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Intersection homology Poincaré spaces and the characteristic variety theorem

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In this paper we affirm a conjecture of R. M. Goresky and P. Siegel [GS] which states that L-groups $L^*(\mathbb{Z})$ defined by A. Mishchenko and A. Ranicki [R] may be realized geometrically as bordism groups Ω_*^{IP} of pseudomanifolds whose intersection homology groups satisfy Poincaré duality. We then apply this result to formulate and prove a "characteristic variety theorem," which uses intrinsic signature and Arf invariants of these pseudomanifolds, as was proposed by D. Sullivan in [Su, pp. 59 and 230]. As the problem posed by Sullivan was a primary motivation for the introduction of intersection homology (cf. [GM1, p. 137]), and since the characteristic variety theorem is central to the classification scheme of manifolds by surgery theory, we will (reversing the order of exposition in the paper) describe in some detail the idea of a characteristic variety and how intersection homology is used to construct it.

Sullivan's approach to understanding the space G/PL (or G/TOP or $G/0$), which classifies normal maps into a fixed manifold N,

$$\{\text{Normal maps } M \to N\} \xrightarrow{\cong} [N, G/PL],$$

was to construct what he called a characteristic variety $\mathscr{V} = \cup V \to N$. Roughly stated, this is a disjoint union of spaces V having two properties:

(1) they are sufficiently singular to represent KO-homology and ordinary homology of N (with appropriate coefficients), and
(2) they are sufficiently smooth to be put into transverse position to submanifolds of N and to carry basic integral invariants like the index, de Rham invariant, or even the Arf invariant (which depends also on tangent bundle information).

Sullivan used these properties to determine the homotopy type of G/PL and G/TOP and, even more, to list the (index and Arf) obstructions to deforming a homotopy equivalence to a homeomorphism (thus proving a special case of the Hauptvermutung).

However, the spaces V only satisfy a formal version of property (2), in that the signature they carry is not intrinsically constructed, as, for instance, it is from the intersection of cycles when V is a closed manifold. And the representation in (1) introduces both indeterminacy and redundancy in the list of numerical invariants associated to a homotopy equivalence (cf. [Su, p. 59]).

In 1974, Goresky and MacPherson discovered the intersection homology groups of a p.l. pseudomanifold. These groups and the intersection pairings they carry are topological invariants, and from them was constructed a signature invariant generalizing that for closed manifolds to pseudomanifolds with only even codimensional strata. These spaces could not, however, be used to define a bordism theory dual to G/TOP because

(1) one needs operations (e.g., coning) which introduce odd-codimensional strata and
(2) the intersection pairing, from which a signature was extracted, was nonsingular over $\mathbb{Q}$, but not in general over $\mathbb{Z}$: this meant essentially that one could only hope for a rational characteristic variety theorem.

With this in mind, J. Morgan showed in 1975 (unpublished) how to construct a bordism theory of singular spaces which carry a signature and whose dual cohomology theory is that based on the periodic Ω-spectrum G/TOP. This is the optimal formulation of a characteristic variety theorem. His idea for intersection theory was an interesting and non-trivial variant of Goresky and MacPherson's: the singular spaces V are defined inductively as stratified spaces together with a choice of self-annihilating subspace for an intersection pairing on the middle homology of the link of each stratum. Each such space then supports a graded group (depending on the choices) satisfying Poincaré duality, which in turn gives rise to a signature. It is, however, possible for a stratified space to support different intersection structures in its links and thus differ as representatives in this bordism theory: the signature is, in particular, not topologically intrinsic to the spaces V.

Back on the intersection homology side, the first difficulty was overcome by P. Siegel [Si] whose Witt spaces underlay a bordism theory dual to $KO[\frac{1}{2}]$, which by Sullivan's work is $G/TOP[\frac{1}{2}]$. Finally, the second problem was solved by Goresky and Siegel in [GS], where a class of pseudomanifolds was defined whose intersection homology group satisfy Poincaré duality over $\mathbb{Z}$ (see (1.4) below for the definition).

These spaces are called *intersection homology Poincaré spaces*, or *IP* spaces, in this paper, and their bordism groups are denoted Ω_*^{IP}. The first main theorem is proved in §2–4.

THEOREM. *There are isomorphisms*

$$\Omega_n^{IP} \xrightarrow{\cong} \begin{cases} \mathbb{Z}, & n \equiv 0(4) \\ \mathbb{Z}/2, & n \equiv 1(4),\ n > 1 \\ 0, & otherwise \end{cases}$$

induced by the index and deRham invariant.

The bordism groups $L^n(\mathbb{Z})$ of chain complexes over $\mathbb{Z}$ studied by Mishchenko and Ranicki are also isomorphic to $\mathbb{Z}$ for $n \equiv 0(4)$; to $\mathbb{Z}/2$ for $n \equiv 1(4)$, $n > 1$; and are trivial otherwise. The isomorphisms are given by the index and deRham invariants. It seems likely that sending an *IP* space to its intersection homology chain complex would induce directly an isomorphism

$$\Omega_*^{IP} \to L^*(\mathbb{Z}).$$

In any case, since

$$\pi_n(G/TOP) \xrightarrow{\cong} \begin{cases} \mathbb{Z}, & n \equiv 0(4) \\ \mathbb{Z}/2, & n \equiv 2(4) \\ 0, & \text{otherwise} \end{cases}$$

the (putative) universal coefficient theorem in this setting leads one to guess that the cohomology theory dual to $\Omega_*^{IP}(-)$ is $[\Sigma^*(-), G/TOP]$. Actually, since G/TOP is naturally 4-periodic, the dual cohomology theory must be made naturally 4-periodic. The details of this construction (in §5 below) were shown to me by John Morgan, as was the idea for the construction of the map μ in the following, second main theorem of this paper.

CHARACTERISTIC VARIETY THEOREM. *There is a* 4-*periodic reduced cohomology theory* $\tilde{\Omega}_{IP}^*(X)_{per}$ *which in degree zero detects homotopy classes of maps of a finite CW complex X to G/TOP: there is an isomorphism of sets*

$$\mu : [X, G/TOP] \to \tilde{\Omega}_{IP}^0(X)_{per}.$$

The additive structure on $[X, G/TOP]$ induced by μ is called the characteristic variety addition. When X is a sphere, the domain of μ is well-known (as has been recalled above) and its range is easily computed from Ω_*^{IP}. So the hard part is the construction of μ. For this, enough intersection homology machinery (functionality for normally nonsingular maps, characteristic classes) has been developed, so that the real difficulties are isolated around the Arf invariant, which must appear in some form in any characteristic variety theorem. Here it takes the form of the θ-invariant of Morgan and Sullivan, which they showed ([MS, 6.1]) is essentially equivalent to the Arf invariant. The construction of the θ-invariant is the core of §5 and of the characteristic variety theorem. It is essentially this delicate "two-torsion information" which advances Siegel's bordism representation of $G/TOP[\frac{1}{2}]$ to one of G/TOP itself.

Acknowledgement

John Morgan showed me in 1975 how the "right" singular spaces would lead to the characteristic variety theorem proved here and it is with his generous permission that his ideas are reproduced in (5.1)–(5.5) below. The main results of this paper were proved some time ago. It is mainly due to the much appreciated encouragement from, and collaboration on [GP] with, Mark Goresky that this work has finally been completed.

§1. Definitions, notation and preliminary results

(1.1) DEFINITION-PROPOSITION. An n-dimensional pseudomanifold X is a finite n-dimensional simplicial complex such that

(i) Every simplex is the face of an n-simplex

(ii) every $(n-1)$-simplex is the face of at most two n-simplices. X is *closed* if "at most" is replaced by "exactly" in (ii); otherwise its *boundary* ∂X is subcomplex generated by those $(n-1)$-simplices which are the faces of exactly one n-simplex, and ∂X is a closed $(n-1)$-dimensional pseudomanifold. If there is a p.l. homeomorphism of $\partial X \times I$ onto a neighborhood of ∂X in X, the pair $(X, \partial X)$ is called a *pseudomanifold with boundary*. X is called *irreducible* if

(iii) for each pair σ, σ' of n-simplices there is a finite sequence $\sigma = \sigma_1, \ldots, \sigma_m = \sigma'$ of n-simplices such that σ_i and σ_{i+1} have an $(n-1)$-face in common, $i = 1, \ldots, m-1$.

This is equivalent (in the presence of (i) and (ii)) to

$$H_n(X, \partial X; \mathbb{Z}/2) = \mathbb{Z}/2.$$

If in addition $H_n(X, \partial X) = \mathbb{Z}$, then X is called *orientable*.

Each pseudomanifold with boundary $(X, \partial X)$ admits a filtration by closed p.l. subspaces (a *stratification*)

$$X_0 \subseteq X_1 \subseteq \cdots X_{n-2} = X_{n-1} \nsubseteq X_n$$

such that each i-*stratum*, $X_i - X_{i-1}$, is an i-dimensional p.l. manifold and such that for each $x \in X_i - X_{i-1}$, there is a closed stratified pseudomanifold L and a p.l. stratum-preserving homeomorphism of $\mathbb{R}^i \times \mathring{c}L$ or of $\mathbb{R}^i_+ \times \mathring{c}L$ onto an open neighborhood of x in X, where $\mathbb{R}^i \times \mathring{c}L$ and $\mathbb{R}^i_+ \times \mathring{c}L$ are given the product stratifications and $\mathbb{R}^i$ and $\mathbb{R}^i_+$ have only one stratum.

(1.2) In this paper we use intersection homology with respect to middle-perversity only; for a pseudomanifold X it will be denoted

$$IH_*(X).$$

A chain in X is *allowable* if it is so in the sense of [GMI, 1.3]. The so-called *local calculation* of this middle perversity group is:

(1.3) PROPOSITION. *$IH_k(cX) = 0$ if $k \geq \dim X/2$ and the inclusion $X \to cX$ induces an isomorphism $IH_k(X) \xrightarrow{\cong} IH_k(cX)$ if $k < \dim X/2$.*

(1.4) An *intersection homology Poincaré space*, or *IP space*, is a stratified pseudomanifold satisfying the conditions of [GS, §7]:

(a) if $L = L^{2c}$ is the link of a point in the $(2c+1)$-codimensional stratum, then

$$IH_c(L) = 0$$

(b) if $L = L^{2c+1}$ is the link of a point in the $(2c+2)$-codimensional stratum, then

$$\operatorname{Tors} IH_c(L) = 0.$$

(The validity of the conditions is independent of the stratification of X.)

The reason for our terminology is:

(1.5) PROPOSITION [GS]. *If X is an n-dimensional IP space with boundary ∂X, then intersection and linking of allowable cycles induce nonsingular ("duality") pairings for each i,*

$$IH_i(X, \partial X)/\mathrm{Tors} \times IH_{n-i}(X)/\mathrm{Tors} \to \mathbb{Z}$$

and

$$\operatorname{Tors} IH_i(X, \partial X) \times \operatorname{Tors} IH_{n-i-1}(X) \to \mathbb{Q}/\mathbb{Z}.$$

The algebraic duals which arise in this theorem are denoted

$$M^* := \operatorname{Hom}(M, \mathbb{Z}) \text{ and } M\hat{} := \operatorname{Hom}(M, \mathbb{Q}/\mathbb{Z}).$$

Two closed oriented n-dimensional IP spaces X and X' are called *cobordant* if there is an oriented $(n+1)$-dimensional IP space W whose boundary is $X \cup -X'$. The resulting group of cobordism classes is denoted

$$\Omega_n^{IP}.$$

(1.6) PROPOSITION. *In case* $n = 4k$, $i = 2k$ *and* $\partial X = \phi$, *the index of the intersection pairing in* (1.4) *is a cobordism invariant and induces a homomorphism*

$$\sigma : \Omega_{4k}^{IP} \to \mathbb{Z}.$$

In case $n = 4k + 1$, *the number* mod 2 *of* $\mathbb{Z}/2$*-summands in* $\operatorname{Tors} IH_{2k}(X)$ *is a cobordism invariant, the* de Rham invariant, *and induces a homomorphism*

$$dR : \Omega_{4k+1}^{IP} \to \mathbb{Z}/2.$$

If M^{4l} *is a closed oriented p.l. manifold, then*

$$\sigma(X \times M) = \sigma(X) \cdot \sigma(M) \quad \text{and} \quad dR(X \times M) = dR(X) \cdot \sigma(M).$$

Proof. The proofs in the manifold case use only duality, so are equally valid here.

