gamma)]$. The jacobi field that vanishes at a and has derivative 1 at a is

$$J_\gamma^a(t) = J_\gamma(a) \cdot J_\gamma(t) \cdot \int_a^t \frac{ds}{J_\gamma(s)^2} = J_\gamma(a) \cdot J_\gamma^0(t) + J_\gamma(a) \cdot \int_a^0 \frac{ds}{J_\gamma(s)^2} \cdot J_\gamma(t).$$

Thus by the linearity of Φ, $\Phi(J_\gamma^a)(t)$ must be

$$J_\gamma(a) \cdot \Phi(J_\gamma^0)(t) + J_\gamma(a) \cdot \int_a^0 \frac{ds}{J_\gamma(s)^2} \cdot \Phi(J_\gamma)(t).$$

It is important to note that $\Phi(J^a_\gamma)(t)$ is not necessarily the jacobi field along $F(\gamma)$ that vanishes at a (i.e. $\Phi(J^a_\gamma) \neq \Phi(J_\gamma)^a$). We will let $f_\gamma(a) \in [0, 1(t)]$ be the place where it does vanish. We will later think of f as a function on the unit sphere bundle and write $f(\gamma'(a))$ for $f_\gamma(a)$. Similarly for $u = \gamma'(a)$ we will write J^u and $\Phi(J^u)$ for J^a_γ and $\Phi(J^a_\gamma)$. The above formulas give:

LEMMA 2.1

$$\int_0^{f_\gamma(a)} \frac{ds}{\Phi(J_\gamma)(s)^2} = \int_0^a \frac{ds}{J_\gamma(s)^2} \tag{1}$$

$$\Phi(J^a_\gamma)'(f_\gamma(a)) = \frac{J_\gamma(a)}{\Phi(J_\gamma)(f_\gamma(a))}. \tag{2}$$

Proof. Putting the formula for $\Phi(J^0_\gamma)$ into the formula for $\Phi(J^a_\gamma)$ gives:

$$\Phi(J^a_\gamma)(t) = J_\gamma(a) \cdot \Phi(J_\gamma)(t) \cdot \left\{ \int_0^t \frac{ds}{\Phi(J_\gamma)(s)^2} - \int_0^a \frac{ds}{J_\gamma(s)^2} \right\}.$$

(1) follows from the fact that J_γ is positive and $\Phi(J^a_\gamma)(f_\gamma(a)) = 0$.
(2) comes from differentiating the above and using 1). ■

If x is a generic point of M_1 it will be called regular if only a finite number of the geodesics through x graze the boundary. For a regular x let $D_x =$ closure of $\{p \in M_1 | \text{There is a geodesic segment from } p \text{ to } x\}$. The boundary ∂D_x is a circle consisting of two parts: B, which is the union of intervals B_i of N, and a union of geodesic segments τ_i. We will let $F(\partial D_x)$ be the corresponding circle in M_0, i.e. it consists of B and the $F(\tau_i)$'s. We will let $F(D_x)$ be the closure of $\{q \in M_0 | q \text{ lies on a } F(\gamma) \text{ where } \gamma \text{ is a geodesic segment passing through } x\}$.

LEMMA 2.2. *If x is a regular point of M_1, then $F(D_x)$ is domain of M_0 with boundary $F(\partial D_x)$.*

Proof. We first show that $F(\partial D_x)$ is an imbedded circle, i.e. that if $i \neq j$ then $F(\tau_i) \cap F(\tau_j)$ is empty. Let p_i and q_i be the points on N such that τ_i is the geodesic segment from p_i to q_i (note that p_i is closer to x than q_i and that τ_i is tangent to N at p_i). If $F(\tau_i)$ intersects $F(\tau_j)$ then triangle inequalities show $d_0(p_i, q_j) + d_0(q_i, p_j) < d_0(p_i, q_i) + d_0(p_j, q_j)$. On the other hand, $d_1(q_i, p_j) \geq d_1(q_i, x) - d_1(p_j, x)$ and $d_1(q_j, p_i) \geq d_1(q_j, x) - d_1(p_i, x)$. Adding these two inequalities and using $d_1(p_i, q_i) = d_1(q_i, x) - d_1(p_i, x)$ yields $d_1(q_i, p_j) + d_1(q_j, p_i) \geq d_1(p_i, q_i) + d_1(p_j, q_j)$. However the p's and q's lie on N hence $d_0 = d_1$ and we get a contradiction. Thus $F(\partial D_x)$ is imbedded.

