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## The maximum principle at infinity for minimal surfaces in flat three manifolds

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### 1. Introduction

Maximum principles are used as basic analytic tools for studying properties of functions defined on domains in  $\mathbb{R}^n$  and satisfying certain equations (e.g. elliptic). In general these maximum principles play a fundamental role in analysis on complete Riemannian manifolds, especially in the study of variational problems. For example, the well-known maximum principle for harmonic functions has had both a simplifying and unifying effect on the fields of harmonic and complex analysis.

H. Hopf [18] gave an important general maximum principle for second order linear elliptic partial differential equations. The Hopf maximum principle easily yields a maximum principle for solutions of the minimal surface equation. In this context the principle states that if  $D \subset \mathbb{R}^2$  is a smooth connected domain and  $f_1, f_2$  are two smooth functions on  $D$  that satisfy the minimal surface equation, then the difference  $f_1 - f_2$  cannot have an interior maximum or minimum unless the difference is constant.

The maximum principle for minimal graphs gives rise to the following geometric result for minimal surfaces in Riemannian three-manifolds: *If  $M_1$  and  $M_2$  are minimal surfaces in a Riemannian three-manifold that intersect at a common interior point  $p$  and  $M_1$  is on one side of  $M_2$  near  $p$ , then  $M_1$  intersects  $M_2$  in an open surface containing  $p$ .* In particular it follows that two differential minimal surfaces cannot intersect in their interiors at an isolated point. This geometric version of the maximum principle has many important applications to the general theory of minimal surfaces and, in its higher dimensional formulation, to the study of minimal hypersurfaces in  $n$ -dimensional Riemannian manifolds.

Recently Hoffman and Meeks [7] proved a theorem, called the Strong Halfspace Theorem, that is related to the maximum principle for minimal surfaces. Their

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theorem can be interpreted as a kind of maximum principle at infinity for minimal surfaces. This theorem is based on the next fundamental result.

**HALFSPACE THEOREM.** *If  $f : M \rightarrow \mathbb{R}^3$  is a proper connected minimal immersion that is contained in a halfspace, then  $f(M)$  is a flat plane.*

The Halfspace Theorem fails for minimal hypersurfaces in  $\mathbb{R}^n$ ,  $n \geq 4$ . In fact the  $n$ -dimensional  $SO(n)$ -invariant version of the catenoid,  $C^n \subset \mathbb{R}^{n+1}$ , is a properly embedded minimal surface with bounded  $x_{n+1}$ -coordinates.

Earlier using the work of do Carmo and Peng [2] and of Fischer–Colbrie and Schoen [4] on the geometry of stable minimal surfaces in  $\mathbb{R}^3$ , Meeks, Simon and Yau [14] showed that two properly immersed minimal surfaces in  $\mathbb{R}^3$  either intersect at some point or each is contained in a halfspace. This result together with the Halfspace Theorem yielded the following [7].

**STRONG HALFSPACE THEOREM.** *Suppose  $M_1, M_2$  are connected properly immersed minimal surfaces in  $\mathbb{R}^3$ . If  $M_1$  and  $M_2$  are disjoint, then  $M_1$  and  $M_2$  are parallel planes.*

It is the above generalized version of the Halfspace Theorem that has had many applications in recent years to global questions in the classical theory of minimal surfaces. However, for some applications of this type of result, there was a need to reformulate the Strong Halfspace Theorem to a more applicable form. Langevin and Rosenberg [9] gave a maximum principle at infinity for minimal surfaces of finite total curvature in  $\mathbb{R}^3$ . Their theorem stated that if  $M_1$  and  $M_2$  are disjoint, connected, properly embedded, minimal surfaces of finite total curvature and the boundaries of  $M_1$  and of  $M_2$  are compact (possibly empty), then  $\text{dist}(M_1, M_2) > 0$ . They also found interesting applications of their maximum principle at infinity to the study of the uniqueness of solutions to the minimal surface equation on the exterior of the unit disk in  $\mathbb{R}^2$ . Choi, Meeks and White [1] gave a generalization that they needed in their study of the isometry group of a properly embedded minimal surface in  $\mathbb{R}^3$ . What they found is the following: *If  $M_1$  and  $M_2$  are two disjoint, connected, properly immersed, minimal surfaces that have compact boundary (possibly empty) and  $M_1$  is asymptotic to a plane, then  $\text{dist}(M_1, M_2) > 0$ .*

