Zeitschrift: Commentarii Mathematici Helvetici

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 65 (1990)

Artikel: The monodromy of a series of hypersurface singularities.

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DOI: https://doi.org/10.5169/seals-49720

Comment. Math. Helvetici 65 (1990) 181–197 0010-2571/90/020181-17$1.50 + 0.20/0

The monodromy of a series of hypersurface singularities

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Abstract. Let $\{f = 0\}$ be a hypersurface in C^{n+1} with a 1-dimensional singular set Σ. We consider the series of hypersurfaces $\{f + \varepsilon x^N = 0\}$ where x is a generic linear form.

We derive a formula, which relates the characteristic polynomials of the monodromies of f and $f + \varepsilon x^N$. Other ingredients in this formula are the horizontal and the vertical monodromies of the transversal (isolated) singularities on each branch of the singular set. We use polar curves and the carrousel method in the proof.

The formula is a generalization of the Iomdin formula for the Milnor numbers:

$$\mu(f + \varepsilon x^N) = \mu_n(f) - \mu_{n-1}(f) + Ne_0(\Sigma).$$

§1. Introduction

Let $f : \mathbf{C}^{n+1} \to \mathbf{C}$ be a germ at $\underline{0} \in \mathbf{C}^{n+1}$ of an analytic function. Let f have a 1-dimensional critical locus Σ. We consider for each $N \in \mathbf{N}$ the functions:

$$f_N = f + \varepsilon x^N \qquad \varepsilon \in \mathbf{C}$$

where x is an admissible linear form, which means that $f^{-1}(0) \cap \{x = 0\}$ has an isolated singularity. We call the series $\{f_N\}$ a *linear series* or *Iomdin series* of hypersurface singularities.

An important invariant of a singularity is the Milnor fibration [Mi]: For $\varepsilon > 0$ small enough there exists $\eta > 0$ such that

$$f : B_\varepsilon \cap f^{-1}(S^1_\eta) \to S^1_\eta$$

is a locally trivial fibre bundle. [B_ε is the closed ε-ball in $\mathbf{C}^{n+1}$, S^1_η is the circle with radius η in $\mathbf{C}$]. A typical fibre $F = f^{-1}(\eta) \cap B_\varepsilon$ is called a *Milnor fibre* of f. Let

$$\mu_k(f) = \dim \tilde{H}_k(F).$$

If $\dim \Sigma = 1$ then it is known (cf [KM]) that $\mu_k(f) = 0$ if $k \neq n - 1, n$. If f has an isolated singularity then $\mu_k(f) = 0$ if $k \neq n$, and $\mu(f) = \mu_n(f)$ is called the Milnor number of f. Iomdin studied the Milnor numbers for the linear series $f_N = f + \varepsilon x^N$ and related them to the non-isolated singularity f.

THEOREM [Iomdin] ([Io], cf. also [Lê-5]). *Let* $f : \mathbf{C}^{n+1} \to \mathbf{C}$ *have a 1-dimensional critical locus* Σ *and let* x *be an admissible linear form. There exists an* N_0 *such that for all* $N > N_0$:

1. f_N *has an isolated singularity for all* $\varepsilon \in \mathbf{C}$, $\varepsilon \neq 0$.
2. $\mu(f_N) = \mu_n(f) - \mu_{n-1}(f) + Ne_0(\Sigma)$ *where* $e_0(\Sigma)$ *is the algebraic multiplicity of* Σ *at* $\underline{0}$

In this paper we study the *monodromy* of such a linear series in connection with the mondromy of f.

A geometric monodromy is a diffeomorphism $h : F \to F$, which is a characteristic map for the Milnor fibration over the circle S^1_η. It has the property that there exists an diffeotopy $H : F \times [0, 2\pi] \to B_\varepsilon \cap f^{-1}(S^1_\eta)$ such that:

$$f(H(x, t)) = \eta e^{it}$$

$$H(x, 0) = \text{identity}$$

$$H(x, 2\pi) = h(x).$$

The induced map:

$$\mathbb{T} : \tilde{H}_*(F) \to \tilde{H}_*(F)$$

is called the algebraic monodromy.