(1.7) REMARKS. (a) In §2 and §4 we prove σ and dR are isomorphisms $(k \geq 1)$ and that the other cobordism groups Ω_*^{IP} are trivial.

(b) Every IP space is cobordant to its normalization: the cobordism is the mapping cylinder of the normalization (see [GM 1, (4.1)]). Two cobordant normal IP spaces are cobordant via a normal IP space. Hence it is no loss of generality to assume all IP spaces are normal. A normal pseudomanifold has a stratification with empty codimension two stratum. The link of a point in the codimension three stratum is a p.l. 2-sphere by (1.4)(a). Hence we may assume there is no codimension three stratum. This shows that the theorems in part (a) of this remark are valid in dimensions ≤ 3.

In [S, III.2.2], Siegel shows that every $\alpha \in IH_k(X)$, $k \leq \dim X - 2$ admits a representative cycle x such that $|x|$ is a closed, oriented irreducible pseudomanifold and $|x| - |x|^{k-2} \subseteq X - X_{n-2}$. We will need a relative, more precise version of this.

(1.8) PROPOSITION. *Let $(X, \partial X)$ be an n-dimensional pseudomanifold with boundary, and let $\alpha \in IH_k(X, \partial X)$, $k \leq n-2$. Then there is a representative y for α and a regular neighborhood R of $|y|$ in X such that*

(a) *$|y|$ is an oriented k-dimensional pseudomanifold with boundary $|x| = |y| \cap \partial X$ where $x = \partial y$ (the boundary of y as a chain),*

(b) *$|y| - |y|^{k-2} \subseteq X - X_{n-2}$ and $|x| - |x|^{k-3} \subseteq \partial X - (\partial X)_{n-3}$, and*

(c) *$S := R \cap \partial X$ is a regular neighborhood of $|x|$ such that a p.l. retraction induces injections for $l \geq k-1$,*

$$r_* : IH_l(R, S) \to IH_l(|y|, |x|).$$

Proof. The injectivity of r_* is proved in [GP, 17.3]. The rest follows from [Si, III.2.2] or is trivial.

(1.9) COROLLARY. *With the assumptions and notation above,*

(a) $IH_l(R, S) = 0 = IH_l(R)$, $l \geq k+1$

(b) $IH_k(R) = 0$, if $|x| \neq \phi$

(c) $IH_k(R, S) = \mathbb{Z}$, *generated by* $[y] = \alpha$, *and* $IH_{k-1}(S) = \mathbb{Z}$, *generated by* $\partial[y] = [x]$

(d) Tors $IH_{k-1}(R) = 0$.

Proof. (a), (b) and (c) are immediate from the injectivity of r_*. (One has to use the absolute version on $|x| \subseteq S$). To prove (d), let $|x|^{k-3} = \Sigma$, $|y|^{k-2} = \bar{\Sigma}$. Then $\bar{\Sigma} \cap |x| = \Sigma$ and if $N(\bar{\Sigma})$ denotes a regular neighborhood of $\bar{\Sigma}$ in $|y|$, then $|y| - \mathring{N}(\bar{\Sigma})$ is a k-manifold with boundary $\partial_r N \cup (|x| - \mathring{N}(\Sigma))$ where $N(\Sigma) = N(\bar{\Sigma}) \cap |x|$ is a regular neighborhood of Σ in $|x|$, and $\partial_r N \cap (|x| - \mathring{N}(\Sigma)) = \partial N(\Sigma)$.

Since $\dim \bar{\Sigma} < k-2$, we have an injection

$$H_{k-1}(|y|) \to H_{k-1}(|y|, \bar{\Sigma}).$$

By excision the group on the right is $H_{k-1}(|y| - N(\bar{\Sigma}), \partial_r N)$, and by duality this is $H^1(|y| - \mathring{N}(\bar{\Sigma}), |x| - \mathring{N}(\Sigma))$, which is torsion-free. Consequently $H_{k-1}(|y|)$ is torsion free. Since $IH_k(R, S) \xrightarrow{\cong} IH_{k-1}(S) = \mathbb{Z}$ by (b), the injectivity of r_* in (1.8)(c) implies $r_* : IH_{k-1}(R) \to IH_{k-1}(|y|)$ is injective. Hence $IH_{k-1}(R)$ is also torsion-free.

§2. Even-dimensional *IP*-cobordism

Let X^n be an intersection homology Poincaré (*IP*) space and Ω_n^{IP}, the group of their cobordism classes.

(2.1) THEOREM. (a) $\Omega_{2k}^{IP} = 0$, *k odd.*
(b) $\sigma : \Omega_{2k}^{IP} \xrightarrow{\cong} \mathbb{Z}$, *induced by the signature* σ *when k is even.*

Proof. We have observed in §1 that the signature of a boundary is zero. It remains therefore to show:

(2.2) PROPOSITION. *For any* $k > 0$, *an oriented IP space* X^{2k} *bounds if k is odd, or if k is even and* $\sigma(X^{2k}) = 0$.

Proof. If $k = 1$, then $X = X^2$ is an oriented p.l. 2-manifold, hence the boundary of an oriented 3-manifold. Hence we assume $k \geq 2$.

Under either hypothesis there is $\alpha \in IH_k(X)$ such that $\alpha \cdot \alpha = 0$. Represent α by a cycle y satisfying the conditions of (1.8), (1.9) with $|x| = \phi$; let U be its regular neighborhood.

(2.3) LEMMA. (a) $X' := (X - \mathring{U}) \cap c(\partial U)$ *is an IP-space.*
(b) $W := X \times I \cup c(U \times \{1\} \cup c(\partial U \times \{1\}))$ *is an IP-space with boundary* $X \cup X'$ $(:= X \times \{0\} \cup [(X - \mathring{U}) \times \{1\} \cup c(\partial U \times \{1\})].)$

Proof. X' is a stratified pseudomanifold, since $X - \mathring{U}$ and $c(\partial U)$ are pseudomanifolds with collared boundaries ($= \partial U$) respecting the stratifications: for $c(\partial U)$ this defines the stratification and for $X - \mathring{U}$ it is immediate from the fact that ∂U is transverse to the stratification of X. To check the link conditions defining IP-spaces, observe that the only links of strata in X' or W not already appearing (up to homeomorphism) in X are at interior cone points. Thus, we need to show

$$(2.4)\quad \begin{aligned} &\text{(a) Tors } IH_{k-1}(\partial U) = 0 \\ &\text{(b) } IH_k(U \cup c(\partial U)) = 0. \end{aligned}$$

For this consider the exact sequence

$$(2.5)\quad IH_k(U) \xrightarrow{j_*} IH_k(U, \partial U) \to IH_{k-1}(\partial U) \to IH_{k-1}(U).$$

By (1.9), $IH_{k-1}(U)$ is torsion-free, while by duality $IH_k(U, \partial U) \cong IH_k(U)^* \oplus$ Tors $IH_{k-1}(U)^\wedge = IH_k(U)^*$ (by (1.9) again). Composing j_* with this isomorphism gives the adjoint of the intersection pairing

$$IH_k(U) \times IH_k(U) \to \mathbb{Z}$$

which is zero by assumption. Thus $j_* = 0$ and $IH_k(\partial U)$ is torsion-free.

To prove (2.4)(b), we use the Mayer–Vietoris sequence, where $\hat{U} := U \cup c(\partial U)$:

$$IH_k(\partial U) \to \left\{\begin{matrix} IH_k(U) \\ \oplus \\ IH_k(c\,\partial U) \end{matrix}\right\} \to IH_k(\hat{U}) \to IH_{k-1}(\partial U) \to \left\{\begin{matrix} IH_{k-1}(U) \\ \oplus \\ IH_{k-1}(c\,\partial U) \end{matrix}\right\}.$$

By (1.3), $IH_{k-1}(\partial U) \xrightarrow{\cong} IH_{k-1}(c\,\partial U)$ and $IH_k(c\,\partial U) = 0$. So it suffices to show $IH_k(\partial U) \to IH_k(U)$ is surjective. But this follows from j_* being zero in (2.5). This completes the proof of (2.3).

To compute $IH_k(X')$, consider the Mayer–Vietoris sequence

$$IH_k(\partial U) \to \left\{\begin{matrix} IH_k(X-U) \\ \oplus \\ IH_k(c\,\partial U) \end{matrix}\right\} \to IH_k(X') \to IH_{k-1}(\partial U) \to \left\{\begin{matrix} IH_{k-1}(X-U) \\ \oplus \\ IH_{k-1}(c\,\partial U) \end{matrix}\right\}.$$

Again using (1.3) and the injectivity of $IH_{k-1}(\partial U) \to IH_{k-1}(U)$, extract from the Mayer–Vietoris sequence for $X' = (X - \mathring{U}) \cup c\,\partial U$, the vertical exact sequence in the commutative diagram

$$\begin{array}{lccccccc}
 & IH_{k+1}(U, \partial U) & \xrightarrow{\partial_*} & IH_k(\partial U) & \longrightarrow & IH_k(U) & \longrightarrow & IH_k(U, \partial U) \\
 & \downarrow{\scriptstyle\cong} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\cong} \\
(2.6) & IH_{k+1}(X, X-U) & \longrightarrow & IH_k(X-U) & \longrightarrow & IH_k(X) & \xrightarrow{k_*} & IH_k(X, X-U) \\
 & & & \downarrow & & & & \\
 & & & IH_k(X') & & & & \\
 & & & \downarrow & & & & \\
 & & & 0 & & & &
\end{array}$$

The vertical isomorphisms are from excision; ∂_* is injective because $IH_{k+1}(U) = 0$ (1.9); and k_* satisfies the following:

(2.7) LEMMA. *$k_*(\omega) = \omega \cdot \alpha[t]$, where t is any k-disc transverse to a k-simplex of $|y|$ in $X - X_{2k-2}$, with boundary $\partial t \subseteq \partial U$, and oriented so that $t \cdot y = 1$.*

Proof. $IH_k(X, X-U) \xleftarrow{\cong} IH_k(U, \partial U) \xrightarrow{\cong} IH_k(U)^* = \mathbb{Z}$. Since the second isomorphism is the adjoint of an intersection pairing, $IH_k(U)$ is generated by $\alpha = [y]$

and $t \cdot y = 1$, it follows that any such $[t]$ generates $IH_k(X, X - U)$. But if w represents ω, then by definition [GM1], the intersection number $\omega \cdot \alpha$ is computed by putting w transverse to y so that they meet only interior points of k-dimensional faces of $|y|$ in $X - X_{2k-2}$; while $k_*\omega$ is then just represented by the classes in $IH_k(X, X - U)$ corresponding to the k-simplices of w which meet y (for U sufficiently small). Each such class has been seen to be $\pm[t]$, where the sign is determined by the intersection number with y.

Now an easy (and standard) diagram chase in (2.6) shows that if α has finite order, then

$$IH_k(X') \cong IH_k(X)/\langle\alpha\rangle$$

and if α has infinite order, then

$$\operatorname{rank} IH_k(X') = \operatorname{rank} IH_k(X) - 2.$$

Hence we may find a corbordism V of X to Y where $IH_k(Y) = 0$; $N := V \cup cY$ is then a null-cobordism of X.

For application to the odd-dimensional case we append here some more precise results about special, even-dimensional IP-spaces X^{2k}.

Namely, suppose in the foregoing that $IH_k(X) = \mathbb{Z}^2$, generated by α, β such that $\alpha^2 = 0$ and $\alpha \cdot \beta = 1$. Choose a cycle y such that $\alpha = [y]$, $|y|$ is a pseudomanifold, and U is a regular neighborhood of $|y|$ satisfying the conditions of (1.8), (1.9) with $|x| = \phi$. Let N be the corresponding null-cobordism of X constructed above: $N = X \times I \cup c(U \cup c(\partial U)) \cup cX'$.