These maximum principles at infinity for minimal surfaces now play a fundamental role in virtually every aspect of the classical theory of minimal surfaces. In this paper we shall prove the following maximum principle at infinity for minimal surfaces in flat three-manifolds.

**THEOREM 2 (Strong Maximum Principle at Infinity).** *Suppose  $N$  is a complete flat three-dimensional manifold and  $M_1$  and  $M_2$  are disjoint, connected, properly*

*immersed minimal surfaces in  $N$  with compact boundary (possibly empty). Then:*

1. *If  $\partial M_1$  or  $\partial M_2$  is nonempty, then, after possibly reindexing, there exists a point  $x \in \partial M_1$  and a point  $y \in M_2$ , such that  $\text{dist}(x, y) = \text{dist}(M_1, M_2)$ .*
2. *If  $\partial M_1$  and  $\partial M_2$  are empty, then  $M_1$  and  $M_2$  are flat.*

When  $N = \mathbb{R}^3$ , the strong maximum principle at infinity is a simple consequence of the following weaker version (see the proof of Theorem 3 in Section 2.)

**THEOREM 1 (Weak Maximum Principle at Infinity).** *Suppose  $N$  is a complete flat three-dimensional manifold and  $M_1$  and  $M_2$  are connected properly immersed minimal surfaces in  $N$  with compact boundary (possibly empty). If  $M_1$  and  $M_2$  are disjoint, then  $\text{dist}(M_1, M_2) > 0$ .*

The proofs of the above maximum principles at infinity are informative and give some insight into the asymptotic behavior of minimal surfaces. Also their proofs introduce new constructions that are themselves useful in making nontrivial applications of the maximum principle at infinity. We refer the reader to [5], [8], [12], [13] for such applications.

The paper is arranged as follows. In Section 2 we prove the strong maximum principle at infinity for embedded minimal surfaces in  $\mathbb{R}^3$ . In Section 3 we reduce the weak maximum principle at infinity to the case where  $M_1$  and  $M_2$  are stable embedded minimal annuli of finite total curvature in  $\mathbb{R}^3/S_\theta$  where  $S_\theta$  is a screw-motion which is a nontrivial vertical translation composed with a rotation around the  $x_3$ -axis by  $\theta$ ,  $0 \leq \theta < \pi$ . In Section 4 we complete the proof of the weak maximum principle at infinity. Finally in Section 5 we show that the weak maximum principle at infinity implies the strong one.

## 2. The Strong Maximum Principle at Infinity for minimal surfaces in $\mathbb{R}^3$

**LEMMA 1.** *Suppose  $M_1$  and  $M_2$  are two disjoint connected minimally immersed hypersurfaces in a complete flat  $n$ -manifold. If the distance between the surfaces is realizable by a point in  $\text{Int}(M_1)$  and a point in  $\text{Int}(M_2)$ , then  $M_1$  and  $M_2$  are totally geodesic.*