We can also use the above to see that $F(\gamma_u) \cap F(\tau_i) \subset B$. Assume that for some v and i we had $F(\gamma_v)$ intersect $F(\tau_i)$ in the interior. Let $I \subset U_x$ be $\{u | F(\gamma_u)$ intersects $F(\tau_i)$ in the interior$\}$. $F(\tau_i)$ is the extension of some $F(\gamma_w)$. Variations near w make it clear that for u in a neighborhood of w, u is not in I. Thus there is a $u \neq w$ on the boundary of I. But it is clear that this can only happen if $F(\gamma_u)$ does not intersect $F(\tau_i)$ but an extension $F(\tau_j)$ of it does. This contradicts the previous paragraph and gives $F(\gamma_u)$ never intersects $F(\tau_i)$ in the interior.

A similar argument says $F(\gamma_u)$ does intersect $F(\gamma_v)$ for all u and v. Let $I = \{u | F(\gamma_u)$ intersects $F(\gamma_v)\}$. Variations near v show that I includes a neighborhood of v. If $I \neq U_x$ then there is a boundary point w. But as before $F(\gamma_w)$ must extend to $F(\tau_i)$ which intersects $F(\gamma_v)$ which contradicts the previous paragraph.

We can use this to show that $F(\partial D_x)$ is contractible in M_0 and hence separates M_0. Fix $v \in U_x$. Since each $F(\gamma_u)$ intersects $F(\gamma_v)$ we can homotop in the obvious way $F(\partial D_x)$ along the $F(\gamma_u)$ so that the image lies in a small neighborhood of $F(\gamma_v)$ and hence can be contracted. Since the homotopy was made through points of $F(D_x)$ we see that $F(D_x)$ contains one of the components of $M_0 - F(\partial D_x)$. On the other hand since the $F(\gamma_u)$ do not intersect $F(\partial D_x)$ except at endpoints we see that $F(D_x)$ is the closure of this component and the lemma follows. ■

Remark. The above proof shows that $F(D_x)$ is in fact a disk.

LEMMA 2.3. *If x is a generic point in the interior of M_1 then*

$$2\pi \geq \int_{U_x} \Phi(J^u)'(f(u))\, du$$

where U_x is the circle of unit tangent vectors at x with the usual measure du.

Proof. This is an applicaiton of Lemma 1.3. For $u \in U_x$ there is a number $t(u)$ and a geodesic γ_u such that $\gamma_u'(t(u)) = u$. The functions a and b in the definition of Q are given by $a(u) = f(u) - t(u)$, $b(u) = 1(\gamma_u) - t(u)$. The reason for introducing $t(u)$ is that for our choice of parameter for geodesics (i.e. $\gamma(0) \in N$) f is not a smooth (or even continuous) function of u, however $f(u) - t(u)$ will be smooth when x is generic. The fact that x is generic also guarantees that the map $H : Q \to M_0$ defined by $H(u, s) = F(\gamma_u)(s + t(u))$ is smooth. It is an immersion on the interior since the variation field vanishes only for $s = a(u)$. We may assume that there are finitely many places where b is not smooth since if not we need only look at nearby regular x where there are finitely many, prove that lemma for this x and then take limits. Let D_x, ∂D_x, $F(D_x)$, and $F(\partial D_x)$ be as in Lemma 2.2. It is clear that $F(\partial D_x)$ is the image under H of ∂_1. It may appear that only part of $F(\gamma_u)$ is in the image $H(Q)$ but the other part shows up as part of $F(\gamma_{-u})$ since $F(\gamma_u)(f(u)) = F(\gamma_{-u})(f(-u))$.

Hence $H(Q)$ is $F(D_x)$, the only thing left to show in order to apply Lemma 1.3 is that $H(\partial_0) \cap \partial F(D_x)$ is empty. But every point of $H(\partial_0)$ is an interior point of a $F(\gamma_u)$ and hence cannot intersect the boundary. ■

Proof of Theorem A. Let (M_1, g_1) be a surface with boundary with the same boundary distance function as (M_0, g_0) (i.e. $d_1 = d_0$.) We have seen in Lemma 2.3 that for all but a set of measure 0 points x in M_1

$$\int_{U_x} \Phi(J^u)'(f(u))\,du \leq 2\pi.$$

Integrating this over all x in M_1 leads to

$$\int_{UM_1} \Phi(J^u)'(f(u))\,du \leq 2\pi \cdot \mathrm{Vol}\,(M_1)$$

where here du represents the standard measure on UM_1. Let Γ represent the space of geodesic segments on M_1 with standard measure $d\gamma$. Then using Santaló's formula (see [Sa] pp. 336–338 or [C] sec. III) the above says

$$\int_\Gamma \int_0^{1(\gamma)} \Phi(J_{\gamma'(t)})'(f(\gamma'(t)))\,dt\,d\gamma \leq \mathrm{Vol}\,(UM_1).$$