We now consider the case of a 1-dimensional critical locus in more detail. As we mentioned above: $\tilde{H}_k(F) = 0$ if $k \neq n-1, n$. So the algebraic monodromy can only act non trivially at the levels n-1 and n;

$$\mathbb{T} \mid H_n(F) : H_n(F) \to H_n(F)$$

$$\mathbb{T} \mid H_{n-1}(F) : H_{n-1}(F) \to H_{n-1}(F)$$

For every irreducible branch Σ_i of Σ we have on $\Sigma_i - \{\underline{0}\}$ a local system of transversal singularities: Take at any $x \in \Sigma_i - \{\underline{0}\}$ the germ of a generic transversal section. This gives an isolated singularity, whose μ-constant class is well defined. We denote a typical Milnor fibre of this transversal singularity by F_i'. The only non-vanishing homology group is $\tilde{H}_{n-1}(F_i')$. This is a special case of Deligne's sheaf of vanishing cycles (cf. [De]). On the level of homology we have got in this way a local system, which has two different monodromies:

(a) the *vertical monodromy*

$$A_i : \tilde{H}_{n-1}(F'_i) \to \tilde{H}_{n-1}(F'_i)$$

which is the characteristic mapping of the local system over the punctured disc $\Sigma_i - \{\underline{0}\}$

(b) the *horizontal monodromy*

$$T_i : \tilde{H}_{n-1}(F'_i) \to \tilde{H}_{n-1}(F'_i)$$

which is the Milnor fibration monodromy, when we restrict f to a transversal slice through $x \in \Sigma_i$

Note that the two monodromies A_i and T_i commute, since they are defined on

$$(\Sigma_i - \{\underline{0}\}) \times S^1_n$$

which is homotopy equivalent to a torus.

The two monodromies play an important role in the relation between the monodromies of f and f_N.

MAIN THEOREM. *Let $f : \mathbf{C}^{n+1} \to \mathbf{C}$ have a 1-dimensional critical locus $\Sigma = \Sigma_1 \cup \ldots \cup \Sigma_n$ (irreducible components). Let x be an admissible linear form. Let $M[f](\lambda)$ be the alternating product of the characteristic polynomials of the monodromy $\mathbb{T}$ of f in dimensions n and $n-1$. Let $M[f + \varepsilon x^N](\lambda)$ be the characteristic polynomial of the monodromy of $f + \varepsilon x^N$ in dimension n. Then for all N sufficiently large $N \geq N_0$*

$$M[f + \varepsilon^N](\lambda) = M[f](\lambda) \prod_{l=1}^{r} det(\lambda^{Nd_i} I - A_i T_i^{Nd_i}),$$

where:

$$A_i : \tilde{H}_{n-1}(F'_i) \to \tilde{H}_{n-1}(F'_i) \quad \textit{vertical monodromy}$$

$$T_i : \tilde{H}_{n-1}(F'_i) \to \tilde{H}_{n-1}(F'_i) \quad \textit{horizontal monodromy}$$

$$d_i = e_0(\Sigma_i^{red}), \quad \textit{the multiplicity of } \Sigma_i^{red}.$$

Remarks

1. $m_i = d_i \mu_i$ where $m_i = e_0(\Sigma_i)$ and $\mu_i = \dim \tilde{H}_{n-1}(F'_i)$
2. $M[f](\lambda) = \det(\lambda I - \mathbb{T}|H_n(F)).\ [\det(\lambda I - \mathbb{T} \mid H_{n-1}(F))]^{-1}$
3. $M[f]$ is related to ζ_f, the zeta function of the monodromy which is defined as follows: $\zeta_f(\lambda) = \Pi_{q>0} \{\det(I - \lambda \mathbb{T}|H^q(F))\}^{(-1)^{i+1}}$ cf. A'Campo [Ac]

COROLLARY. *The eigenvalues of the monodromy satisfy Steenbrink's spectrum conjecture* [*St*] (2.2).

The spectrum of a singularity, which is defined in [*St*] is a set of a real numbers $Sp(f)$. A spectrum number $\sigma \in Sp(f)$ is via

$$\lambda = e^{2\pi i \sigma}$$

related to the eigenvalues of the monodromy (including multiplicities). The multi-valuedness of

$$\sigma = \frac{1}{2\pi i} \log \lambda$$

is normalized with the Hodge filtration on the Milnor fibre. The semi simple part of the monodromy respects the Hodge filtration and the level of the log λ.

Our main theorem can be written as

$$e^{2\pi i [\text{Spectrum conjecture}]}.$$

As a general reference for hypersurface singularities we refer to the book of Arnol'd-Guzein Zade-Varchenko [AGV].

The proof of the main theorem arose from a discussion with Jozef Steenbrink in the train from Nancy to Maastricht.

Almost simultaneously M. Saito [Sa] announced a proof of Steenbrink's spectrum conjecture. His proof appears to be completely different from ours and uses the deep theory of Mixed Hodge Modules.