*Proof.* This proof appears in [11] but for completeness we repeat the proof here. Suppose  $p \in M_1$  and  $q \in M_2$  are points where the distance between  $M_1$  and  $M_2$  is realized. Let  $l$  be a line segment in  $N$  with end points  $p, q$  that realizes the distance. Note  $l$  is orthogonal to  $M_1$  and  $M_2$ . Choose embedded disk neighborhoods  $U_p \subset M_1$  and  $V_q \subset M_2$  that are small enough so that  $U_p \cup l \cup V_q$  is simply connected and lift this set to the universal cover  $\mathbb{R}^n$ . In  $\mathbb{R}^n$  let  $\tilde{U}_p$  denote the translate of  $U_p$



along  $l$  so that  $p$  gets translated to  $q$ . Since  $l$  minimizes distance between  $p$  and  $q$ ,  $\tilde{U}_p$  lies on one side of  $V_q$  at  $q$ . The maximum principle implies that a smaller neighborhood  $\hat{U}_p \subset \tilde{U}_p$  actually is contained in  $V_q$ . In particular any small parallel translate  $l'$  near  $p$  of  $l$  with one end point on  $U_p$  has its other end point on  $V_q$ . Since  $l'$  minimize the length between  $U_p$  and  $V_q$ , it is orthogonal to both surfaces. Hence the unit normal to  $U_p$  and  $V_q$  is parallel near  $p$  and  $q$ . This implies that  $U_p$  and  $V_q$  are totally geodesic and hence by analyticity  $M_1$  and  $M_2$  are also.  $\square$

**COROLLARY 1.** *Suppose  $M_1$  and  $M_2$  are disjoint proper minimally immersed hypersurfaces in a complete flat  $n$ -manifold. If  $M_1$  is compact, then  $\text{dist}(M_1, M_2) = \min \{ \text{dist}(\partial M_1, M_2), \text{dist}(\partial M_2, M_1) \}$ .*

*Proof.* Since  $M_1$  is compact and  $M_2$  is proper, there exists points  $p \in M_1$  and  $q \in M_2$  such that  $\text{dist}(M_1, M_2) = \text{dist}(p, q)$ . If  $p \in \partial M_1$  or  $q \in \partial M_2$ , then we are finished. If  $p \in \text{Int}(M_1)$  and  $q \in \text{Int}(M_2)$ , then Lemma 1 states that  $M_1$  and  $M_2$  are totally geodesic. In this case the proof of the corollary is immediate.  $\square$

**LEMMA 2.** *The weak maximum principle at infinity holds for properly embedded minimal surfaces of finite total curvature and compact boundary in  $\mathbb{R}^3$ . In other words, if  $M_1$  and  $M_2$  are two such disjoint surfaces, then  $\text{dist}(M_1, M_2) > 0$ .*

*Proof.* Suppose  $M_1$  and  $M_2$  are two disjoint properly embedded minimal surfaces of finite total curvature in  $\mathbb{R}^3$  with compact boundary. In this case  $M_1 \cup M_2$  has a finite number of annular ends, each of which is asymptotic to a catenoid or to a plane [20]. Suppose  $\text{dist}(M_1, M_2) = 0$ . This implies there exist annular ends  $E_1$  of  $M_1$  and  $E_2$  of  $M_2$ , each asymptotic to a half-catenoid which we may assume is  $C = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 = (\cosh x_3), x_3 \geq 0\}$ , or to a plane, that we may assume is  $\mathbb{R}^2$ . Clearly we could choose  $E_1$  and  $E_2$  to be graphs over the exterior of a large disk  $D$  in  $\mathbb{R}^2$ . Since  $E_1 \cap E_2 = \emptyset$ , we may assume without loss of generality that  $E_1$  lies above  $E_2$ . After a small vertical downward translation  $E'_1$  of  $E_1$ ,  $\partial E'_1$  still lies above  $E_2$  but outside of a large ball,  $E'_1$  lies below  $E_2$ . It follows that  $E'_1 \cap E_2$  is a compact nonempty one-dimensional analytic subset of both  $E'_1$  and  $E_2$ .