Lemma 2.1 tells us that

$$\int_\Gamma \int_0^{1(\gamma)} \frac{J_\gamma(t)}{\Phi(J_\gamma)(f(\gamma'(t)))}\,dt\,d\gamma \leq \mathrm{Vol}(UM_1).$$

Now for each fixed γ use Lemma 1.1 with $j = J_\gamma$, $\bar{j} = \Phi(J_\gamma)$, $f(t) = f(\gamma'(t))$, $C_1 = 0$, $C_2 = 1$, $a = \bar{a} = 0$, and $b = \bar{b} = 1(\gamma)$ (note that Lemma 2.1 says that i holds.) This yields $\int_\Gamma 1(\gamma)\,d\gamma \leq \mathrm{Vol}\,(UM_1)$ and hence equality must hold in all the inequalities. In particular by Lemma 1.1 we have $f(\gamma'(t)) = t$ and $J_\gamma(t) = \Phi(J_\gamma)(t)$ for all γ and t and hence the spaces are isometric. ■

III. The proof of Theorem B

In this section we will assume that M_0 is a compact surface of genus ≥ 2 with a Riemannian metric of nonpositive curvature. $F: UM_1 \to UM_0$ will be a C^1 diffeomorphism which induces a conjugacy of geodesic flows where M_1 is a Riemannian surface. All geodesics will be parameterized by arclength unless otherwise stated. If γ is an oriented geodesic in M_1 then F will take its tangent vector field, $T\gamma$, to the tangent vector field of a geodesic in M_0 which we will denote by $F(\gamma)$. For $i = 0, 1$ we will let Z_i be the vector field on UM_i generating the geodesic flows ζ_i^t. So $F_*(Z_1) = Z_0$.

It was pointed out in [B–K] that for orientable surfaces of genus ≥ 2 if K_i is the subgroup of $\pi_1(UM_i)$ generated by the fiber (i.e. the kernel of the projection map) then K_i is the center of $\pi_1(UM_i)$. In the nonorientable case $K_i = \{a \in \pi_1(UM_i) \mid bab^{-1} = a \text{ or } a^{-1} \text{ for all } b \in \pi_1(UM_i)\}$. In either case F_* must take K_1 to K_0. In particular we see:

LEMMA 3.1. *F lifts to a map from $U\tilde{M}_1$ to $U\tilde{M}_0$ where $\tilde{M}_i$ is the universal covering space of M_i.*

By abuse of notation we will also refer to this lifted map as F.

Proof. The only requirement for the existence of such a lift is that $(\pi_0 \circ F)_*(K_1)$ is trivial but this follows from the above remarks. ∎

Remark. In Section IV we will see that this is false for flat tori.

The fact that $F_* K_1 = K_0$ also implies that the map F induces an isomorphism from $\pi_1(M_1)$ to $\pi_1(M_0)$ and hence the map F on closed geodesics induces an isomorphism of free homotopy classes. In particular since two freely homotopic closed geodesics in M_0 have the same length the same is true for M_1.

LEMMA 3.2. *M_1 has no conjugate points.*

Proof. By the above, every closed geodesic γ is the shortest curve in its free homotopy class. Since this applies as well to all iterates of γ we see that the lift $\tilde{\gamma}$ of γ to $\tilde{M}_1$ is minimizing and hence has no conjugate pairs. But by [B] the set of closed geodesics is dense in UM_0 and hence via F^{-1} in UM_1. Thus there are no conjugate points in M_1. ∎

The space of jacobi fields Ψ along a geodesic γ splits naturally as $\Psi = \Psi^\perp + \Psi^t + \Psi^b$ where $\Psi^\perp$ consists of those jacobi fields that are perpendicular to γ, Ψ^t is spanned by γ', and Ψ^b is spanned by $t\gamma'$. Although all jacobi fields arise from variations of geodesics only those in $\Psi^\perp + \Psi^t$ come from variations of geodesics γ_s which are all parameterized by arclength.

Let J be a jacobi field along a geodesic γ in $\Psi^\perp + \Psi^t$. We define a vector field TJ along $T\gamma$ in the unit tangent bundle as the variation field of the variation $T\gamma_s$ where γ_s is a variation of geodesics whose variation field is J. TJ is determined by the fact that $\pi_*(TJ) = J$ and that the vertical (with respect to the usual connection) component $v(TJ)$ of TJ is equal to J', the covariant derivative of J with respect to γ' where $v(TJ)$ and J' are thought of as tangent vectors perpendicular to γ'. (Note that J' is perpendicular to γ' since $J \in \Psi^{\perp} + \Psi'$.) In particular $|TJ(t)|^2 = |J(t)|^2 + |J'(t)|^2$. $\Psi^{\perp}$ is thus the subspace of J where TJ is perpendicular to Z.