§2. The polar curve and the carrousel

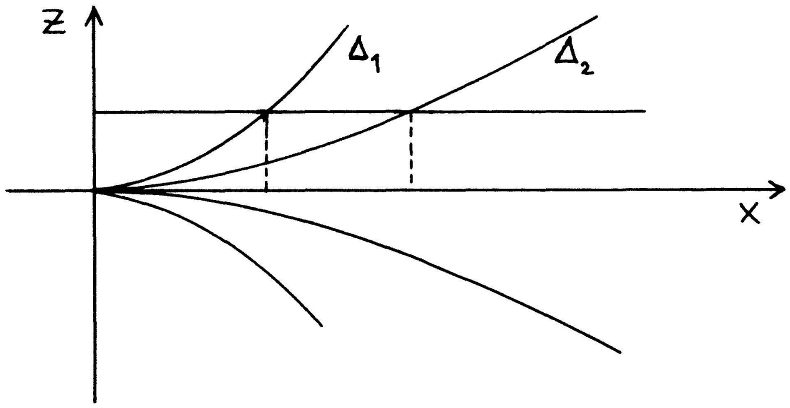
We start with a summary about the polar filtration and the carrousel method. For details we refer to [Lê-4].

Let $f: \mathbf{C}^{n+1} \to \mathbf{C}$ be a germ of an analytic function. Let $x: \mathbf{C}^{n+1} \to \mathbf{C}$ be an admissible linear form. After a change of coordinates we can assume that x is the first coordinate in $\mathbf{C}^{n+1}$. We denote points in $\mathbf{C}^{n+1}$ by $(x, y) \in \mathbf{C} \times \mathbf{C}^n = \mathbf{C}^{n+1}$.

Let $\Phi : \mathbf{C}^{n+1} \xrightarrow{(f,x)} \mathbf{C} \times \mathbf{C}$. We denote the singular locus of Φ by C. The polar curve of f (relative to x) is the closed set:

$$\Gamma = \overline{C \backslash f^1(0)}.$$

The image of Γ in $\mathbf{C}^2$ is called the *Cerf diagram* Δ. Let $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_s$ (decomposition in irreducible components) and $\Delta = \Delta_1 \cup \cdots \cup \Delta_s$ with $F(\Gamma_i) = \Delta_i$. According to [Lê-1] the components of Δ are tangent to $z = 0$ in $\mathbf{C}^2$. [We use (z, x) as coordinates in the target].



The curves Δ_i have Puiseux expansions of the form

$$x = a_i z^{r_i} + \cdots \qquad \text{with } a_i \neq 0 \ r_i \in \mathbf{Q}.$$

The numbers $r_1, \ldots, r_s$ are called the *polar ratios* (of f at $\underline{0}$). Note that several branches of Δ can have the same polar ratio. Now let $\rho_1, \ldots, \rho_l$ be the different polar ratios and assume

$$\rho_1 > \rho_2 > \cdots > \rho_l.$$

Let Δ_i^1 be the first approximation of Δ_i, defined by

$$x = a_i z_i^{r_i}.$$

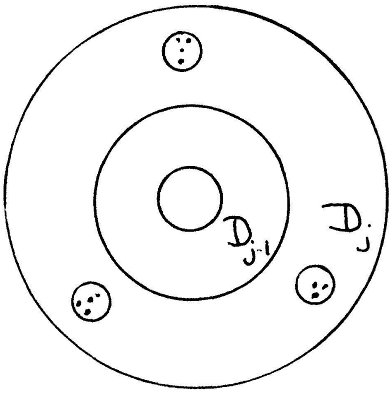
Δ_i^1 has the same polar ratio as Δ_i. With the help of the polar ratios one defines a filtration by concentric discs in the x-plane

$$0 \subset D_1 \subset \cdots \subset D_l$$

such that $\{\eta\} \times (D_j \backslash D_{j-1})$ contains the intersections

$$\Delta_i \cap \{z = \eta\} \text{ and } \Delta_i^1 \cap \{z = \eta\}$$

for all branches with polar ratio ρ_i.



Let $F' = \Phi^{-1}(\eta, 0)$ and $F_j = \Phi^{-1}(\{\eta\} \times D_j)$. The filtration:

$$F' \subset F_1 \subset \cdots \subset F_l$$

is called the *polar filtration* of the Milnor fibre F of f.

Lê constructed a geometric monodromy of f, which keeps the polar filtration invariant. As mentioned in [Lê-4] the geometric monodromy can be constructed as a carrousel. Lê used this construction to show that:

—there exists a geometric monodromy without fixed points [Lê-2],
—the algebraic monodromy is quasi-unipotent [Lê-4].