(2.8) PROPOSITION. (a) *There is a split exact sequence*

$$0 \to IH_k(U) \to IH_k(X) \to IH_k(N) \to 0$$

where the homomorphisms are induced by inclusions.

(b) $IH_k(N, X) = 0$.

(c) $IH_{k+1}(N, X) \cong \mathbb{Z}$, *generated by* cy' (*the cone on* y') *where* y' *is a* k-*cycle in* ∂U *such that* $i_*[y'] = [y]$, $i_* : IH_k(\partial U) \to IH_k(U)$ *and* $|cy'| = c|y'| \subseteq c(\partial U)$ *in* N.

Proof. Let W be the cobordism of Lemma (2.3) and let $\hat{U} = U \cup c(\partial U)$. Then there is a Mayer–Vietoris sequence

$$(2.9)\quad IH_k(U) \to \left\{\begin{matrix} IH_k(X) \\ \oplus \\ IH_k(c\hat{U}) \end{matrix}\right\} \to IH_k(W) \to IH_{k-1}(U) \to \left\{\begin{matrix} IH_{k-1}(X) \\ \oplus \\ IH_{k-1}(c\hat{U}) \end{matrix}\right\}.$$

First, $IH_k(c\hat{U}) = 0$ by (1.3). Second, $IH_{k-1}(U) \to IH_{k-1}(c\hat{U})$ is injective (proof below). From these two facts will follow part (a) with W in place of N, since $IH_k(U) \cong \mathbb{Z} \to IH_k(X)$ is a split injection by construction. To show $IH_{k-1}(U) \to IH_{k-1}(c\hat{U})$ is injective, recall $IH_j(c\hat{U}, \hat{U}) = 0, j \leq k+1$ by (1.3), so it is enough to show $IH_k(\hat{U}, U) = 0$, which by excision is $IH_k(c(\partial U), \partial U) = 0$. Once again, this follows from (1.3).

Now to complete the proof of (a) we need $IH_k(W) \cong IH_k(N)$. By excision $IH_j(N, W) \cong IH_j(cX', X')$, which vanishes for $j \leq k+1$ by (1.3). This finishes (a).

To prove (b), first observe that $IH_k(W, X) \cong IH_k(N, X)$, again because $IH_j(N, W) = 0, j \leq k+1$. By excision $IH_j(W, X) \cong IH_j(c\hat{U}, U)$ (for all j), which we saw above is isomorphic to $IH_j(\hat{U}, U) \cong IH_j(c(\partial U, \partial U), j \leq k+1$. For $j = k$ this vanishes, proving (b).

To prove (c) consider the commutative square

$$\begin{array}{ccc} IH_{k+2}(cX', X') & \xrightarrow{\cong} & IH_{k+1}(X') \\ \cong \downarrow \text{excision} & & \downarrow \\ IH_{k+2}(N, W) & \longrightarrow & IH_{k+1}(W, X) \end{array}$$

As we have seen above that $IH_{k+1}(N, W) = 0$, and in the proof of (2.2) that $IH_k(X') = 0$ it follows that

$$\begin{aligned} IH_{k+1}(N, X) &\cong \operatorname{cok}(IH_{k+2}(N, W) \to IH_{k+1}(W, X)) \\ &\cong \operatorname{cok}(IH_{k+1}(X') \to IH_{k+1}(W, X)) \\ &\cong IH_{k+1}(W, \partial W). \end{aligned}$$

By duality, $IH_{k+1}(W, \partial W) \cong IH_k(W)^* \oplus \operatorname{Tors} IH_{k-1}(W)\hat{}$. By (a), $IH_k(N) \cong \mathbb{Z}$ and in its proof we showed $IH_k(N) \cong IH_k(W)$; hence $IH_k(W)^* \cong \mathbb{Z}$. On the other hand, in the exact sequence

$$IH_k(W, X) \to IH_{k-1}(X) \to IH_{k-1}(W) \to IH_{k-1}(W, X)$$

we have seen that the extreme terms vanish, while $\operatorname{Tors} IH_{k-1}(X) \cong \operatorname{Tors} IH_k(X)\hat{}$ which vanishes by assumption. Hence $\operatorname{Tors} IH_{k-1}(W) = 0$, so $IH_{k+1}(W) \cong IH_k(W)^* \cong \mathbb{Z}$. The fact that it is generated by cy' is left as an exercise to the reader.

§3. The linking pairing in odd-dimensional IP-spaces

Let X^{2k+1} be an oriented IP-space. Following the muster of surgery theory we want to cap off cycles to kill classes in Tors $IH_k(X)$. Naturally the linking pairing on this group comes into the discussion, just as the intersection pairing did in §2. But here some second-order, non-torsion information must be included to formulate the obstruction to surgery efficiently. We begin with a review of properties of the linking form adapted to our needs. Much of this material is adapted from [W′], the only real difference being our lack of framing data from a normal map. This causes the algebra and topology to be somewhat different; e.g., the de Rham invariant appears here but not in [W′].

Let $z_1, \dots, z_n$ be allowable, oriented k-cycles in X such that each $[z_i]$ is in Tors $IH_k(X)$ and $|z_1|, \dots, |z_n| \subseteq X$ are disjoint pseudomanifolds. Let U be the disjoint union of regular neighborhoods U_i of the $|z_i|$ such that each ∂U_i is transverse to the stratification of X. To define $l([z_i], [z_j])$ choose r_i so that $r_i[z_i] = 0$, then Z_i so that $\partial Z_i = r_i z_i$, and set

$$(3.1)\quad l([z_i], [z_j]) = r_i^{-1}(Z_i \cdot z_j) \in \mathbb{Q}/\mathbb{Z}$$

where Z_i has been chosen to meet z_j admissibly, hence in isolated points. To make this work when $i = j$ and to adapt the procedure to the neighborhood U we use special Z_i's.

Since $IH_{k+1}(U) = 0 = \text{Tors}\, IH_k(U)$ by (1.9), duality says $IH_k(U, \partial U) = 0$. Hence $i_* : IH_k(\partial U) \to IH_k(U)$ is surjective and so split (because $IH_k(U)$ is free according to (1.9)). For each i, choose a cycle z_i' in ∂U such that $i_*[z_i'] = [z_i]$. Suppose $z_1, \dots, z_n$ generate a subgroup $T \subset \text{Tors}\, IH_k(X)$ and let

$$(3.2)\quad j : \mathbb{Z}^n \to T$$

be the surjection with $j(e_i) = [z_i]$ where $\{e_i\}$ is the standard basis of $\mathbb{Z}^n$. Choose a basis of ker j,

$$(3.3)\quad f_i = \sum \alpha_{ij} e_j; \quad \alpha_{ij} \in \mathbb{Z}, \quad i = 1, \dots, n.$$

Since $0 = j(f_i) = \Sigma\, \alpha_{ij}[z_j]$ in $IH_k(X)$ there are allowable $(k+1)$-chains $Z_1', \dots, Z_n'$ in X such that

$$(3.4)\quad \partial Z_i' = \sum \alpha_{ij} z_j', \quad i = 1, \dots, n,$$

and Z_i' meets U only along the $|z_j'|$ for which $\alpha_{ij} \neq 0$ and in transverse discs to the $|z_j|$ in U_j, one for each isolated transverse intersection point of Z_i' with z_j. (Since ∂U is transverse to the stratification of X, there is an inward pointing normal field along the boundary ∂U of $X - \mathring{U}$. Hence there is an allowable imbedding

$|z_j'| \times I \to X - \mathring{U}$ with $|z_j'| \times \{0\}$ sent to ∂U and its complement to the interior of $X - \mathring{U}$. Now choose Z_i'' with $\partial Z_i'' = \Sigma\, \alpha_{ij}(z_j' \times \{1\})$ and so that Z_i'' meets each z_j in only isolated points. Take $Z_i' = Z_i'' + \Sigma\, \alpha_{ij}(z_j' \times I)$.)

Pick a k-simplex in $|z_j|$. Then the dual (in X) $(k+1)$-cell defines a relative cycle t_j in $U \bmod \partial U$. (Any such k-simplex is in the n-manifold $X - X_{n-2}$ and U is assumed chosen so that $|t_j| \cap \partial U$ is the boundary of t_j.) It is obvious that $t_j \cdot z_j = \pm 1$ and t_j will be oriented so as to get the plus sign.

Suppose $|Z_i'| \cap |Uz_j| = \{P_1, \dots, P_S\}$ and set

$$(3.5)\quad \mathring{Z}_i' = Z_i' - \bigcup_{m=1}^{S} t_{P_m}$$

where t_{P_m} is the transverse cell through P_m.

Then $\mathring{Z}_i'$ is a relative $(k+1)$-cycle in $X - \mathring{U} \bmod \partial U$; let d_* be the boundary

$$(3.6)\quad d_* : IH_{k+1}(X - \mathring{U}, \partial U) \to IH_k(\partial U).$$

(3.7) PROPOSITION. *$IH_k(U)$ and $IH_{k+1}(U, \partial U)$ are free of rank n on $\{[z_1], \dots, [z_n]\}$ and $\{[t_1], \dots, [t_n]\}$ respectively, where t_j is any dual $(k+1)$-cell to a k-simplex of $|z_j|$, $j = 1, \dots, n$, oriented so that $t_i \cdot z_j = \delta_{ij}$. The portion of the exact sequence of $(U, \partial U)$*

$$IH_{k+1}(U, \partial U) \xrightarrow{\partial_*} IH_k(\partial U) \xrightarrow{i_*} IH_k(U)$$

is short exact, so $\{\partial_*[t_i] = [\partial t_i], [z_j'] \mid i, j = 1, \dots, n\}$ is a basis of $IH_k(\partial U)$. We have

$$(3.8)\quad d_*[\mathring{Z}_i'] = \sum \gamma_{ij}[\partial t_j] + \sum \alpha_{ij}[z_j']$$

where $\gamma = (\gamma_{ij})$ satisfies

$$(3.9)\quad l([z_l], [z_m]) = (\alpha^{-1}\gamma)_{lm} \bmod \mathbb{Z}.$$

Finally, if δ is the diagonal matrix with $\delta_{ii} = z_i' \cdot z_i'$ (in ∂U), then

$$\gamma\alpha^t + (-1)^k \alpha\gamma^t + \alpha\delta\alpha^t = 0.$$

In particular, if $\delta = 0$, then $\gamma\alpha^t$ is $(-1)^{k+1}$-symmetric.

Proof. It was shown above that i_* is surjective; ∂_* is injective by (1.9) and $IH_k(U)$ is free by (1.9). Since $\operatorname{Tors} IH_{k+1}(U, \partial U)$ is Poincaré dual to

Tors $IH_{k-1}(U)=0$ (by (1.9)), $IH_{k+1}(U,\partial U)$ is free and hence isomorphic to $IH_k(U)^*$ by duality; the fact that $t_i\cdot z_j=\delta_{ij}$ makes $\{[t_1],\ldots,[t_n]\}$ a basis.

Since α is invertible over $\mathbb{Q}$, there is an $n\times n$ matrix $\beta=(\beta_{ij})$ and a positive integer r such that

$$\beta\alpha=rI_n.$$

Consequently, $\partial(\Sigma\,\beta_{li}Z'_i)=rz_l$, $l=1,\ldots,n$, so by definition

$$\begin{aligned} l([z_l],[z_m]) &= \frac{1}{r}\left(\sum \beta_{li}Z'_i\right)\cdot z_m \quad (\mathrm{mod}\ \mathbb{Z})\\ &= \frac{1}{r}\sum \beta_{li}Z'_i\cdot z_m\\ &= \frac{1}{r}\,\beta_{li}\gamma_{im}. \end{aligned}$$

Finally, the intersection number $d_*[\dot{Z}'_i]\cdot d_*[\dot{Z}'_j]$ is zero in ∂U. Hence, for each i and j,

$$\begin{aligned} 0 &= \left(\sum \gamma_{ik}[\partial t_k]+\sum \alpha_{ik}[z'_k]\right)\cdot\left(\sum \gamma_{jl}[\partial t_l]+\sum \alpha_{jl}[z'_l]\right)\\ &= \sum_{k,j}\gamma_{ik}\alpha_{jl}[\partial t_k]\cdot[z'_l]+\sum_{k,j}\alpha_{ik}\gamma_{jl}[z'_k][\partial t_l]+\sum_{k,j}\alpha_{ik}\alpha_{jl}[z'_k][z'_l]\\ &= (\gamma\alpha^t)_{ij}+(-1)^k(\alpha\gamma^t)_{ij}+(\alpha\delta\alpha^t)_{ij}. \end{aligned}$$

(3.10) COROLLARY. *If k is odd, then $\gamma\alpha^t$ is symmetric. If k is even and $l([z_i],[z_i])=0$, $i=1,\ldots,n$, then admissible k-cycles z_i can be chosen in ∂U such that $i_*[z'_i]=[z_i]$ and $z'_i\cdot z'_i=0$ in ∂U. Consequently, $\gamma\alpha^t$ is skew-symmetric.*

Proof. If k is odd, the intersection pairing on $IH_k(\partial U)$ is skew-symmetric so $\delta\equiv 0$. If k is even and $l([z_i],[z_i])=0$ for all i, we will show that for any choice of $z'_i\subset\partial U$ with $i_*[z'_i]=[z_i]$,

$$(3.11)\qquad z'_i\cdot z'_i=2p_i,\qquad p_i\in\mathbb{Z}.$$

Changing z'_i to $z'_i-p_i(\partial t_i)$ gives the desired conclusion.