The subspace of Ψ consisting of jacobi fields J such that $|TJ(t)|$ goes to 0 as t goes to ∞ (resp. $-\infty$) will be denoted Ψ^s (resp. Ψ^u.) It is easy to see that Ψ^s (resp. Ψ^u) $\subset \Psi^{\perp}$ since if it had any component in $\Psi' + \Psi^b$ it could not vanish at ∞. We will let Ψ^{ws} (resp. Ψ^{wu}) be those $J \in \Psi^{\perp}$ such that $|TJ(t)|$ stays bounded as t goes to ∞ (resp. $-\infty$.) By definition $\Psi^s \subset \Psi^{ws} \subset \Psi^{\perp}$.

Since F takes geodesics to geodesics (actually tangent fields to tangent fields) then F induces a map, Φ, from the jacobi fields along a geodesic γ in $\Psi_1^{\perp} + \Psi_1'$ to the jacobi fields along $F(\gamma)$ in $\Psi_0^{\perp} + \Psi_0'$ by taking variations to variations. We thus see that $F_*(TJ) = T\Phi(J)$ and hence $\pi_{0*}(F_*(TJ)) = \Phi(J)$. In particular Φ is a linear isomorphism.

LEMMA. 3.3. *Along every geodesic γ of M_1 Φ takes the sets $\Psi_1^{\perp}$, Ψ_1^s, Ψ_1^{ws}, Ψ_1^u and Ψ_1^{wu} to the corresponding sets $\Psi_0^{\perp}$, Ψ_0^s, Ψ_0^{ws}, Ψ_0^u, and Ψ_0^{wu} along $F(\gamma)$.*

Proof. Since F is a C^1 map between compact manifolds there is a number $a > 1$ such that $1/a|V| < |F_*(V)| < a|V|$ for all $V \in TUM_1$ where all norms are with respect to the usual metric. In particular for a jacobi field J, $|TJ(t)|$ goes to zero at ∞ if and only if $|T\Phi(J)(t)|$ goes to zero at ∞. Thus we see $\Phi(\Psi_1^s) = \Psi_0^s$ and similarly $\Phi(\Psi_1^u) = \Psi_0^u$. Along a dense set of geodesics in M_0 (for example those closed geodesics that pass through a region of negative curvature — see [B]) Ψ_0^s and Ψ_0^u span $\Psi_0^{\perp}$ and hence $\Phi^{-1}\Psi_0^{\perp} \subset \Psi_1^{\perp}$. For dimension reasons $\Phi\Psi_1^{\perp} = \Psi_0^{\perp}$. By continuity this holds for all geodesics. The fact that $\Phi\Psi_1^{ws} = \Psi_0^{ws}$ (resp. Ψ^{wu}) follows from the same argument as for Ψ^s along with the fact that $\Phi\Psi_1^{\perp} = \Psi_0^{\perp}$. ∎

In particular the lemma says that dF takes $Z_1^{\perp}$ to $Z_0^{\perp}$ at each point of UM_1 and hence preserves the cannonical contact form θ and thus the canonical volume form $\theta \wedge d\theta$ (and thus as well the orientation.) This yields:

LEMMA 3.4. *F is orientation and volume preserving.*

Along a geodesic γ where no pair of points on γ are conjugate along γ it is natural to look at $\Psi^{\perp} = \Psi^n \cup \Psi^z$ where Ψ^n consists of those jacobi fields that never vanish and Ψ^z those that do.

By [Gre] or [E] along a geodesic without conjugate points a jacobi field that vanishes must be unbounded at ∞ and $-\infty$ and hence Ψ^{ws} and Ψ^{wu} are contained in Ψ^n. Along a geodesic where $K \leq 0$ it is easy to find nontrivial elements of Ψ^{ws}

and Ψ^{wu} (these elements may coincide if the curvature is identically 0.) Via Φ^{-1} we thus see there are nontrivial elements of Ψ_1^{ws} and Ψ_1^{wu}.

For a geodesic on a surface we can choose a parallel unit field X normal to γ' along γ. Every jacobi field $J(t)$ in $\Psi^\perp$ can (and will) be written as $J(t) = j(t) \cdot X(t)$ where $j(t)$ is a function. We will sometimes confuse the jacobi field with the function j.