The construction of the geometric monodromy involves a vectorfield v on $S_\eta^1 \times D$ which lifts the unit vectorfield on S_η^1.

The x-component integrates in a first approximation to a rotation by an angle $2\pi\rho_j$; on $D_j \backslash D_{j-1}$ and to the identity on ∂D. Moreover the vectorfield is tangent to the intersection of the Cerf diagram Δ with $S_\eta^1 \times D$. Near the intersection points of the approximated branch $\Delta_i^1 : x = a_i z^{r_i}$ one uses a similar construction, which involves step by step also the next Puiseux pairs.

The geometric monodromy of F now is a lift by Φ of the vectorfield. For more details see the original papers. [Lê-2], [Lê-3] and [Lê-4].

§3. The geometric monodromy

Let now as in the introduction $f: \mathbf{C}^{n+1} \to \mathbf{C}$ have a 1-dimensional critical locus Σ and x be an admissible linear coordinate.

We want to compare the polar filtrations and the monodromy operators, associated to f and to $f_N = f + \varepsilon x^N$. We studied f already in §2, we now treat f_N. Let

$$\Phi_N = (f + \varepsilon x^N, x) : \mathbf{C}^{n+1} \to \mathbf{C}^2.$$

Φ_N has the same critical locus as Φ, namely $C = \{\partial f/\partial y = 0\}$. This is no surprise since the diffeomorphism

$$h : \mathbf{C}^2 \to \mathbf{C}^2$$

defined by $h(z, x) = (z + \varepsilon x^N, x)$ has the property

$$\Phi h = \Phi_N.$$

Next we consider the question when f_N has an isolated singularity. (cf. [Pe], p. 106). Consider the composition

$$f_N : \mathbf{C}^{n+1} \xrightarrow{\Phi_N} \mathbf{C}^2 \xrightarrow{\pi} \mathbf{C}$$
$$(w, x) \mapsto w.$$

Note that π is submersive, so we can restrict ourselves to the critical set C of Φ_N. The condition we have to satisfy is:

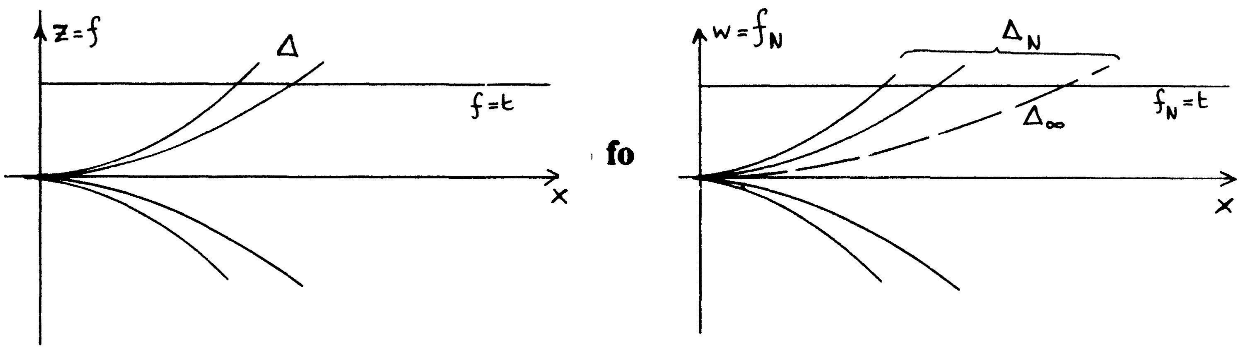
$$\text{image}(d\Phi_N) \subset \text{Ker}(d\pi) = \{w = 0\}.$$

Outside the origin image $(d\phi_N)$ is the tangent space to $\Phi_N(C)$. Since $\Phi(C)$ is given by fractional power series $z = f = \alpha(x)$ we need for each branch

$$w = f + \varepsilon^N = \alpha(x) + \varepsilon x^N = 0.$$

This has a non-isolated solution at $x = 0$ only if $\alpha(x)$ is identical to $-\varepsilon x^N$. This is a strong condition! A necessary condition is that the polar ratio of the branch is $1/N$. Moreover if for a given n we have $\alpha(x) \equiv -\varepsilon x^N$ then the number of ε-solutions is finite.

Next assume N and ε are chosen such that f_N has an isolated singularity. Observe that f and f_N have almost the same polar curves Γ and Γ_N, since $\Gamma_N = \Gamma \cup \Sigma$. As a consequence the Cerf diagram Δ_N of f_N has one component more than the Cerf diagram Δ of f. More precisely h maps $\Delta \cup \{z = 0\}$ bijectively onto Δ_N. The extra component can be non reduced.