We now show that  $E'_1 \cap E_2$  is a simple closed curve  $\gamma$  and  $E'_1$  is transverse to  $E_2$  along  $\gamma$ . Since  $E'_1$  is a graph over  $\mathbb{R}^2 - D$ , the projection  $\Pi : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  of  $E'_1 \cap E_2$  is a compact nonempty one-dimensional analytic variety in  $\mathbb{R}^2$ . If  $\Pi(E'_1 \cap E_2)$  is not a connected homotopically nontrivial simple closed curve in  $\mathbb{R}^2 - D$ , then  $\mathbb{R}^2 - \Pi(E'_1 \cap E_2)$  contains a compact component disjoint from  $D$ . This is impossible since the lifts of this component to  $E_2$  and  $E'_1$  correspond to different solutions to the minimal surface equation with the same boundary values. Hence,  $E'_1$  intersects  $E_2$  transversely in a single curve  $\gamma$  that is homotopically nontrivial on both  $E'_1$  and  $E_2$ . Let  $\tilde{E}_1$  and  $\tilde{E}_2$  denote the ends of  $E'_1$ ,  $E_2$ , respectively with boundary  $\gamma$ .

The surfaces  $\tilde{E}_1$  and  $\tilde{E}_2$  represents distinct solutions to the minimal surface equation over the unbounded region  $\Delta$  of  $\mathbb{R}^2$  with boundary curve  $\Pi(\gamma)$  and they have the same boundary values along  $\Pi(\gamma)$ . Since  $\tilde{E}_1$  and  $\tilde{E}_2$  are asymptotic to translates of a fixed vertical catenoid, they have the same signed logarithmic growth in terms of  $|\mathbf{x}|$ ,  $\mathbf{x} \in \Delta$ . (By the logarithmic growth rate of such an end  $\tilde{E}$  we mean the following:  $\tilde{E}$  is the graph of a function  $F$  on the exterior domain  $|\mathbf{x}| > R$ . The fact that  $F$  satisfies the minimal surface equation implies that  $F$  has an asymptotic expansion at infinity of the form:  $F(\mathbf{x}) = a \log(|\mathbf{x}|) + \mathcal{O}(1)$ ,  $a$  is the logarithmic growth rate of  $\tilde{E}$  [17]. Notice that the catenoid  $C$  to which we are assuming  $E_1$  and  $E_2$  are asymptotic, has logarithmic growth one.)

We will now give a simple geometric flux calculation to show that  $\tilde{E}_1 = \tilde{E}_2$ . (This proof easily generalizes to deal with similar uniqueness questions that arise in the proof of Theorem 1.)

First consider a simple closed homotopically nontrivial curve  $\alpha$  on the half-catenoid  $C$  defined above. Suppose  $X$  is the gradient of the third coordinate function on  $C$ . Let  $\eta$  be the conormal of the unbounded component of  $C - \alpha$ . This means the unit vector field normal to  $\alpha$ , tangent to  $C$  and pointing into the unbounded component of  $C - \alpha$ . The flux of  $X$  across  $\alpha$  is

$$F(\alpha, X) = \int_{\alpha} X \cdot \eta = 2\pi.$$

This is clear if  $\alpha = \partial C$  and follows for any  $\alpha$  by the divergence theorem applied to the harmonic function  $x_3$  on  $C$ . Similarly if  $\tilde{C}$  is a minimal annulus that is a graph asymptotic to  $C$ , then the associated flux across the boundary of  $\tilde{C}$  is also  $2\pi$ . This follows from the Weierstrass Representation (see [20]).

Let  $X_1$  and  $X_2$  denote the gradient of the third coordinate functions of  $\tilde{E}_1$  and  $\tilde{E}_2$ , respectively. From the above discussion we conclude that the flux of these vectors fields across their common boundary curve  $\gamma$  are equal. But since  $\tilde{E}_1$  lies below  $\tilde{E}_2$  along  $\gamma$ ,  $X_1 \cdot \eta_1 < X_2 \cdot \eta_2$  at every point of  $\gamma$ . Integrating this inequality along  $\gamma$  contradicts the fact that the flux of  $\tilde{E}_1$  equals that of  $\tilde{E}_2$ . This contradiction proves Lemma 2. □

The following corollary to Lemma 2 was first proved by Langevin and Rosenberg [9] using a different method.