For a fixed geodesic γ of M_1 we will from now on denote by J_1^s the element of Ψ^{ws} with $J_1^s(0) = 1$. Similarly define J_1^u (which may coincide with J_1^s). By the above J_1^s never vanishes and so we can define a new jacobi field J_1^z by

$$j_1^z(t) = j_1^s(t) \cdot \int_0^t \frac{ds}{j_1^s(s)^2}. \tag{1}$$

Any jacobi field J in $\Psi^\perp$ is a linear combination of J_1^s and J_1^z. For $v = \gamma'(x)$ we let J_1^v be the jacobi field along γ such that $j_1^v(x) = 0$ and $j_1^{v\prime}(x) = 1$. We see that

$$J_1^v(t) = j_1^s(x) \cdot J_1^z(t) + j_1^s(x) \cdot \int_x^0 \frac{ds}{j_1^s(s)^2} \cdot J_1^s(t). \tag{2}$$

Along the geodesic $\Phi(\gamma)$ we will let $J_0^s = \Phi(J_1^s)$. By Lemma 3.3 we know that $J_0^s \in \Psi_0^{ws} \subset \Psi^n$. We define J_0^z from J_0^s in the same way that J_1^z was defined from J_1^s. We know there are constants c_1 and c_2 such that

$$\Phi(J_1^z) = c_1 \cdot J_0^s + c_2 \cdot J_0^z. \tag{3}$$

LEMMA 3.5. *In the above $c_2 = 1$.*

Proof. The fact that F is measure preserving, and takes Z_1 to Z_0 and $Z_1^\perp$ to $Z_0^\perp$, implies that is it measure preserving on $Z^\perp$. This translates to the fact that $j_1^s(t) \cdot j_1^{z\prime}(t) - j_1^{s\prime}(t) \cdot j_1^z(t) = \Phi(j_1^s)(t) \cdot \Phi(j_1^z)'(t) - \Phi(j_1^s)'(t) \cdot \Phi(j_1^z)(t)$ for all t. Using the formula for $j_1^{z\prime}$ in terms of j_1^s at $t = 0$ yields the left hand side to be 1. For the right hand side we get: $j_0^s(0) \cdot \{c_1 j_0^{s\prime}(0) + c_2 j_0^{z\prime}(0)\} - j_0^{s\prime}(0) \cdot \{c_1 j_0^s(0) + c_2 j_0^z(0)\} = c_2\{j_0^s(0) \cdot j_0^{z\prime}(0) - j_0^{s\prime}(0) \cdot j_0^z(0)\}$. Using the definition of j_0^z in terms of j_0^s we find that the left hand side is c_2 at $t = 0$ and the lemma follows. ■

If (as we shall show next) $\Phi(\Psi_1^z) \subset \Psi_0^z$ for all geodesics in M_1 then we define a function $g : UM_1 \to \mathbb{R}$ as follows: for $v \in UM_1$ let γ_v be the geodesic determined by v. Then $\Phi(J_1^v)$ will vanish once along $\Phi(\gamma_v)$ say at $\Phi(\gamma_v)(t_0)$. We let $g(v) = t_0$. (In the above we thought of $v = \gamma'(0)$ if instead $v = \gamma'(t_1)$ then take $g(v) = t_0 - t_1$ to be consistent with different choices of parameter for γ_v.) We also define $G : UM_1 \to UM_0$ by $G(v) = \Phi(\gamma_v)'(g(v))$.

LEMMA 3.6. *We have* $\Phi(\Psi_1^n) = \Psi_0^n$ *and* $\Phi(\Psi_1^z) = \Psi_0^z$.
Further the maps G and g are continuous and hence g is bounded (say $|g(v)| \le g_0$*).*

Proof. We first show $\Phi(\Psi_1^z) \subset \Psi_0^z$ and hence that we can define G and g. We need to show that $\Phi(J_1^v)$ vanishes for all $v = UM_1$. Let γ be a geodesic in M_1. Using equations 1 (with 0 subscripts), 2, 3 and Lemma 3.5 we see that

$$\Phi(J_1^v)(t) = j_1^s(x) \cdot \left\{ c_1 j_0^s(t) + j_0^s(t) \int_0^t \frac{ds}{j_0^s(s)^2} \right\} + j_1^s(x) \int_x^0 \frac{ds}{j_1^s(s)^2} j_0^s(t).$$

Hence

$$\Phi(J_1^v)(t) = j_1^s(x) j_0^s(t) \cdot \left\{ c_1 + \int_0^t \frac{ds}{j_0^s(s)^2} - \int_0^x \frac{ds}{j_1^s(s)^2} \right\}. \tag{4}$$

Thus $\Phi(J_1^v)$ will vanish somewhere if and only if there is a t_x such that

$$c_1 + \int_0^{t_x} \frac{ds}{j_0^s(s)^2} = \int_0^x \frac{ds}{j_1^s(s)^2} \tag{5}$$