Remember that the polar ratios of f and $\rho_1 > \cdots > \rho_l$. Let now $N_0 = 1/\rho_l$. If $N \geq N_0$ then ρ_N has polar ratios $\rho_1 > \cdots > \rho_l \geq 1/N$. N.B. If $N < N_0$ the polar ratio's become $\max\{\rho_i, 1/N\}$. We have the following propositions:

PROPOSITION. *Let $N \geq N_0$. There is a map $g : F \to F^N$ from a representative of the Milnor fibre F of f into the Milnor fibre F^N of f_N which induces a diffeomorphism on the polar filtrations up to level l:*

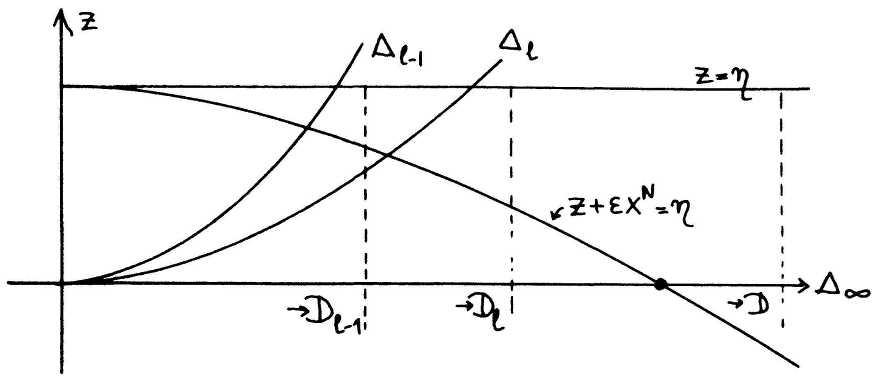
$$\begin{array}{ccccccccc} F' & \subset & F_1 & \subset & F_2 & \subset \cdots \subset & F_l & = & F \\ \| & & \downarrow h & & \downarrow h & & \downarrow h & & \\ F' & \subset & F_1^N & \subset & F_2^N & \subset \cdots \subset & F_l^N & \subset & F^N. \end{array}$$

So f and f_N have the same polar filtration, except f_N has one level more.

Proof. We take a representative of the map germ

$$(f + \varepsilon x^N, x, \varepsilon) : \mathbf{C}^{n+1} \times \mathbf{C} \to \mathbf{C} \times \mathbf{C} \times \mathbf{C}.$$

We use polydisc neighborhoods; let D be the corresponding disc in the x-coordinate plane. We take an admissible $\varepsilon > 0$ and choose next $\eta > 0$ such that the solutions of $x^N = \eta/\varepsilon$ are contained in D. We put the information for $\varepsilon = 0$ and $\varepsilon \neq 0$ in one picture.



We use the filtration $0 \subset D_1 \subset \cdots \subset D_l$ (associated to f) to define the first l steps in the polar filtrations of both f and f_N. For the Milnor fibres we have:

$$F = \Phi^{-1}[\{z = \eta\} \cap x^{-1}(D_l)]$$
$$F^N = \Phi^{-1}[\{z + \varepsilon x^N = \eta\} \cap x^{-1}(D)]$$

and for the polar filtrations

$$F_j = \Phi^{-1}[\{z = \eta\} \cap x^{-1}(D_j)] \qquad j = 1, \ldots, l$$
$$F_j^N = \Phi^{-1}[\{z + \varepsilon x^N = \eta\} \cap x^{-1}(D_j)] \qquad j = 1, \ldots, l.$$

We next use ε as a parameter, and we lift the isotopy

$$\{z = \eta\} \cap x^{-1}(D_j) \to \{z + \varepsilon x^N = \eta\} \cap x^{-1}(D_j)$$

with the help of the Thom isotopy-lemma to a diffeomorphism

$$h : F_l \to F_l^N$$

which induces diffeomorphisms on each level

$$h \mid F_j : F_j \to F_j^N.$$

Remark. We can make the construction and the arguments finer such that also the whole carrousels of F_l and F_l^N are diffeomorphic. Since the carrousel in connection with the Puiseux data determine a geometric monodromy it follows that F_l and F_l^N have the same carrousel monodromy.

Remark. From now on we use h to identify $F_l^N = F$.

Next we want to complete the monodromy over $D \backslash D_l$.