Set $z_1=z$ and $z'_1=z'$. Recall that $l([z],[z])\equiv(1/r)(Z'\cdot z)$, where Z' is a $(k+1)$-chain such that $\partial Z'=rz'$ and $|Z'|$ intersects U only along dual $(k+1)$-cells, say $|t_{p_1}|,\ldots,|t_{p_s}|$, to smooth points of z in $X-X_{n-2}$, at which Z' and z intersect transversely. Thus there is an integer in such that $r \mid m$ and

$$\sum \epsilon(p_i) = m$$

where $\epsilon(p_i)(=\pm 1)$ is the intersection number at p_i. Consequently, in $X - \mathring{U}$,

$$\partial(Z' - \cup t_{p_i}) = rz' - \sum \epsilon(p_i)\, \partial t_i,$$

so

$$\left(rz' - \sum \epsilon(p_i)\, \partial t_i\right) \cdot \left(rz' - \sum \epsilon(p_i)\, \partial t_i\right) = 0$$

in ∂U. Multiplying out, we get

$$r^2 z' \cdot z' = 2r \sum \epsilon(p_i) = 2rm.$$

Since $r \mid m$, $z' \cdot z'$ is even, as claimed.

(3.12) PROPOSITION. *Let X^{2k+1} be an oriented IP space. Let $[z_1], \ldots, [z_n] \in IH_k(X)$ be torsion classes, $U = \cup U_i$ a regular neighborhood of $\cup |z_i|$, and $\mathring{Z}'_i$ relative $(k+1)$-cycles of $X - \mathring{U}$ mod ∂U satisfying* (3.1). *Given any $(-1)^{k+1}$-symmetric $\rho \in M_n(\mathbb{Z})$, there are k-cycles w_i in X with $[w_i] = [z_i]$ in $IH_k(X)$, a regular neighborhood $V = \cup V_i$ of $\cup |w_i|$ and relative $(k+1)$-cycles $\mathring{W}'_i$ of $X - \mathring{V}$ mod ∂V such that*

$$d_*[\mathring{W}'_i] = \sum (\gamma + \alpha\rho)_{ij}[\partial t_j] + \sum \alpha_{ij}[w'_j].$$

REMARK. Now that $\alpha^{-1}(\gamma + \alpha\rho) = \alpha^{-1}\gamma + \rho \equiv \alpha^{-1}\gamma \bmod M_n(\mathbb{Z})$, so that the linking form is unchanged (see (3.9)), as it must be.

Proof. In case ρ has the form $\mu + (-1)^{k+1}\mu^t$, it is evidently sufficient to make the construction in the special case

$$\rho = e_{pq} + (-1)^{k+1} e_{qp}$$

where e_{pq} is the matrix with (p, q)-entry equal to one and all other entries zero.

Pick distinct points x and y in the interiors of k-simplices of $|z_p|$ and $|z_q|$, respectively, and an imbedded path in $X - X_{n-2}$ connecting them and meeting them only at its endpoints.

Let $|Y_p|$ be the track of an ambient isotopy which drags (only) a small neighborhood of y in $|z_p|$ across (and past) x in $|z_p|$ along the path connecting y and x; let $|Y'_p|$ be the track of the resulting isotopy of $|Z'_p|$. Then $|Y_p|$ and $|Y'_p|$ define allowable $(k+1)$-chains Y_p and Y'_p oriented so that

$$\partial Y_p = \bar{z}_p - z_p$$

$$\partial Y'_p = \bar{z}'_p - z'_p$$

where $|\bar{z}_p|$ and $|\bar{z}'_p|$ are the results of the above isotopies of $|z_p|$ and $|z'_p|$. Furthermore, by construction

$$Y_p \cdot z'_q = Y'_p \cdot z_q = 1.$$

We set

$$w_j = \begin{cases} \bar{z}_p, & j = p \\ z_j, & j \neq p \end{cases}$$

$$W'_i = Z'_i + \alpha_{ip} Y'_p.$$

It is clear from the construction that it is sufficient to show

$$(3.13) \quad W'_i \cdot w_j - Z'_i \cdot z_j = (\alpha\rho)_{ij}.$$

Since by the construction of W'_i and w_j and the choice of ρ, both sides of (3.13) are zero if $j \neq p$ or q, we assume to begin with that $j = q$. If $p \neq q$ then $w_q = z_q$ so the left side of (3.13) is $\alpha_{ip} Y'_p \cdot z_q = \alpha_{ip} = (\alpha\rho)_{iq}$. If $p = q$, then $w_q = \bar{z}_q$, so the left side of (3.13) is

$$\begin{aligned} \alpha_{ip} Y'_p \cdot w_q + Z'_i \cdot (\bar{z}_q - z_q) &= \alpha_{ip} + Z'_i \cdot \partial Y_p \\ &= \alpha_{ip} + (-1)^{k+1} \partial Z'_i \cdot Y_p \\ &= \alpha_{ip} + (-1)^{k+1} \alpha_{iq} = \alpha_{ip} + (-1)^{k+1} \alpha_{ip}. \end{aligned}$$

If $j = p$, the proof of (3.13) is similar.

Every skew-symmetric matrix is a sum of matrices ρ as above; and every symmetric one is such a sum, plus a diagonal matrix. Hence, to finish the proof we need the case where ρ is diagonal. This uses a homological version of twisting the "normal field (which gave us z'_i) on z_i."

Inside a Euclidean neighborhood N, in $X - X_{2k-1}$ and disjoint from all the Z''s and z's, pick a standardly imbedded k-sphere $\sigma = \sigma^k$. In the boundary of a tubular neighborhood T of σ there is an imbedded k-sphere σ', homologous in T to σ, and in N a $(k+1)$-chain Σ' such that $\Sigma' \cap \sigma = 1$ and $\partial\Sigma' = \sigma'$. Let w_1 be the connected sum (through $X - X_{2k-1}$) of the cycles z_1 and σ. There is a regular neighborhood U_1 of w_1 in X and a cycle w_1' in ∂U_1 which is the connected sum of z_1' and σ'. Connecting up $\alpha_{i1}\Sigma'$ with Z_i', for each i, yields a $(k+1)$-chain W_i' with $\partial W_i' = \alpha_{i1} w_1' + \Sigma_{j>1} \alpha_{ij} z_j'$.

Evidently, setting $w_i = z_i$, $w_i' = z_i'$ and $W_i' = Z_i'$ for $i \neq 1$,

$$W_i' \cdot w_j = \begin{cases} Z_i' \cdot z_j, & j \neq 1 \\ Z_i' \cdot z_j + \alpha_{i1}, & j = 1 \end{cases}.$$

Hence, if $\rho = \operatorname{diag}(1, 0, \ldots, 0)$,

$$W_i' \cdot w_j - Z_i' \cdot z_j = (\alpha\rho)_{ij},$$

as required.

(3.14) PROPOSITION. *Let X^{2k+1} be an IP-space.*

(a) *If k is odd then one may choose generators $[z_1], \ldots, [z_n]$ for Tors $IH_k(X)$ and admissible relative $(k+1)$-cycles $\check{Z}_1', \ldots, \check{Z}_n'$ in $(X_0, \partial U)$ as in* (3.5) *such that γ is invertible in* (3.8).

(*b*) *If k is even and the deRham invariant of X^{2k+1} is trivial, then one may choose generators $[z_1], \ldots, [z_n]$ for a subgroup $S \subseteq$ Tors $IH_k(X)$, where $|S|^2 = |$Tors $IH_k(X)|$ and $l(S, S) \equiv 0$, and admissible relative $(k+1)$-cycles $\check{Z}_1', \ldots, \check{Z}_n'$ in $(X_0, \partial U)$ as in* (3.5) *such that $\gamma \equiv 0$ in* (3.8).

Proof. (a) It is well-known ([*W*]) that every nonsingular linking form

$$l : T \times T \to \mathbb{Q}/\mathbb{Z}$$

admits a resolution: there is an $(n \times n)$ symmetric matrix $\alpha \in M_n(\mathbb{Z})$ so that

(a) $\operatorname{cok}(\alpha : \mathbb{Z}^n \to \mathbb{Z}^n) \cong T$

(b) if $j : \mathbb{Z}^n \to \operatorname{cok}\alpha \cong T$ is the surjection induced from (a) and $\alpha^{-1} \in M_n(\mathbb{Q})$ is the inverse of α, then $(\alpha^{-1})_{pq} = l(j(e_p), j(e_q))$, mod $\mathbb{Z}$ where $e_1, \ldots, e_n$ is the standard basis of $\mathbb{Z}^n$.

Use such an α in (3.4) to construct the Z_i''s. Then from (3.9) we also have

$$(\alpha^{-1}\gamma)_{pq} = l(j(e_p), j(e_q)), \ (\text{mod } \mathbb{Z}) \quad \text{where } p, q = 1, \ldots, n$$

and $\alpha^{-1}\gamma$ is symmetric by (3.10). Hence

$$\rho := \alpha^{-1} - \alpha^{-1}\rho$$

is an $n \times n$ symmetric matrix in $M_n(\mathbb{Z})$. Thus our conclusion follows from (3.12).

(b) If the deRham invariant is trivial, then there is a submodule $S \subseteq \text{Tors}\, IH_k(X)$ having the desired properties. Choose generators $[z_1], \ldots, [z_n]$ for it. Since $l \mid S \times S \equiv 0$, (3.9) says $\alpha^{-1}\gamma$ has integral entries; and (3.10) says we can choose the z_i' so that $\alpha^{-1}\gamma$ is skew-symmetric. Taking $\rho = -\alpha^{-1}\gamma$ in (3.12) gives the conclusion.

§4. Odd-dimensional *IP*-cobordism

It is not possible to reduce the size of $IH_k(X^{2k+1})$ by coning off the boundary of a regular neighborhood U of a cycle (as was done in the even-dimensional case): for the cone to be an IP-space its base ∂U (= the link of the cone point) would have to satisfy $IH_k(\partial U) = 0$. This is never the case, even if the cycle in question is null-homologous.

However, according to (2.8), ∂U does bound an IP-space N^{2k+1}. We begin by checking that $(X - \mathring{U}) \cup N$ is IP-cobordant to X. Then we examine the effect of this operation on IH_k.

Let $z_1, \ldots, z_n$ be k-cycles in X and $z_1', \ldots, z_n'$, k-cycles in ∂U satisfying the conclusions of (3.7). Let $U = \dot{\cup}\, U_i$ where U_i is a regular neighborhood of $|z_i|$. Let N_i be the null-cobordism of ∂U_i constructed in (2.8) by coning off a regular neighborhood of z_i' in ∂U.