If such a t_{x_0} exists for some x_0 then it must exist for all $x > x_0$ since both sides of the equation are monotone increasing to ∞ (since j_0^s and j_1^s are bounded at ∞). Since we can also pick x so that the right hand side is $> c_1$ we see that such t_x exists for all large x. Now we could have gone through the whole process above (starting just before equation (1) starting with j_1^u in place of j_1^s to derive the equations corresponding to 4 and 5 only with j_1^s and j_0^s replaced with j_i^u and j_0^u (where c_1 may be different) since the only property of j_1^s that we used was that it never vanished. In this case since j_1^u and j_0^u are bounded at $-\infty$ we see that $\Phi(J^v)$ must vanish somewhere for all small (near $-\infty$) x and hence for all x by our previous discussion. Thus we see that there is a t_x for all x so that equation 5 is satisfied and $\Phi(\Psi_1^z) \subset \Psi_0^z$.

As v varies continuously the jacobi equations (thought of as an equation on the reals) $j''(t) + K_v(t) \cdot j(t) = 0$, where $K_v(t)$ represents the curvature of the surface M_0 at $\Phi(\gamma)(t)$, will vary continuously. Also $T\Phi(J^v)(0) = F_*(TJ^v(0))$ varies continuously with v and hence so do the initial conditions $\Phi(J^v)(0)$ and $\Phi(J^v)'(0)$. Thus by the theory of ordinary differential equations $\Phi(J^v)(t)$ varies continuously with v. On a surface without conjugate points jacobi fields $\Phi(J^v)(t)$ that vanish are 0 at exactly one point and they cross the t axis transversely and hence the 0 varies continuously with v. Thus $G(v)$ and $g(v)$ are continuous and in particular $g(v)$ is bounded.

The boundedness and continuity of g imply that for any γ as t varies from $-\infty$ to ∞ so does $g(\gamma'(t)) + t$ and hence $\Phi(\Psi_1^z) = \Psi_0^z$. ∎

LEMMA 3.7. *There is a number $R > 0$ such that if γ and σ are geodesics in $\tilde{M}_1$ such that $\gamma(0) = \sigma(0)$ and $\gamma \neq \sigma$ then $F(\sigma)(R) \notin F(\gamma)[-g_0, \infty)$.*

Proof. Fix $p \in M_0$ and $v \in U_p$. For any $\tilde{p} \in \tilde{M}_0$ which projects to p and any $w \in U_p$ we let $\tilde{\gamma}_v$ and $\tilde{\gamma}_w$ be the geodesics in $\tilde{M}_0$ starting at $\tilde{p}$ with initial tangents that project to v and w respectively. Lemma 3.6 guarantees the existence of a $\theta_v > 0$ such that if w makes an angle less than θ_v with v then the geodesics $F^{-1}(\tilde{\gamma}_v)$ and $F^{-1}(\tilde{\gamma}_w)$ intersect at some $F^{-1}(\tilde{\gamma}_v)(t)$ for $t < g_0 + 1$ and hence never intersect for $t > g_0 + 1$ as long as they do not coincide. Easy continuity arguments along with the compactness of UM_0 allow us to choose a θ with $\theta_v > \theta > 0$ for all v in UM_0.

Now let γ and σ be as in the statement of the lemma and let R be greater than $\max\{g_0 + 1, \pi a/\sin(\theta), g_0 + \pi a\}$ where a is as in the proof of Lemma 3.3. Assume $F(\gamma)(t_0) = F(\sigma)(R)$ then the first paragraph says that $F(\sigma)'(R)$ and $F(\gamma)'(t_0)$ make an angle greater than θ (since $R > g_0 + 1$.) If $t_0 \geq 0$ then $d(F(\sigma)(0), F(\gamma)(0)) \geq R \sin(\theta) > \pi a$ since $\tilde{M}_0$ has nonpositive curvature. If $t_0 < 0$ (here we need to worry about the angle close to π) the triangle inequality gives again $d(F(\sigma)(0), F(\gamma)(0)) \geq R - g_0 > \pi a$. On the other hand there is a path in $U\tilde{M}_1$ from $\gamma'(0)$ to $\sigma'(0)$ of length $\leq \pi$. By the definition of a its image in $U\tilde{M}_0$ is a curve of length $\leq \pi a$ which when projected to $\tilde{M}_0$ becomes a curve of length $\leq \pi a$ from $F(\gamma)(0)$ to $F(\sigma)(0)$. This contradiction yields the lemma.