For f we make the x-component of the carrousel such that it becomes the identity on $D \backslash D_l$ except near ∂D_l, where we interpolate it with the rotation on $D_l \backslash D_{l-1}$. This has the advantage that over $D \backslash D_l$ the geometric monodromy preserves the x-coordinate.

Moreover this monodromy extends from $S^l_n \times (D \backslash D_l)$ over $D_\eta \times (D \backslash D_l)$ and gives rise to a geometric monodromy T of Φ.

For f_N we have one extra level in the polar filtration. The Cerf-diagram has the extra branch

$$w = \varepsilon x^N,$$

which has exactly N intersection points with the Milnor fibre $\{f_N = \eta\}$.

These points are of the form $x_k = x_1 e^{2\pi i k/N}$ $(k = 1, \dots, N)$, where x_1 is one intersection point. They all have the same absolute value and are contained in $D \backslash D_l$.

The x-component of the carrousel for f_N is a rotation by $2\pi/N$ on $D \backslash D_l$.

Let S be the diffeomorphism $F^N \backslash F \to F^N \backslash F$ which integrates a lift of the x-component of the carrousel vector field.

CLAIM. *The geometric local (relative) monodromy on $F \backslash F_l$ is just $T \cdot S$.*

Proof. Let S_t be the integral of the lifted x-component over the interval $[0, t]$, and T_t be similar for the z-component.

$$S_{2\pi} = S, \qquad T_{2\pi} = T,$$

S_t preserves the levels $f = c$; T_t preserves the x-coordinate.

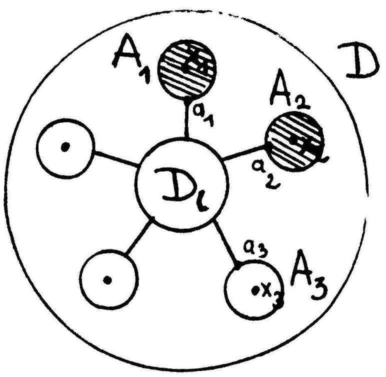
So

$$T_t S_t \{f + \varepsilon x^N\} = T_t \{f + \varepsilon x^N e^{it}\} = f e^{it} + \varepsilon x^N e^{it} = (f + \varepsilon x^N) e^{it}$$

which tells us that TS is the monodromy of f_N on $F^N \backslash F$.

§4. The algebraic monodromy

Next we study the algebraic monodromy.



Let $x_1, \ldots, x_N$ be the intersection points of $z + \varepsilon x^N = \eta$ with $z = 0$. At these points the projection

$$g : \Phi \mid F^N \backslash F \rightarrow D \backslash D_l$$

fails to be a submersion. Around $x_1, \ldots, x_N$ we choose small discs $A_1, \ldots, A_N$ and points $a_1, \ldots, a_N$ on $\partial A_1, \ldots, \partial A_N$.

By deformation retraction and excision we have

$$H_*(F^N, F) \cong \bigoplus_{k=1}^{n} H_*(g^{-1}(A_k), g^{-1}(A_k)).$$

Before doing more excision we remember that the projections

$$\Phi \mid \Sigma_i : \Sigma_i \rightarrow \{z = 0\}$$

are branched coverings of topological degree d_i.

So each a_k has d_i preimages under this map. Let $\{b_{i,1}, \ldots, b_{i,Nd_i}\}$ be the preimages of $a_1, \ldots, a_k, \ldots, a_N$ numbered in such a way that

$$\Phi(b_{i,k}) = a_{k(\mathrm{mod}N)} \text{ where } 1 \leq k(\mathrm{mod}N) \leq N \text{ and } k(\mathrm{mod}N) \equiv k \bmod N.$$

Next we choose $E_{i,k}$ as a Milnor ball and $F_{i,k} = g^{-1}(a_j) \cap E_{i,k}$ as Milnor fibre of the isolated singularity $g : (F^N \backslash F_l^N, b_{i,k}) \rightarrow (\mathbf{C}, a_{k(\mathrm{mod}N)})$. Further deformation and excision gives us

$$H_q(F^N, F_l^N) = \bigoplus_{i=1}^{r} \bigoplus_{k=1}^{Nd_i} H_q(E_{i,k}, F_{i,k}) = \bigoplus_{i=1}^{r} \bigoplus_{k=1}^{Nd_i} \tilde{H}_{q-1}(F_{i,k}).$$

These local Milnor fibres $F_{i,k}$ are nothing other than the Milnor fibre of the transversal singularity on Σ_i, taken at the point $b_{i,k}$.