**COROLLARY 2.** *Suppose  $E_1$  and  $E_2$  are graphical solutions to the exterior Plateau problem for a compact domain in  $\mathbb{R}^2$ . If  $E_1$  and  $E_2$  each have the same limiting vertical normal vector, the same logarithmic growth and the same boundary, then  $E_1 = E_2$ .*

*Proof.* Suppose  $E_1$  and  $E_2$  satisfy the hypotheses of the corollary and  $E_1 \neq E_2$ . In this case  $E_i$  is asymptotic to an end-representative  $C_i$  of a catenoid or a horizontal plane. Note that  $C_1$  and  $C_2$  have the same logarithmic growth and limiting vertical normal vector. Hence,  $C_1$  and  $C_2$  can be chosen to be translates of each other.

By the flux argument in the proof of Lemma 2,  $E_1$  does not lie above  $E_2$  near their common boundary. Hence any small upward vertical translation of  $E_1$  yields a  $E'_1$  such that  $E'_1$  intersects  $E_2$  near  $\partial E_2$ . Since a large vertical upward translation of  $C_1$  produces a surface that is a positive distance from  $C_2$ , a large upward translation of  $E_1$  produces a surface that is disjoint from  $E_2$ . The maximum principle for minimal surfaces implies there exists a smallest  $T > 0$  such that  $(E_1 + (0, 0, T)) \cap E_2 = \emptyset$ . Clearly  $\text{dist}(E_1 + (0, 0, T), E_2) = 0$ , which contradicts Lemma 2.  $\square$

Recall that a noncompact surface in a Riemannian manifold is said to have *least-area* if compact subdomains have least-area with respect to their boundaries.

**LEMMA 3.** *The weak maximum principle at infinity holds for properly embedded minimal surfaces with compact boundary in  $\mathbb{R}^3$ .*

*Proof.* Suppose  $M_1$  and  $M_2$  are properly embedded disjoint minimal surfaces in  $\mathbb{R}^3$  with compact boundary and suppose that  $\text{dist}(M_1, M_2) = 0$ . Suppose  $B$  is a large ball that contains  $\partial M_1 \cup \partial M_2$  in its interior and such that  $\partial B$  is transverse to  $M_1 \cup M_2$ . In this case  $M_i - B$  consists of a finite number of components for  $i = 1, 2$ . Since  $\text{dist}(M_1, M_2) = 0$ , it follows that a component of  $M_1 - \text{Int}(B)$  is a distance zero from a component of  $M_2 - \text{Int}(B)$ . Hence, replacing  $M_1$  and  $M_2$  by these components we may assume that  $\partial M_i = M_i \cap B \subset \partial B$  for  $i = 1, 2$ . By Corollary 1 we may assume that  $M_1$  and  $M_2$  are noncompact.

Our basic approach to proving the lemma will be to show that  $M_1$  and  $M_2$  can be separated by a pair of disjoint complete embedded minimal surfaces with compact boundary on  $\partial B$  and of finite total curvature. By Lemma 2 these finite total curvature surfaces are separated by a distance  $\varepsilon > 0$ , which gives a lower bound on the distance between  $M_1$  and  $M_2$ . We now construct these finite total curvature surfaces.