PROPOSITION 3.8. *In the situation of Theorem B we have for every $p \in M_1$ (we parameterize geodesics γ_v so that $\gamma_v'(0) = v$ for all $v \in U_p$)*

$$2\pi \geq \int_{U_p} \Phi(J^v)'(g(v))\, dv$$

Proof. The inequality is an application of Lemma 1.3. We can parameterize U_p as usual by θ in $[0, 2\pi]$ then $a(\theta)$ will be $g(\theta)$ and $b(\theta) = R$ where R comes from Lemma 3.7. We define the map $H(\theta, s) = F(\gamma_\theta)(s)$ into $\tilde{M}_0$. We need only show that H has all the right properties from the fact that the jacobi fields $\Phi(J^\theta)$ vanish only at $g(\theta)$. By Lemma 3.7 H maps ∂_1 in a 1-1 fashion to an imbedded circle ∂ in $\tilde{M}_0$ which will bound a disk D.

Since $\Phi(J^\theta)$ is perpendicular to $F(\gamma_\theta)$ and $\Phi(J^\theta)(R)$ is tangent to ∂ we see that $F(\gamma_\theta)$ is the geodesic perpendicular to ∂ at $\partial(\theta)$.

As s goes to ∞ $F(\gamma_\theta)(s)$ goes to ∞ and hence eventually lies outside D. By Lemma 3.7 $F(\gamma_\theta)(R, \infty) \cap \partial = \varnothing$ and hence $F(\gamma_\theta)(R, \infty)$ lies outside D and since $F(\gamma_\theta)[-g_0, R) \cap \partial = \varnothing$ we have $F(\gamma_\theta)[-g_0, R)$ lies in D. In particular $H(\partial_0)$ lies in the interior of D and property iv is satisfied.

For any $p \in D$ let τ be a minimizing geodesic from p to ∂. Then τ is perpendicular to ∂ so $p = \Phi(\gamma_\theta)(t)$ for some θ and t. We need to show that $g(\theta) \le t \le R$. By the previous paragraph $t \le R$. Since $\Phi(J^\theta)(g(\theta)) = 0$ and $\Phi(J^\theta)$ is the variation field of the variation of normal geodesics the usual variation argument will say that, since τ is the shortest path from p to ∂, t cannot be $< g(\theta)$. Hence D is the image of H and property iii is satisfied.

We can thus apply Lemma 1.3 to yield the inequality. ■

Proof of Theorem B. Integrating the inequality of Lemma 3.8 over M_1 we get:

$$2\pi \cdot \mathrm{Vol}\,(M_1) \ge \int_{UM_1} \Phi(J^v)'(g(v))\, dv.$$

From the invariance of the canonical measure under the geodesic flow we get for each $L > 0$:

$$2\pi L \cdot \mathrm{Vol}\,(M_1) \ge \int_{UM_1} \int_0^L \Phi(J^{\zeta^t v})'(g(\zeta^t v))\, dt\, dv.$$

For fixed v let $\gamma(t)$ be the geodesic with $\gamma'(0) = v$ so that $\zeta^t(v) = \gamma'(t)$. Differentiating Equation 4 with respect to t, plugging in $g(\gamma'(t))$, and using Equation 5 yields:

$$\Phi(J^{\zeta^t v})'(g(\zeta^t v)) = \frac{j_1^s(t)}{j_0^s(g(\gamma'(t)) + t)}.$$

(In the above one must be careful with parameters since $\Phi(j^{\gamma'(t)})$ is a jacobi field along the geodesic $F(\gamma)$ with the parameter shifted by t.)

Apply Lemma 1.1 with $f(t) = g(\gamma'(t)) + t$, $j = j_1^s$, and $\bar{j} = j_0^s$ we find that

$$2\pi L \cdot \mathrm{Vol}\,(M_1) \ge \int_{UM_1} \frac{L^{3/2}}{(L + g(\zeta^L v) - g(v))^{1/2}}\, dv.$$

Rearranging terms we see:

$$1 \ge \frac{1}{\mathrm{Vol}\,(UM_1)} \cdot \int_{UM_1} \frac{1}{\left(1 + \dfrac{g(\zeta^L v) - g(v)}{L}\right)^{1/2}}\, dv.$$

Using a Jensen inequality for the function $x^{-1/2}$ we see that

$$1 \geq \left[\frac{1}{\operatorname{Vol}(UM_1)} \cdot \int_{UM_1} \left(1 + \frac{g(\zeta^L v) - g(v)}{L}\right) dv\right]^{-1/2}$$

with equality holding only if $g(\zeta^L(v)) = g(v) + c(L)$ where $c(L)$ is a constant depending at most on L. On the other hand the invariance of dv under ζ^t says

$$\int_{UM_1} g(\zeta^t v)\, dv = \int_{UM_1} g(v)\, dv$$

and hence we get equality everywhere and further $c(L) = 0$ and hence $g(v) = g(\zeta^L v)$ for all v and L. Since there are dense geodesics (see [B–B–E]) we see that $g(v)$ is a constant K. By composing with ζ^K we can assume that $g(v) = 0$ (i.e. we consider $\zeta^K \circ F$ instead of F and will show it is dI.)