They are all isomorphic with a typical fibre F_i'. Later on this isomorphism will be important; for now:

$$H_q(F^N, F) \cong \bigoplus_{i=1}^{r} [\tilde{H}_{q-1}(F_i')]^{Ndi}.$$

Note also that only non vanishing dimension is $q = N$.

Let $T_{\text{rel}}: H_n(F^N, F) \to H_n(F^N, F)$ denote the induced action of the geometric monodromy on $H_n(F^N, F)$. We call it the *relative monodromy*.

The definitions, the excisions and the deformations from above, can be made compatible with the actions of S and T. We suppose we have done so. The construction of S was related to the cyclic permutation:

$$x_1 \to x_2 \to \cdots \to x_N \to x_1$$

and we can assume the same permutation on

$$a_1 \to a_2 \to \cdots \to a_N \to a_1.$$

In the same way S permutes the $g^{-1}(A_k)$. $k = 1, \ldots, N$

We split $H_n(F^N, F)$ according to the branches $\Sigma_1, \ldots, \Sigma_r$. From the above arguments, it follows that $\mathbb{T}_{\text{rel}}$ acts diagonal on this decomposition

$$H_n(F^N, F) = \bigoplus_{i=1}^{r} \left[\bigoplus_{k=1}^{Nd_i} \tilde{H}_{n-1}(F_{i,k}) \right].$$

So we can treat each branch separately. Since Σ_i is irreducible and the branched covering

$$\Sigma_i \to \{z = 0\}$$

has degree d_i the induced action of S is also a full cyclic permutation of $\{b_{i,1}, \ldots, b_{i,Nd_i}\}$. We assume that we numbered our $b_{i,j}$ such that

$$b_{i,l} \to b_{i,2} \to \cdots \to b_{i,Nd_i} \to b_{i,1}.$$

So S induces permutations:

$$E_{i,1} \to E_{i,2} \to \cdots \to E_{i,Nd_i} \to E_{i,1}$$

$$F_{i,l} \to F_{i,2} \to \cdots \to F_{i,Nd_i} \to F_{i,l}.$$

Note also that T preserves the x-coordinate and can be assumed to be the identity outside a neighborhood of the polar curve, more precisely on:

$$g^{-1}(A_k)\Big\backslash \bigcup_{k'} \bigcup_{i=1,\dots r} E_{i,k'}$$

with $k'(\mathrm{mod}N) = k$. We denote the resulting actions on the homology as follows:

$$T_{i,k} : \tilde{H}_{n-1}(F_{i,k}) \to \tilde{H}_{n-1}(F_{i,k})$$
$$S_{i,k} : \tilde{H}_{n-1}(F_{i,k}) \to \tilde{H}_{n-1}(F_{i,k+1}).$$

The description of the monodromy in the block decomposition

$$\bigoplus_{k=1}^{Nd_i} \tilde{H}_{n-1}(F_{i,k})$$

is as follows:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & & S_{i,Nd_i}T_{i,Nd_i} \\ S_{i,1}T_{i,1} & 0 & 0 & . & . & \\ 0 & S_{i,2}T_{i,2} & 0 & . & . & \\ . & . & . & . & . & . \\ . & & . & . & 0 & . \\ 0 & & & 0 & S.T. & 0 \end{bmatrix}.$$

LEMMA.

$$(S_{i,Nd_i}T_{i,Nd_i}) \cdot \ldots \cdot (S_{i,2}T_{i,2}) \cdot (S_{i,1}T_{i,1}) = A_i(T_i)^{Nd_i}$$

where $A_i = \tilde{H}(F_{i,1}) \to \tilde{H}(F_{i,1})$ *is the vertical monodromy of the transversal local system and* $T_i = T_{i,1} : \tilde{H} \to \tilde{H}(F_{i,1})$ *is the horizontal monodromy of a transversal section.*

Proof. This is clear since Φ is a submersion over the torus

$$|x| = |a_j|,\ |z| = \eta.$$

Moreover:

$$A_i = A_{i,1} = S_{i,Nd_i} \cdot S_{i,2} \cdot S_{i,1}.$$

LEMMA. *Let*

$$B = \begin{bmatrix} 0 & 0 & & & 0 & B_m \\ B_1 & 0 & & & & 0 \\ 0 & B_2 & 0 & & & \\ 0 & 0 & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ 0 & & & & B_{m-1} & 0 \end{bmatrix}$$

a $m \times m$ matrix consisting of blocks of size $v \times v$, then

$$det(\lambda I - B) = det(\lambda^m I - B_1 \cdot \ldots \cdot B_m).$$

Proof. As shown to me by Rob Schrauwen, the proof is an elementary exercise in block matrix computation.