(4.1) PROPOSITION. *$Y^{2k+2} := X \times I \cup_{U \times \{1\}} (\cup_i c(U_i \cup N_i))$ is an IP-cobordism between $X = X \times \{0\}$ and $X' := (X \times \{1\} - \mathring{U} \times \{1\}) \cup (\cup_i N_i)$.*

Proof. It is clear from the fact that $X - \mathring{U}$ and $\cup N_i$ are IP-spaces with boundary ∂U that X' is an IP-space. Similarly, to see Y is an IP-space, it suffices to check the link conditions (1.4) at the cone points. This means showing

(4.2) $\quad \text{Tors}\, IH_k(U_i \cup N_i) = 0, \quad$ for each i.

Dropping the subscripts, look at the Mayer-Vietoris sequence:

$$IH_k(\partial U) \xrightarrow{a} \begin{Bmatrix} IH_k(U) \\ \oplus \\ IH_k(N) \end{Bmatrix} \to IH_k(U \cup N) \to IH_{k-1}(\partial U) \xrightarrow{b} \begin{Bmatrix} IH_{k-1}(U) \\ \oplus \\ IH_{k-1}(N) \end{Bmatrix}$$

where $N := \cup N_i$. By (2.8)(a), a is an isomorphism and by (2.8)(b) b is injective, giving (4.2).

(4.3) THEOREM. *An IP-space X^{2k+1} bounds if k is odd, or if k is even and its deRham invariant is trivial.*

Proof. If k is odd choose by (3.14)(a) generators $[z_1], \ldots, [z_n]$ of Tors $IH_k(X)$, a regular neighborhood U of $\cup|z_i|$, and k-cycles $z'_1, \ldots, z'_n$ in ∂U so that γ is an isomorphism in (3.8). If k is even and $dR(X) = 0$, apply (3.14)(b) to get $\gamma \equiv 0$ in (3.8).

Using the Y and X' constructed in (4.1), consider the braid diagram

(4.4)
$$\begin{tikzcd}[column sep=small]
 & IH_{k+1}(X') \arrow[dr, "l"] & & IH_{k+1}(Y,X) \arrow[dr] \arrow[rr, bend left, "\partial"] & & IH_k(X) \arrow[dr] & & \\
IH_{k+2}(Y, X\cup X') \arrow[ur] \arrow[dr] & & IH_{k+1}(Y) \arrow[ur] \arrow[dr, "h"] & & IH_{k+1}(Y, X\cup X') \arrow[ur] \arrow[dr] & & IH_k(Y) \arrow[dr] & \\
 & IH_{k+1}(X) \arrow[ur] & & IH_{k+1}(Y,X') \arrow[ur] & & IH_k(X') \arrow[ur] & & IH_k(Y,X)
\end{tikzcd}$$

(4.5) CLAIM. *$IH_k(X')$ is torsion free.*

This will complete the proof, for then $Y \cup cX'$ will be an IP-null-cobordism of X. (4.5) will be proved, following some preliminary work, in (4.8) and (4.9) below.

Recall from (2.8) that to construct the null-cobordism of ∂U, the cycle z'_i was "pushed out to the boundary of a regular neighborhood". Call this cycle z''_i. Then the boundary of the regular neighborhood was coned off. Thus, the cone on z''_i, cz''_i, is a $(k+1)$-chain in W_i (cf. (2.3) and the proof of (2.8) for W), hence in V_i; it is easily checked to be admissible.

(4.6) LEMMA. *In (4.4), $IH_{k+1}(Y, X)$ is free on the classes $[cz''_1], \ldots, [cz''_n]$ and $IH_k(Y, X) = 0$. The homomorphism $IH_{k+1}(X) \to IH_{k+1}(Y, X')$ is trivial.*

Proof. For $i = 1, \ldots, n$, $IH_j(c(U_i \cup N_i), U_i \cup V_i) = 0$ if $j \leq k+1$. Hence $IH_j(Y, X) \cong IH_j(Y, X \times I) \cong IH_j(X \times I \cup \cup_i N_i, X \times I) \cong \oplus IH_j(N_i, \partial U_i)$, where the last isomorphism is by excision. The first conclusion now follows from (2.8)(b) and (2.8)(c).

Tors $IH_{k+1}(Y, X') \cong$ Tors $IH_k(Y, X)\hat{}\ = 0$ by the first part; now the commutative diagram

$$\begin{tikzcd}
IH_{k+1}(X)/\mathrm{Tors}\, IH_{k+1}(X) \arrow[r] \arrow[d, "\cong"] & IH_{k+1}(Y, X') \arrow[d, "\cong"] \\
IH_k(X)^* \arrow[r, "\partial^*"] & IH_{k+1}(Y, X)^*
\end{tikzcd}$$

shows the triviality of the top homomorphism because the image of ∂ was assumed torsion. This gives the desired result.

(4.7) LEMMA. *There is a commutative diagram (up to sign)*

$$\begin{tikzcd}
IH_{k+1}(X) \arrow[r] \arrow[ddd, "="'] & IH_{k+1}(Y) \arrow[r, "h"] \arrow[d, "\cong"'] & IH_{k+1}(Y, X') \\
& IH_{k+1}(Y, \cup(c(U_i \cup N_i)) & IH_{k+1}(\cup(c(U_i \cup N_i)), \cup N_i) \arrow[u, "\cong"'] \\
& & IH_{k+1}(\cup(U_i \cup N_i), \cup N_i) \arrow[u, "\cong"'] \\
IH_{k+1}(X) \arrow[r] & IH_{k+1}(X - \mathring{U}, \partial U) \arrow[r, "g"] \arrow[uu, "\cong"'] & IH_{k+1}(U, \partial U) \arrow[u, "\cong"']
\end{tikzcd}$$

where g is the homomorphism $d_* : IH_{k+1}(X - \mathring{U}, \partial U) \to IH_k(\partial U)$ of (3.8) *composed with the splitting $\sigma : IH_k(\partial U) \to IH_{k+1}(U, \partial U)$ of $IH_{k+1}(U, \partial U) \to IH_k(\partial U)$, induced by the basis of $IH_k(\partial U)$ given in* (3.7).

Proof. Two of the verticals are isomorphisms by (1.3); the others are excision isomorphisms.

Let $\mathring{Z}$ be a relative $(k+1)$-cycle in $X - \mathring{U} \bmod \partial U$. Let $[\partial t_1], \ldots, [\partial t_n]$, $[z'_1], \ldots, [z'_n]$ be a basis of $IH_k(\partial U)$ satisfying the conclusions of (3.7). Since $[z''_i] = [z'_i]$ in $IH_k(\partial U)$ (cf. the paragraph preceding (4.6)) it is possible to take z''_i as representatives of these classes. Let

$$d_*[\mathring{Z}] = \sum \gamma_i [\partial t_i] + \sum \alpha_i [z''_i]$$

in $IH_k(\partial U)$. Using the collaring of ∂U in $X - \mathring{U}$ it is easy to arrange that $\mathring{Z}$ meet ∂U only along the sets $|z''_i|$ and $|\partial t_i|$ (as was done in §3).

Let Z be the cycle in $Y := X \times I \cup (\cup c(U_i \cup N_i))$ obtained from $\mathring{Z}$ (viewed in $X \times \{1\} \subset Y$) as follows:

$$Z = \mathring{Z} + \sum -\gamma_i t_i + \sum -\alpha_i (cz''_i)$$

where cz_i'' is the $(k+1)$-cycle in N_i appearing in Lemma (4.6). Evidently Z is a $(k+1)$-cycle in Y whose class corresponds in the diagram to $[\dot{Z}]$ in $IH_{k+1}(X-U, \partial U)$. As $|\dot{Z}|$ and $|cz_i''|$ are in X', the image of $[Z]$ in $IH_{k+1}(Y, X')$ is $\Sigma -\gamma_i[t_i]$ which clearly corresponds (under the right vertical sequence of isomorphisms) to the class in $IH_{k+1}(U, \partial U)$ denoted the same way. But by definition of g, the above formula for $d_*[\dot{Z}]$ shows $g[\dot{Z}] = \Sigma\, \gamma_i[t_i]$. So the right side of the diagram commutes up to sign. It is obvious that the left side also commutes.

(4.8) PROPOSITION. *Let k be odd. If U is chosen to satisfy the conclusion of* (3.14)(*a*) *then $IH_k(X')$ is free in* (4.4).

Proof. $IH_k(Y, X) = 0$ by (4.6). Since it was assumed, in case k is odd, that $\operatorname{im} \partial = \operatorname{Tors} IH_k(X)$ in (4.4), $IH_k(Y)$ must be free.

Since γ in (3.8) is an isomorphism, it is immediate that g is surjective in (4.7) so by commutativity in (4.7), $IH_{k+1}(Y) \to IH_{k+1}(Y, X')$ is as well. Thus $IH_k(X') \to IH_k(Y)$ is injective, so we are done.

(4.9) PROPOSITION. *Let k be even. If U is chosen to satisfy the conclusion of* (3.14)(*b*), *then $IH_k(X')$ is free in* (4.4).

Proof. First consider the modified braid

$$\begin{tikzcd}[column sep=small]
\overline{IH_{k+1}(X')} \arrow[dr, "\bar{l}"] \arrow[rr, bend left] & & IH_{k+1}(Y,X) \arrow[dr, "i"] \arrow[rr, bend left, "\partial"] & & IH_k(X) \arrow[dr] & \\
& \overline{IH_{k+1}(Y)} \arrow[ur] \arrow[dr, "\bar{h}"] & & IH_{k+1}(Y, X\cup X') \arrow[ur, "f"] \arrow[dr, "\delta"] & & IH_k(Y) \\
& & IH_{k+1}(Y,X') \arrow[ur] \arrow[rr, bend right] & & IH_k(X') \arrow[ur] &
\end{tikzcd}$$

where

$$\overline{IH_{k+1}(X')} = IH_{k+1}(X')/\operatorname{im} IH_{k+2}(Y, X\cup X')$$

and

$$\overline{IH_{k+1}(Y)} = IH_{k+1}(Y)/\operatorname{im} IH_{k+1}(X)$$

and $\bar{l}$ and $\bar{h}$ are the induced maps (using the second statement of (4.6) to define $\bar{h}$).

First, $\bar{h} \equiv 0$: this is immediate from (4.7), (3.14)(b), and the identification of g with γ upon passage to $\overline{IH_{k+1}(X - \mathring{U}, \partial U)}$. Hence $\bar{l}$ is an isomorphism so f induces

$$\operatorname{im} i \cong \operatorname{im} \partial.$$

Second, from the diagram

$$\begin{tikzcd}
IH_{k+2}(Y, X \cup X') \arrow[r] \arrow[d] & IH_{k+1}(X) \arrow[d] \\
IH_{k+1}(X') \arrow[r, "l"] \arrow[d] & IH_{k+1}(Y) \arrow[d] \\
\overline{IH_{k+1}(X')} \arrow[r, "\bar{l}"] & \overline{IH_{k+1}(Y)}
\end{tikzcd}$$

and the surjectivity of the top map (by (4.6)), it follows that l is itself surjective.
Now from above, $\operatorname{im} i \subseteq \operatorname{Tors} IH_{k+1}(Y, X \cup X')$. Also, since

$$|\operatorname{im} \partial|^2 = |\operatorname{Tors} IH_k(X)|,$$

$$|\operatorname{im} \partial| = |\operatorname{Tors} IH_k(Y)| = |\operatorname{Tors} IH_{k+1}(Y, X \cup X')\hat{\ }\,| = |\operatorname{Tors} IH_{k+1}(Y, X \cup X')|.$$

Now using $|\operatorname{im} i| = |\operatorname{im} \partial|$,

(4.10) $\quad \operatorname{im} i = \operatorname{Tors} IH_{k+1}(Y, X \cup X')$.

Finally consider the ladder

$$\begin{tikzcd}
\operatorname{Tors} IH_{k+1}(Y, X \cup X') \arrow[r, "\delta \mid \mathrm{Tors}"] \arrow[d] & \operatorname{Tors} IH_k(X') \arrow[d] \\
IH_{k+1}(Y, X \cup X') \arrow[r, "\delta"] \arrow[d] & IH_k(X') \arrow[d] \\
IH_{k+1}(Y, X \cup X')/\mathrm{Tors} \arrow[r] \arrow[d, "\cong"] & IH_k(X')/\mathrm{Tors} \arrow[d, "\cong"] \\
IH_{k+1}(Y)^* \arrow[r, "l^*"] & IH_{k+1}(X')^*
\end{tikzcd}$$

From above, l^* is injective; and δ is surjective because $IH_k(Y, X) = 0$ (4.6). Thus $\delta \mid \text{Tors}$ is surjective. But it is also trivial by (4.10). This implies $IH_k(X')$ is free.