The curves  $\partial M_1 \cup \partial M_2$  bound a subdomain  $\Delta$  of  $\partial B$  with at least one component having boundary in both  $\partial M_1$  and  $\partial M_2$ . It follows that  $M_1 \cup M_2 \cup \Delta$  is a connected properly embedded piecewise smooth surface in  $\mathbb{R}^3$ . This surface disconnects  $\mathbb{R}^3$  into two components  $C, D$  where  $D$  is the closure of the component that contains  $\text{Int}(B)$ . Note that  $\partial M_1 \subset \partial D$  is homologous to zero in  $B \subset D$ . Since  $M_1$  and  $M_2$  are both noncompact and proper, there exists a proper arc  $\delta : \mathbb{R} \rightarrow \partial C$  that intersects

$\partial M_1$  transversely in a single point. If  $\partial M_1$  bounded a compact surface  $E_1$  in  $C$ , then since  $\partial M_1$  bounds a compact surface  $E_2$  in  $D$ ,  $\delta$  has odd intersection number with the cycle  $E_1 \cup E_2$ , which is impossible. Hence  $\partial M_1$  is not homologous to zero mod 2 in  $C$ .

Notice that  $C$  has an analytic triangulation since it is an analytic manifold except along a finite number of compact transverse intersection curves. Change the metric in a compact neighborhood in  $C$  of  $\Delta \subset \partial C$  in  $C$  so that the new metric satisfies (see the proof of Theorem 1 in [16]):

1. The 2-simplices of  $\partial C$  have nonnegative mean curvature and the edges of two adjacent simplices meet in an angle less than or equal to  $\pi$ .
2. If  $\sigma_1$  is a 2-simplex in  $\Delta$  and  $\sigma_2$  is a 2-simplex in  $M_1 \cup M_2$ ,  $\sigma_1$  and  $\sigma_2$  adjacent, then the angle between  $\sigma_1$  and  $\sigma_2$  is less than  $\pi$  along their common boundary.

We make this change of metric so that the least-area Plateau problem can be solved in  $C$ , i.e. any smooth 1-cycle in  $C$  that is null homologous in  $C$  is the boundary of a least-area surface  $\Sigma \subset C$  and  $\text{Int}(\Sigma)$  is smooth and embedded. Moreover if  $\Sigma$  meets  $\partial C$  at a point  $x$ , then the maximum principle implies that the connected component of  $\Sigma$  containing  $x$  is contained in  $\partial C$  (see Theorem 2 in [16]). Let  $\Sigma_1 \subset \Sigma_2 \subset \dots$  be a compact exhaustion of  $M_1$  by subdomains with smooth boundary and  $\partial M_1 \subset \partial \Sigma_1$ . Let  $\tilde{\Sigma}_i$  be a least-area surface in  $C$  with  $\partial \tilde{\Sigma}_i = \partial \Sigma_i$  and so that  $\tilde{\Sigma}_i$  is  $\mathbb{Z}_2$ -homologous to  $\Sigma_i$  (rel  $(\partial \Sigma_i)$ ). In this case  $\tilde{\Sigma}_i \cup \Sigma_i$  is a boundary in  $C$  and hence  $\tilde{\Sigma}_i$  is orientable.

We will now prove that a subsequence of the  $\tilde{\Sigma}_i$  converge. This follows by showing that this family of surfaces satisfy uniform area and curvature estimates that we will now describe in detail.

Let  $B$  be a ball in  $C$  and  $W$  a least-area surface embedded in  $C$ ,  $\partial W$  disjoint from  $B$  and  $W$  transverse to  $\partial B$ . (If  $B \cap \partial C \neq \emptyset$ , then assume  $\partial B \cap \partial C$  is a disk.) Then  $W \cap \partial B$  is the boundary of a region in  $\partial B$  of area at most half the area of  $\partial B$ . Consequently, there is a uniform local area bound for the  $\tilde{\Sigma}_i$  (since  $\tilde{\Sigma}_i$  minimizes in its  $\mathbb{Z}_2$ -homology class as a relative class.) Curvature estimates of Schoen [19] state that there exists a universal constant  $c$  such that for any stable orientable minimal surface  $T$  in a flat orientable three-manifold and  $p \in T$  of distance  $d$  from  $\partial T$ , the Gaussian curvature is estimated by  $|K(p)| \leq c/d^2$ . This estimate leads to uniform curvature estimates for the family  $\tilde{\Sigma}_i$  away from  $\partial M_1$ .