COROLLARY. *The characteristic polynomial of the monodromy on*

$$\bigoplus_{k=1}^{d_i N} \tilde{H}_{n-1}(F_{i,k})$$

is equal to:

$$\det[\lambda^{Nd_i} I - A_i(T_i)^{Nd_i}].$$

We have shown:

PROPOSITION. *For the relative monodromy we have*:

$$\det(\lambda I - \mathbb{T}_{\text{rel}}) = \prod_{i=1}^{r} \det[\lambda^N I - A_i(T_i)^{Nd_i}].$$

As a last step in the proof of the main theorem, we consider the action of the monodromy $\mathbb{T}$ on the long exact sequence of the pair (F^N, F):

$$\begin{array}{ccccccccc} 0 \to & H_n(F) & \to & H_n(F^N) & \to & H_N(F^N, F) & \to & H_{n-1}(F) & \to 0 \\ & \downarrow \mathbb{T} & & \downarrow \mathbb{T}_N & & \downarrow \mathbb{T}_{\text{rel}} & & \downarrow \mathbb{T} & \\ 0 \to & H_n(F) & \to & H_n(F^N) & \to & H_N(F^N, F) & \to & H_{n-1}(F) & \to 0. \end{array}$$

It follows that:

$$M[f_N](\lambda) = \det(\lambda I - \mathbb{T}_N) = M[f](\lambda) \cdot \det(\lambda I - \mathbb{T}_{\text{rel}})$$
$$= M[f](\lambda) \cdot \prod_{i=1}^{r} \det[\lambda^{Nd_i} I - A_i(T_i)^{Nd_i}]$$

$\left(\text{since } M[f](\lambda) \text{ is the quotient: } \dfrac{\det(\lambda - \mathbb{T} \mid H_n(F))}{\det(\lambda I\text{-}\mathbb{T} \mid H_{n-1}(F))}\right).$

EXAMPLE. Consider the case that f is homogeneous of degree d. Notice that:

- $d_i = 1$ for all i.
- the only polar ratio is $1/d$,
- the horizontal and vertical algebraic monodromies are related by $A_i = T_i^{-d}$ (cf. [St])

So if $N \geq d$ the formula of the main theorem reduces to

$$M[f + \varepsilon x^N](\lambda) = M[f](\lambda) \cdot \prod_{i=1}^{r} \det(\lambda^N I - T_i^{N-d}).$$

If $N = d$ it follows that

$$M[f + \varepsilon x^d](\lambda) = M[f](\lambda) \cdot \prod_{i=1}^{r} (\lambda^d - 1)^{\mu_i}.$$

Observe that $f + \varepsilon x^d$ is homogeneous of degree d. For almost all ε it has an isolated singularity and its monodromy depends only on the degree d:

$$M[f + \varepsilon x^d](\lambda) = M_d^{\text{reg}}(\lambda),$$

which is described in detail in [Mi], p. 71. We have now an expression for $M[f]$! After substitution we find: ($N \geq d$):

$$M[f + \varepsilon x^N](\lambda) = \frac{M_d^{\text{reg}}(\lambda)}{(\lambda^d - 1)^{\Sigma \mu_i}} \cdot \prod_{i=1}^{r} \det(\lambda^N I - T_i^{N-d})$$

which is similar to the formula in [Ste].

So it is possible to compute $M[f+\varepsilon x^N]$ if we know the degree d and the eigenvalues of the vertical algebraic monodromies. For the Milnor numbers it follows that:

$$\mu(f+\varepsilon x^N) = (d-1)^n + \Sigma(N-d)\mu_i.$$

Remark. Linear series $f+\varepsilon x^N$ as studied by Yomdin, sometimes correspond to the series in Arnol'd's lists of singularities:

$$A_{N-1}\colon y^2+\varepsilon x^N$$

$$D_{N+1}\colon xy^2x^N.$$

But in other cases one does not get the full series; examples are:

$$Y_{r,s}\colon x^2y^2+x^{r+4}+y^{s+4}$$

$$\begin{cases} W^{\#}_{1,2q-1}\colon & (y^2-x^3)^2+x^{4+q}y \qquad q\geq 1 \\ W^{\#}_{1,2q}\ \colon & (y^2-x^3)^2+x^{3+q}y \qquad q\geq 1\cdot \end{cases}$$

This especially occurs when the multiplicity of Σ is different from one.

Schrauwen [Sch] gave in the case of a plane curve a definition of a topological series of singularities and derived [in the case of transversal type A_1] a formula for the monodromy in such a series, which is similar to our main formula.

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Received November 1, 1988