§5. The characteristic variety theorem

Let $\Omega_n^{IP}(X)$ be the bordism of maps $N^n \to X$, where N is an n-dimensional IP-space, defined á lá [CF]; the proof given there that $\Omega_*^{IP}(X)$ is a generalized homology theory works here with one addendum: to replace [CF, (3.1)], given closed disjoint subsets P and Q of an IP space N, one needs an IP space N'^n (with non-empty boundary, in general) such that $P \subset N' \subset N$ and $Q \cap N' = \phi$. For this, one may assume P and Q are subcomplexes and take N' to be a regular neighborhood of P. Then the boundary of N' can be taken transverse to the stratification (in fact to a triangulation) of N, so N' is an IP space with collared IP space boundary. Likewise we denote the bordism group of maps $\mathcal{N}^n \to X$, where $\mathcal{N}^n$ is a $\mathbb{Z}/k$-IP space (defined following [MS, §1]), by $\Omega_n^{IP}(X; \mathbb{Z}/k)$; this defines a generalized homology theory $\Omega_*^{IP}(X; \mathbb{Z}/k)$, for each $k \geq 0$.

For a fixed integer $k > 0$ let

$$(5.1) \quad \{h_i, h_i(\mathbb{Z}/k)\}_{\text{per}} := \left\{ \begin{array}{ccc} \bigoplus\limits_{i \geq 0} \tilde{\Omega}_{4i}^{IP}(X) & \xrightarrow{\oplus h_i} & \mathbb{Z} \\ \downarrow & & \downarrow \\ \bigoplus\limits_{i \geq 0} \tilde{\Omega}_{4i}^{IP}(X; \mathbb{Z}/k) & \xrightarrow{\oplus h_i(\mathbb{Z}/k)} & \mathbb{Z}/k \end{array} \right\}$$

be a commutative diagram in which the verticals are the natural maps, $\tilde{\Omega}_*^{IP}$ denotes reduced bordism, and such that for all i, the *periodicity relations*

$$(5.2) \quad \begin{aligned} &h_{i+4}((N \to X) \times \mathbb{C}P^2) = h_i(N \to X) \\ &h_{i+4}(\mathbb{Z}/k)((\mathcal{N} \to X) \times \mathbb{C}P^2) = h_i(\mathcal{N} \to X) \end{aligned}$$

are satisfied, where N (resp. $\mathcal{N}$) is $4i$-dimensional IP space (resp. $\mathbb{Z}/k - IP$ space). The set of such diagrams forms a group denoted $G(X, k)$.

For each $l > 0$ and each $\{h_i, h_i(\mathbb{Z}/k)\}_{\text{per}} \in G(X, k)$, there is an element of $G(X, kl)$ defined by the outer part of the diagram

$$\begin{array}{ccccccc} \bigoplus \tilde{\Omega}_{4i}(X) & \xrightarrow{=} & \bigoplus \tilde{\Omega}_{4i}(X) & \xrightarrow{\oplus h_i} & \mathbb{Z} & \xrightarrow{\times l} & \mathbb{Z} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \bigoplus \tilde{\Omega}_{4i}(X; \mathbb{Z}/kl) & \longrightarrow & \bigoplus \tilde{\Omega}_{4i}(X; \mathbb{Z}/k) & \xrightarrow{\oplus h_i(\mathbb{Z}/k)} & \mathbb{Z}/k & \xrightarrow{\times l} & \mathbb{Z}/kl \end{array}$$

This defines a homomorphism $G(X, k) \to G(X, kl)$ and we set

$$\Omega_{IP}(X)_{\text{per}} := \varinjlim_k G(X, k).$$

The *characteristic variety theorem*, proved by Morgan (unpublished) for a different bordism theory, is:

(5.3) THEOREM. *For any finite CW complex X, there is a natural isomorphism of sets*

$$\mu : [X, G/TOP] \to \tilde{\Omega}_{IP}(X)_{\text{per}}.$$

(5.4) REMARKS. (a) Define a negatively graded cohomology theory in the usual way:

$$\tilde{\Omega}_{IP}^{-n}(X)_{\text{per}} := \tilde{\Omega}_{IP}(\Sigma^n X)_{\text{per}}, \quad n \geq 0.$$

Then these groups are periodic mod 4. For

$$\tilde{\Omega}_{IP}^{-n}(X)_{\text{per}} := \varinjlim_k G(\Sigma^n X, \mathbb{Z}/k)$$

$$\cong \varinjlim_k \left\{ \begin{array}{ccc} \bigoplus_{j \equiv -n(4)} \tilde{\Omega}_j^{IP}(X) & \longrightarrow & \mathbb{Z} \\ \downarrow & & \downarrow \\ \bigoplus_{j \equiv -n(4)} \tilde{\Omega}_j^{IP}(X; \mathbb{Z}/k) & \longrightarrow & \mathbb{Z}/k \end{array} \right\}$$

The proof of (5.3) will show that this cohomology theory is naturally isomorphic to that of the Ω-spectrum G/TOP.

(b) A standard argument, which uses the Bockstein sequence relating $\Omega_*^{IP}(X)$ to $\Omega_*^{IP}(X; \mathbb{Z}/k)$, shows that

$$(5.5) \quad \tilde{\Omega}_{IP}(X)_{\text{per}} \cong \left\{ \begin{array}{ccc} \bigoplus \tilde{\Omega}_{4i}^{IP}(X; \mathbb{Q}) & \xrightarrow{\oplus h_i(\mathbb{Q})} & \mathbb{Q} \\ \downarrow & & \downarrow \\ \bigoplus \tilde{\Omega}_{4i}^{IP}(X; \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\oplus h_i(\mathbb{Q}/\mathbb{Z})} & \mathbb{Q}/\mathbb{Z} \end{array} \right\}$$

(cf. [MS, §2]).

To prove (5.3) it is enough to construct a natural μ and then show it is bijective for $X = S^n$, $n \geq 0$. Since all of our topology has been (in §2–4) and will be in the p.l. category we need to "reduce to G/PL"; and because of periodicity we can reduce to showing bijectivity on S^n, $n \gg 0$. These things will get us around low-dimensional anomalies (e.g., the non-trivial k-invariant in $(G/PL)_{(2)}$).

(5.6) PROPOSITION. *To prove* (5.3), *it is enough to construct a natural map of sets*

$$\nu : [X, G/PL] \to \tilde{\Omega}_{IP}(X)_{\text{per}},$$

bijective if $X = S^n$, $n \geq 8$.

Proof. Recall that the fibre of the natural map $G/PL \to G/TOP$ is $K(\mathbb{Z}/2, 3)$ and that there is a homotopy equivalence $G/TOP \to \Omega^4(G/TOP)$. (See, for instance, [Sb, p. 327].) Assuming the natural map ν exists, define μ to be the composition

$$\begin{aligned}[X, G/TOP] &\xrightarrow{\cong} [X, \Omega^8(G/TOP)] \xrightarrow{\cong} [\Sigma^8 X, G/TOP] \\ &\xrightarrow{\cong} [\Sigma^8 X, G/PL] \xrightarrow{\nu} \tilde{\Omega}_{IP}(\Sigma^8 X)_{\text{per}} \xrightarrow{\cong} \tilde{\Omega}_{IP}(X)_{\text{per}}.\end{aligned}$$

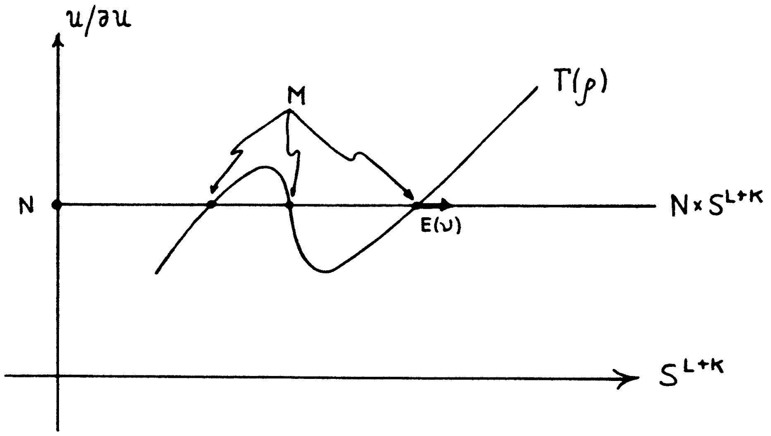
Since all the isomorphisms are natural, so is μ; and its bijectivity will follow from that of ν for S^n, $n \geq 8$.

(5.7) To construct the map ν, suppose given $\alpha : X \to G/PL$. Equivalently, this is a pair (η, t_X) where η is a p.l. bundle over X and t_X is a fibre homotopy equivalence $\eta \xrightarrow{\cong} \epsilon^L$. If $f : N \to X$ (resp. $\varphi : \mathcal{N} \to X$) is a representative for an element of $\Omega_n^{IP}(X)$, $n = 4i$ (resp. $\Omega_n^{IP}(X; \mathbb{Z}/k)$) we need to produce an element of $\mathbb{Q}$ (resp. $\mathbb{Q}/\mathbb{Z}$) so that (5.5) commutes. This will be done in (5.16) and (5.21) below. We work temporarily with $f : N \to X$ only, since the constructions for φ are basically the same.

Pulling (η, t_X) back over N gives a pair (ξ, t_N) over N. Let R be a regular neighborhood of a p.l. imbedding of N in $\mathbb{R}^K$, $K \gg 0$ and let $c : R \to N$ be a p.l. collapse. Let U be the total space of a disc bundle of $c^*\xi$. Using t_N and the obvious collapse $S^K \to R/\partial R$, U admits a spherical reduction

(5.8) $\rho : S^{K+L} \to U/\partial U.$

According to [Mc, §4] the transverse intersection of the graph $\Gamma(\rho)$ and the stratified space $S^{K+L} \times N$ in $S^{K+L} \times (U/\partial U)$ is a stratified space $M = M^{4i}$ with the same local structure as $S^{K+L} \times N$; in particular M is IP space. Transversality in this context implies that M has a p.l. normal block bundle ν in $S^{K+L} \times N$ which is equal to the restriction to M of the normal bundle of $\Gamma(\rho)$ in $S^l \times (U/\partial U)$.



In particular we have a normally non-singular map g (see [GMII, 5.4])

$$(5.9)\qquad \begin{array}{ccc} E(\nu) & \overset{j}{\hookrightarrow} & N \times S^{K+L} \\ \uparrow{\scriptstyle i} & & \downarrow{\scriptstyle \pi_1} \\ M & \xrightarrow{g} & N \end{array}$$

where i is the zero section and j is an open inclusion. We may and will replace S^{K+L} by $\mathbb{R}^{K+L}$ when convenient. Then g induces maps

$$g_*: IH_*(M) \to IH_*(N), \qquad g^*: IH_*(N) \to IH_*(M)$$

essentially given by image and inverse image of cycles, and satisfying the *projection formula*

$$(5.10)\quad g_*(g^*z \cdot y) = z \cdot g_*y.$$

(5.11) PROPOSITION. *The homomorphism g_* is a surjection in each degree, and if $x \in IH_k(M)$, $y \in IH_{n-k}(M)$, $g_*(x) \cdot g_*(y) = x \cdot y$.*

Proof. The first assertion follows by taking $y = 1$ in the projection formula. The second follows by taking $z = g_*x$ in the projection formula and using the fact that $g^*g_*x = x$.

(5.12) REMARKS. (a) The construction of g in the diagram (5.9) evidently generalizes that of a normal map of manifolds $M \to N$ from a map $N \to G/PL$.