We claim that F covers a map $f: M_1 \to M_0$. To see this let $x \in M_1$ and let $c(\theta)$ in UM_1 be the curve of unit vectors at x. Then $c'(\theta)$ corresponds to the jacobi field J^0 along γ_θ and hence $(\pi_0 \circ F)_*(c'(\theta)) = \Phi(J^0)(0) = 0$. Thus $(\pi_0 \circ F)(c(\theta)) = f(x)$ is independent of θ.

To finish the proof we need only note that f is an isometry and $df = F$. But this follows since f takes unit speed geodesics γ_v to unit speed geodesics $\gamma_{F(v)}$. In particular if γ is a minimizing geodesic from p to q then $f(\gamma)$ is a minimizing geodesic of the same length from $f(p)$ to $f(q)$.

IV. The genus one case

In this section we take up the one case of non-positive curvature not covered by Theorem B. This is the case where M_0 is a flat torus. (In the Klein bottle case for algebraic reasons any diffeomorphism of unit tangent bundles will lift to a diffeomorphism of the unit tangent bundles of the oriented double covers.)

EXAMPLE 4.1. If M_0 is a flat two torus, say $M_0 = \mathbb{R}^2/\Gamma$ for a lattice Γ. Let (x, y) be standard parameters for $\mathbb{R}^2$ and θ the angle from the x-axis. Then $UM_0 = \{(x, y, \theta) \in \mathbb{R}^2/\Gamma \times \mathbb{R}^1/2\pi\}$. Note that the geodesic flow vector field at (x, y, θ) is $\cos(\theta) \cdot d/dx + \sin(\theta) \cdot d/dy$. Hence diffeomorphisms $F: (x, y, \theta) \to (x + a(\theta), y + b(\theta), \theta)$ induce a conjugacy of the geodesic flows when a and b are functions of θ such that $a(0) = b(0) = 0$ and $(a(2\pi), b(2\pi)) \in \Gamma$.

It is easy to see that if $(a(2\pi), b(2\pi)) \in \Gamma - (0, 0)$ then F is not homotopic to a fiber preserving map so cannot be of the form $dI \circ \zeta^t$. Even if $(a(2\pi), b(2\pi)) = (0, 0)$ as long as a or b is not identically 0, F is not fiber preserving and (except for special choices $a(\theta) = t(1 - \cos(\theta))$, $b(\theta) = -t\sin(\theta)$) cannot be made so by following by a fixed amount. Hence again F is not $dI \circ \zeta^t$.

Although the above shows that Theorem B does not hold in its strongest form we do have:

THEOREM C. *If the geodesic flow of a closed surface M_1 is conjugate to that of a flat torus M_0 then M_1 is isometric to M_0.*

Proof. We first show that the map on geodesics induced by the conjugacy F induces a 1-1 correspondence between $\pi_1(M_1)$ and $\pi_1(M_0)$. UM_0 is homeomorphic to $S^1 \times S^1 \times S^1$ and $\pi_1(UM_0)$ is isomorphic to $\mathbb{Z}^3$ with generators α, β, γ. We can assume that α and β come from tangent vector fields to closed geodesics on M_0 while γ comes from the fiber. In particular, there is a natural identification between the $\mathbb{Z}^2$ spanned by α and β and $\pi_1(M_0)$ given by lifting a closed geodesic to its tangent vector field in UM_0. Let $P_i : UM_i \to M_i$ be the projection. Then $(P_1 \circ F^{-1})_* : \mathrm{span}\{\alpha, \beta\} \to \pi_1(M_1)$ induces a homomorphism from $\pi_1(M_0)$ to $\pi_1(M_1)$. This homomorphism is onto since each element of $\pi_1(M_1)$ can be represented by a closed geodesic γ_1 and $F^{-1}(T\gamma_1)$ is $T\gamma_0$ for some geodesic γ_0 hence is in the span of α and β. This homomorphism must thus be injective.

We now claim that every closed geodesic γ_1 in M_1 is the shortest in its homotopy class. To see this let τ_1 be a closed geodesic homotopic to γ_1. The corresponding geodesics γ_0 and τ_0 in M_0 must be homotopic by the previous paragraph and hence have the same length (since M_0 is a flat torus.) Thus γ_1 and τ_1 have the same length.

Since closed geodesics are dense in UM_0 they are also in UM_1 via F^{-1}. Proceeding now as in the proof of Lemma 3.2 we see that M_1 has no conjugate points. By E. Hopf's theorem [H] M_1 is flat. It is easy to check that two flat two tori with the same length spectrum are isometric. ■

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