(b) Changing the imbedding of N in $\mathbb{R}^K$, $K \gg 0$, the collapse c, the choice of α in its homotopy class, the choice of N in its class in $\Omega_n^{IP}(X)$, or the perturbation of $\Gamma(\rho)$ to transverse position with $N \times S^{K+L}$, will change the diagram (5.9) by a cobordism in the obvious sense. We refer to such a cobordism as a *normal cobordism.*

(c) Let ξ_*^{-1} denote the bundle $\pi_1^* \xi^{-1}$ over $N \times \mathbb{R}^{K+L}$, where ξ^{-1} is a stable p.l. inverse to ξ over N. Then we claim that the normal bundle to the inclusion $M \xrightarrow{i} E(\nu) \xrightarrow{j} N \times \mathbb{R}^{K+L} \hookrightarrow E(\xi_*^{-1})$ is trivial, hence extends to an open inclusion

$$(5.13) \quad M \times D^J \hookrightarrow E(\xi^{-1}) \qquad (5.13)$$

where ξ^{-1} is a(nother) stable inverse of ξ over N. Evidently this will follow if we can show that ξ restricts (stably) to ν over M.

For this, note that by construction $\rho : S^{K+L} \to U/\partial U$ restricts to the canonical collapse $k : S^K \to R/\partial R$ and that the normal bundle of R in U is ξ. Consequently, the stable normal bundle of $M := \Gamma(\rho) \cap (N \times S^{K+L})$ in $\Gamma(k) \cap (N \times S^K) = N$ is the restriction of ξ to M.

We will refer to this "stabilization" of the normally nonsingular map (5.9)

$$(5.14) \quad \begin{array}{ccc} M \times D^J & \overset{j}{\hookrightarrow} & E(\xi^{-1}) \\ \uparrow{\scriptstyle i} & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow{g} & N \end{array}$$

as a normal map also. From now on we work exclusively with it. The following proposition is the central homological fact about these normal maps.

(5.15) PROPOSITION. *Let $g : M^{4i} \to N^{4i}$ be the normal map in (5.14). If $x \in \ker g_* : IH_{2i}(M) \to IH_{2i}(N)$, then $x \cdot x$ is even.*

Proof. Since ξ^{-1} is fibre homotopically trivial, it has trivial Stiefel–Whitney classes. Let $s : N \to E(\xi^{-1})$ denote the zero section, and let IH_* mean intersection homology with $\mathbb{Z}/2$-coefficients. Then $Sq_{2i} s_* z = s_* Sq_{2i} z$ for each $z \in IH_{2i}(N)$, by [GP, 4.2]. Any $y \in IH_{2i}(E(\xi^{-1}))$ is $s_* z$, for some z since s_* is an isomorphism. Thus we have $\pi_* Sq_{2i} y = \pi_* Sq_{2i} s_* z = Sq_{2i} z = Sq_{2i} \pi_* y$. Again by [GP, 4.2], for any $x \in IH_{2i}(M)$, $i_* j_* Sq_{2i} x = Sq_{2i} i_* j_* x$. Consequently, if $x \in \ker g_*$, $0 = Sq_{2i} g_* x = g_* Sq_{2i} x$. But g_* is the identity on IH_0, so $Sq_{2i} x = 0$. Since $Sq_{2n} x = x \cdot x \bmod 2$, the proof is complete.

(5.16) COROLLARY. *In the notation above, for each homotopy class of maps* $\alpha : X \to G/PL$ *there is a well-defined homomorphism*

$$h_i : \Omega_{4i}^{IP}(X) \to \mathbb{Z}$$

given by

$$h_i(f : N \to X) = \frac{1}{8}\sigma(\ker g_*)$$

where $\sigma(\ker g_*)$ *denotes the signature of the intersection form on*

$$\ker g : IH_{2i}(M) \to IH_{2i}(N).$$

(5.17) Next, given a normal map of $\mathbb{Z}/k$ manifolds

$$\begin{array}{cccc} & \mathcal{M} \times D^J & \longrightarrow & E(\xi^{-1}) \\ (5.18) & \downarrow & & \downarrow \\ & \mathcal{M} & \longrightarrow & \mathcal{N} \end{array}$$

arising from a $\mathbb{Z}/k$-bordism element $\mathcal{N} \to X$ and a map $X \to G/PL$ (as constructed in (5.7) and (5.12)(c) above) we need an element of $\mathbb{Q}/\mathbb{Z}$ to use together with h_i in (5.16) in a commutative diagram (5.5). This will be done, following [MS], by associating to the bockstein of (5.18) an element of $\mathbb{Z}/8$, hence to (5.18) itself an element of $\mathbb{Z}/k \otimes \mathbb{Z}/8 = \mathbb{Z}/(k, 8) \subseteq \mathbb{Q}/\mathbb{Z}$. In more detail, given a $(4n-1)$-dimensional surgery problem $g : M \to N$ (normal data omitted from the notation) of p.l. manifolds, each $(2n-1)$-cycle in the 2-torsion subgroup of $\ker g_* : H_{2n-1}(M) \to H_{2n-1}(N)$ is represented by an imbedded submanifold having a non-zero section of its normal bundle. Using this "normal field", Morgan and Sullivan define a self-linking form on the torsion subgroup of $\ker g_*$; a "Gauss sum" of the values of this form is the $\mathbb{Z}/8$ invariant referred to above.

We will proceed in the same way here. The details are necessarily more complicated because our cycles have no chance to be represented by imbedded submanifolds nor even by normally nonsingular maps (which do have normal bundles) of stratified spaces to M (now an IP space). Hence no normal bundle, let alone normal field, is possible in the usual sense. The substitute is the following.

(5.19) DEFINITION. Let M be a stratified pseudomanifold and let $z \subset M$ be an allowable cycle. A *normal field* on z is an allowable chain $z \times I \subset M$ extending z.

Given such a normal field on a representative x for $\alpha \in \operatorname{Tors} IH_{2n-1}(M^{4n-1})$ let $x' = x \times \{1\}$ and let w be an allowable $2n$-chain with $\partial w = sx$, where $s\alpha = 0$. Define

$$q(\alpha) = w \cdot x'/2s \in \mathbb{Q}/\mathbb{Z}. \tag{5.20}$$

The fact that this can be well-defined is the non-trivial part of the following proposition, whose proof will be given at the end of this chapter.

(5.21) PROPOSITION. *Given a normal map of* $(4n-1)$*-dimensional IP spaces*

$$\begin{array}{ccc} M \times D^J & \longrightarrow & E(\xi^{-1}) \\ \uparrow & & \downarrow \\ M & \xrightarrow{g} & N \end{array} \tag{5.22}$$

there is a function $q : \operatorname{Tors} IK_{2n-1}(g) \to \mathbb{Q}/\mathbb{Z}$, $IK_{2n-1} = \ker g_* : IH_{2n-1}(M) \to IH_{2n-1}(N)$, *such that*

$$2q(\alpha) = l(\alpha, \alpha)$$

and

$$q(\alpha+\beta) - q(\alpha) - q(\beta) = l(\alpha, \beta)$$

where $l : K_{2n-1}(g) \times K_{2n-1}(g) \to \mathbb{Q}/\mathbb{Z}$ *is the linking form. Define*

$$G(q) = \sum_{\alpha \in K_{2n-1}(g)} e^{2\pi i q(\alpha)} \in \mathbb{Z}/8.$$

Then if (5.22) *is the boundary of a normal map of* $4n$*-dimensional IP spaces, its signature* (*in the sense of* (5.16)) *is congruent mod* 8 *to* $G(q)$.

Now just as in [MS, §5], this proposition shows that the maps arising from a homotopy class $\alpha : X \to G/PL$,

$$h_i : \Omega_{4i}(X) \to \mathbb{Q}$$

from (5.16) and

$$\Omega_{4i}(X; \mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

given by the Gauss sum from the bockstein in (5.21) together produce an element $v(\alpha)$ of $\bar{\Omega}_{IP}(X)_{\text{per}}$ as described in (5.4)(b). The fact that this element is well-defined is completely routine, given (5.12)(b).

So given this last proposition, what remains (see (5.6)) is to show that $v : [S^n, G/PL] \to \bar{\Omega}_{IP}(S^n)_{\text{per}}$ is an isomorphism, $n \geq 8$.

Recall that we have shown in §2–4 that

$$\Omega_n^{IP} \cong \begin{cases} \mathbb{Z}, & n \equiv 0(4) \\ \mathbb{Z}/2, & n \equiv 1(4),\ n > 1 \\ 0, & \text{otherwise} \end{cases}$$

Since $\bar{\Omega}_k^{IP}(S^n; R) \cong \bar{\Omega}_{k-1}^{IP}(S^{n-1}; R)$, for $k > 0$, $n > 0$ and $R = \mathbb{Q}$ or $\mathbb{Q}/\mathbb{Z}$, we have

$$(5.23)\quad \bar{\Omega}_{4i}^{IP}(S^n; R) \cong \begin{cases} 0, & 4i < n \\ \Omega_{4i-n}^{IP}(R), & 4i \geq n. \end{cases}$$

It follows easily from this and (5.4)(b) that

$$\bar{\Omega}_{IP}(S^n)_{\text{per}} = \begin{cases} \mathbb{Z}, & n \equiv 0(4) \\ \mathbb{Z}/2, & n \equiv 2(4) \\ 0, & n \text{ odd.} \end{cases}$$

so that $\bar{\Omega}_{IP}(S^n)_{\text{per}}$ and $[S^n, G/PL]$ are abstractly isomorphic for each $n \geq 0$. We need to show v induces an isomorphism for all $n \geq 8$.

Begin with the case $n = 4k + 2$. Let $\mathcal{N}^6$ be a $\mathbb{Z}/2$-manifold with non-trivial de-Rham invariant on its bockstein. For instance, take the non-zero representative of $\Omega_5^{SO} = \mathbb{Z}/2$, SU_3/SO_3; it will have non-trivial deRham invariant, and there will be an oriented 6-manifold N with boundary $SU_3/SO_3 \cup SU_3/SO_3$. Identifying the copies of SU_3/SO_3 by an orientation reversing homeomorphism gives $\mathcal{N}$. This $\mathcal{N}$ also represents the non-trivial element of $\Omega_6^{IP}(\mathbb{Q}/\mathbb{Z}) = \mathbb{Z}/2$, so $(\mathbb{C}P^2)^k \times \mathcal{N}$ represents the non-trivial element of $\Omega_{4k+6}^{IP}(\mathbb{Q}/\mathbb{Z})$, for any $k \geq 0$. The suspension isomorphism for $\Omega_*^{IP}(\text{—}; \mathbb{Q}/\mathbb{Z})$ again says, for $4i \gg 4k + 2$,

$$(5.24)\quad \bar{\Omega}_{4i}^{IP}(S^{4k+2}, \mathbb{Q}/\mathbb{Z}) \cong \Omega_{4i-4k-2}^{IP}(\mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/2$$

where the non-trivial element is represented by

$$\varphi : (\mathbb{C}P^2)^{i-k-2} \times \mathcal{N} \times S^{4k+2} \xrightarrow{\pi_3} S^{4k+2}.$$

(The isomorphism (5.23) is given by sending $\varphi : \mathcal{M} \to S^{4k+2}$ to the $\mathbb{Z}/2 - IP$ space $\varphi^{-1}(p)$, where φ has been put transverse to $p \in S^{4k+2}$.)

Let $\alpha : S^{4k+2} \to G/PL$ represent the non-trivial element of $[S^{4k+2}, G/PL] = \mathbb{Z}/2$. Then carrying out the procedure of (5.7) with this α and φ as above produces the $\mathbb{Z}/2$-surgery problem

$$(\mathbb{C}P^2)^{i-k-2} \times \mathcal{N} \times (K^{4k+2} \to S^{4k+2})$$

where $K^{4k+2} \to S^{4k+2}$ is the Kervaire problem. Since the Gauss sum $G(q)$ (in (5.21)) of its bockstein is non-trivial by [MS, Theorem 6.1], $\nu(\alpha)$ is the non-trivial element of $\bar{\Omega}_{IP}(S^{4k+2})_{\text{per}}$.

Finally we show ν induces an isomorphism $[S^{4k}, G/PL] \to \bar{\Omega}_{IP}(S^{4k})_{\text{per}}$, $k \geq 2$. By a discussion completely analogous to that above, a generator of $\Omega_{4i}(S^{4k}) = \mathbb{Z}$, $i > k$, is $f : (\mathbb{C}P^2)^{i-k} \times S^{4k} \to S^{4k}$. Let $\alpha : S^{4k} \to G/PL$ be a generator of $S^{4k} \to G/PL$. Carrying out the procedure of (5.7), with this α and f, produces the surgery problem

$$(\mathbb{C}P^2)^{i-k} \times (M^{4k} \to S^{4k})$$

where M^{4k} has signature $= 8$. It follows easily from (5.16) that $\nu(\alpha)$ is the generator of $\bar{\Omega}_{IP}(S^{4k})$.

(5.25) To complete the proof of (5.6) and hence of (5.3), it remains to prove (5.21). Changing notation slightly, let

$$\text{(5.26)}\qquad \begin{array}{ccc} M \times I^k & \longrightarrow & E(\xi^{-1}) \\ \uparrow & & \downarrow \\ M & \xrightarrow{g} & N \end{array}$$

be a normal map of IP spaces M and N, where $\dim M = \dim N = 4n - 1$, $I = [-1, 1]$ and $k > 0$. Motivation for the construction to follow can be found in Remark (5.36) below.

Given a $(2n-1)$-cycle x representing $\alpha \in IK_{2n-1}(g) := \ker g_* : IH_{2n-1}(M) \to IH_{2n-1}(N)$, we view it as a cycle in $M \times \{0\} \subseteq M \times I^k$ which is the boundary of an allowable $2n$-chain y in $E(\xi^{-1})$. We may assume that $(|y|, |x|)$ is a pseudomanifold with boundary, where the collaring factor $[0, 1]$ of $|x|$ in $|y|$ is identified with $[0, 1]$ in the first $I = [-1, 1]$ in I^k.

Let $R = R^{k+4n-1}$ be the relative regular neighborhood of $|y|$ in $E(\xi^{-1})$, meaning that $S = S^{k+4n-2} := R \cap (M \times \{0\} \times I^{k-1})$ is the product of $T = T^{4n-1}$, a regular neighborhood of $|x|$ in M, with I^{k-1}, and $R \cap (M \times I^k) = T \times [0, 1] \times I^{k-1}$. Note that S is collared in R and that R is a $(k + 4n - 1)$-dimensional pseudomanifold whose boundary decomposes into $S \cup \partial_r R$ where $S \cap \partial_r R = \partial S$.

We thus have

(5.27) (a) the inclusion of collared pairs $(|y|, |x|) \hookrightarrow (R, S)$ where
(b) $S = T \times [0, 1] \times I^{k-1}$, T is a regular neighborhood of $|x|$ in M and R is a relative regular neighborhood of $|y|$ in $E(\xi^{-1})$, and
(c) $\partial R = S \cup \partial_r R$, $\partial S = S \cap \partial_r R$.

Consider the diagram of vertical and horizontal exact sequences

(5.28)

$$\begin{tikzcd}
IH_{2n+1}(R, \partial_r R) \arrow[r] \arrow[d] & IH_{2n}(\partial_r R) \arrow[d] & & \\
IH_{2n+1}(R, S \cup \partial_r R) \arrow[r] \arrow[d, "\tau"] & IH_{2n}(\partial_r R, \partial S) \arrow[r, "\rho"] \arrow[d] & IH_{2n}(R, S) \arrow[r] \arrow[d] & IH_{2n}(R, S \cup \partial_r R) \\
IH_{2n}(S \cup \partial_r R, \partial_r R) \arrow[r] \arrow[d] & IH_{2n-1}(\partial S) \arrow[r, "\sigma"] & IH_{2n-1}(S) & \\
IH_{2n}(R, \partial_r R) & & &
\end{tikzcd}$$

First of all we know ((1.9)) that

(5.29) $IH_{2n}(R, S) \xrightarrow{\cong} IH_{2n-1}(S) = \mathbb{Z}$

with source generated by $[y]$ and target, by $[x]$. The exact sequences in the diagram give us:

(5.30) $\sigma : IH_{2n-1}(\partial S) \to IH_{2n-1}(S)$ is an isomorphism if $k > 1$ and is surjective if $k = 1$

$$(5.31)\quad \begin{array}{l} \rho : IH_{2n}(\partial_r R, \partial S) \to IH_{2n}(R, S) \text{ is an isomorphism} \\ \text{if } k > 1 \text{ and is surjective if } k = 1. \end{array}$$

The second assertion follows from the vanishing of $IH_{2n}(R, S \cup \partial_r R) = IH_{2n}(R, \partial R)$ for $k \geq 1$ and of $IH_{2n+1}(R, \partial R)$ for $k > 1$. This in turn follows by duality from the vanishing of $IH_j(R)$ for $j \geq 2n$ and its torsion-freeness for $j \geq 2n - 1$ (see (1.9)). The proof of the first assertion is similar.

Assume $k > 1$. Then by (5.30) and (5.31) there is an admissible relative $2n$-cycle y' in $\partial_r R$, unique in $IH_{2n}(\partial_r R, \partial S)$, such that $x' := \partial y'$ is an admissible $(2n-1)$-cycle in ∂S and

$$(5.32)\quad \rho[y'] = [y]$$

$$(5.33)\quad \sigma[x'] = [x].$$

We may assume $(|y'|, |x'|)$ is a (collared) pseudomanifold pair. Now

$$\partial S = \partial(T \times I^{k-1}) = \partial T \times I^{k-1} \cup T \times \partial I^{k-1}$$

and evidently we can also choose $[x']$ satisfying (5.33) to be represented by $x \times (0, \ldots, 0, 1)$ where

$$x := x \times (0, \ldots, 0) \subseteq M \times (0, \ldots, 0) \subseteq M \times I^{k-1}.$$

But by (5.30) the class $[x']$ satisfying (5.33) is unique, so x' is admissibly homologous to $x \times (0, \ldots, 0, 1)$. Using the collar factor $[0, 1]$ in $T \times [0, 1] \times I^{k-1}$, we may append an allowable homology of x' to y' to get $x' = x \times (0, \ldots, 0, 1)$.

Now let R' be a regular neighborhood of y' in $\partial_r R$, let S' be the regular neighborhood $T \times I^{k-2}$ of x' in ∂S, and push S' back along the last factor of I^{k-1} (taking a collared neighborhood of S' in R' with it) so that $R' \cap (M \times I^{k-2}) = T \times I^{k-2}$.

We have now reproduced the data (5.27) used to make the above constructions, but with k replaced by $k - 1$ (assuming $k > 1$). Since $\partial_r R$ and ∂S are transverse to the stratifications of $E(\xi^{-1})$ and $M \times I^k$, what we have also done is to produce a normal field on $|y'|$ in $E(\xi^{-1})$ which restricts on $|x'|$ to the normal field given by the last factor in I^k where $|x'| = |x| \subset M \times [0, 1] \times I^{k-1} \subset M \times I^k$. We may now repeat this process, eventually reaching $k = 1$. This means we have (5.27) where

(5.34) (a) $\dim R = 4n$, $R = R \times \{0, \ldots, 0\} \subset R \times I^{k-1} \subset E(\xi^{-1})$

(b) $\dim S = 4n - 1, S = T \times (0, \ldots, 0) \subseteq T \times [0, 1] \times I^{k-1} \subseteq R \times I^{k-1}$, where the first inclusion is induced by $M \subseteq M \times I^k$ and the second by the collaring $T \times [0, 1] = S \times [0, 1] \subseteq R$.

(5.35) REMARK. Here is the first key homological point. We can push $[x] \in IH_{2n-1}(S)$ (homologically) out to the boundary of its regular neighborhood by (5.30) where $k = 1$. This how the self-linking of $[x]$ in (5.20) is to be defined. *But there are infinitely many ways to do this*: $\ker \sigma : IH_{2n-1}(\partial S) \to IH_{2n-1}(S) \cong IH_{2n}(S \cup \partial_r R, \partial_r R) \cong IH_{2n}(S, \partial S) \cong IH_{2n-1}(S)^* = \mathbb{Z}$, by (1.9). (If S were a normal disc bundle neighborhood of a smoothly imbedded $(2n-1)$-manifold $|x|$ in a smooth M^{4n-1}, a choice of $[x']$ such that $\sigma[x'] = [x]$ corresponds to a reduction of the group of the bundle from GL_{2n} to GL_{2n-1}, or to a nowhere zero section; i.e. a normal field on $|x|$.) The point of our construction is the observation made in [MS, p. 501]: if the choice of $[x']$ is made so as to come from a $[y']$ such that $\rho[y'] = [y]$ (which is also non-unique) then *the choice of* $[x']$ *becomes unique*. This follows from the fact that $\tau = 0$ in (5.28) when $k = 1$. We will not use this uniqueness explicitly, but it helps explain how the "normal bundle information", $M \times D^k \hookrightarrow E(\xi^{-1})$ is being used to construct *the* "normal field" on $|x|$ in the definition of $q(\alpha)$ in (5.20).

The second key point is the framing information in (5.22). It is used to show that any two choices of x' above lead to the same $q[x]$. In fact, let x_0 and x_1 be allowably homologous $(2n-1)$-cycles representing $\alpha \in \ker g_* : IH_{2n-1}(M) \to IH_{2n-1}(N)$. Suppose $x_i = \partial y_i$, $y_i \subset E(\xi^{-1})$. Using the procedure above, we may assume the data of (5.27) and (5.34) for both $(|y_0|, |x_0|)$ and $(|y_1|, |x_1|)$. Since $[x_0] = [x_1]$, there is an admissible $2n$-chain

$$X \subset M \times I$$

such that $\partial X = x_1 - x_0$. Thus, with the obvious notation we have

$$\begin{aligned} y_0 + X - y_1 &\subseteq R_0 \times I^{k-1} \cup M \times [0,1] \times I^{k-1} \cup R_1 \times I^{k-1} \\ &= (R_0 \cup M \times [0,1] \cup R_1) \times I^{k-1} \\ &\subseteq E(\xi^{-1}) \times [0,1]. \end{aligned}$$

Push y_i out to $y_i' \subseteq \partial_r R_i$ using (5.31); let $X' \subseteq M \times [0,1]$ be an admissible homology from x_0' to x_1', $x_i' =: \partial y_i'$.

(5.36) LEMMA. *$X \cdot X'$ is even.*

Proof. In $R_0 \cup M \times [0,1] \cup R_1$, the self-intersection of the class represented by the cycle $y_0 + X - y_1$ is

(5.37) $(y_0 + X - y_1) \cdot (y_0' + X' - y_1')$

which is $X \cdot X'$ since $y_0 \cdot y_0' = 0 = y_1 \cdot y_1'$. But we have the data for a normal map

$$\begin{array}{ccc} (R_0 \cup M \times [0,1] \cup R_1) \times I^{k-1} \times C & \hookrightarrow & E(\xi^{-1}) \times I \\ \uparrow & & \downarrow \\ R_0 \cup M \times [0,1] \cup R_1 & \to & N \times I \end{array}$$

used in the proof of (5.15), where C is a collar factor and $y_0 + X - y_1$ is allowably null-homologous in $E(\xi^{-1}) \times I$. This shows (5.37) is even.

(5.38) PROPOSITION. *Let* $\alpha \in \operatorname{Tors} K_{2n-1}(g)$, *as in* (5.21). *Choose a representative* x *for* α, *a regular neighborhood* S *of* $|x|$ *in* M *and a cycle* x' *in* ∂S *by applying the procedure of* (5.25)–(5.33) k *times. Then if* $s\alpha = 0$ *and* $\partial w = sx$,

$$w \cdot x'/2s \in \mathbb{Q}/\mathbb{Z}$$

depends only on $[x]$.

Proof. For fixed x, independence from the choice of w is well-known and easy. Using the notatin of the lemma, the $2n$-cycle $w_0 + sX - w_1$ in $M \times I$ has self-intersection number zero, as do all $2n$-cycles in $M \times I$. Using the argument for [MS, Fig. 5.8] this number is

$$sw_0 \cdot x_0' + s^2 X \cdot X' - sw_1 \cdot x_1'.$$

Divide by $2s^2$ to finish the proof.

The rest of the proof of (5.21) is routine, in the sense that it follows [MS]. Hence the proof of (5.3) is complete.

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