The above uniform area and curvature estimates for  $\{\tilde{\Sigma}_i\}$  imply the family is compact, i.e., a subsequence of the surfaces  $\tilde{\Sigma}_i$  converges to a proper least-area orientable minimal surface  $\Gamma_1 \subset G$  with  $\partial \Gamma_1 = \partial M_1$ . (See the end of the proof of Theorem 3.1 in [15] for the proof that the smooth limit of least-area surfaces is

again least-area.) This compactness property for  $\{\tilde{\Sigma}_i\}$  is standard and for completeness we outline its proof.

Consider a small ball  $B(r) \subset C - \partial M_1$  of radius  $r$ . By Schoen's curvature estimates, after choosing a possibly smaller  $r$ , every component of  $\tilde{\Sigma}_i \cap B(r)$  that intersects  $B(r/2)$  can be expressed as a graph of small gradient over a plane  $P_i$  in  $B(r)$  passing through the center of the ball and  $P_i$  does not depend on the component. By the uniform area estimates,  $B(r/2) \cap \tilde{\Sigma}_i$  contains a bounded number of components independent of  $i$  and hence there are a bounded number of associated graphs. Suppose for the moment that for every  $i$ ,  $\tilde{\Sigma}_i \cap B(r/2)$  contains one component and corresponding graph  $G(i)$ . Since a subsequence  $P_{i_k}$  converge to a plane  $P$  in  $B(r)$ , the usual compactness theorems for minimal graphs imply that a subsequence  $G(i_k)$  converges to a graph  $G$  over its projection to  $P$ . In the general case a subsequence of the corresponding graphs in  $\tilde{\Sigma}_i \cap B(r)$  converge to a finite number of graphs. Note that  $C - \partial M_1$  has a countable basis of balls  $\{B_j\}$ , where for each  $j$  and for every subsequence  $i_k$  the associated graphs  $G(i_k, j)$  in  $\tilde{\Sigma}_{i_k} \cap B_j$  have a convergent subsequence in  $B_j$ . Suppose that the subsequence  $G(i_k, 1)$  converges in  $B_1$ . Then the associated subsequence of graphs in  $\tilde{\Sigma}_{i_k} \cap B_2$  have a convergent subsequence in  $B_2$  as well as  $B_1$ . Continuing in this manner *ad infinitum* from  $B_i$  to  $B_{i+1}$  and taking a diagonal sequence, yields a subsequence of the  $\tilde{\Sigma}_i$  that converges in each  $B_j$ . The limit  $\Gamma_1$  of this subsequence is a smooth properly embedded minimal surface in  $C - \partial M_1$ , has least area and has boundary  $\partial M_1$ . The boundary regularity theorem in [6] implies  $\Gamma_1$  is smooth along  $\partial M_1$ . This completes our outline of the proof of compactness for the family  $\{\tilde{\Sigma}_i\}$ .

Suppose now that a subsequence of the  $\tilde{\Sigma}_i$  converges to a properly embedded least-area surface  $\Gamma_1$ . Since  $\Gamma_1$  is orientable and stable, it has finite total curvature (see [3] or Theorem 2.1 in [15]). Since  $C - M_1$  is not smooth, the boundary maximum principle (see Theorem 2 in [16]) implies that either  $\Gamma_1 = M_1$  or  $\text{Int}(\Gamma_1) \subset \text{Int}(C)$ . If  $\Gamma_1 = M_1$ , then  $M_1$  has finite total curvature. If  $M_2$  also has finite total curvature, then the lemma follows from Lemma 2. Thus, after possibly interchanging  $M_1$  with  $M_2$  we may assume that  $\text{Int}(\Gamma_1) \subset \text{Int}(C)$ .

The surface  $\Gamma_1$  separates  $C$  into two components where one component contains  $M_1$  and the other contains  $M_2$ . Let  $H$  denote the closure of the component containing  $M_2$ . Arguing as above for  $M_2 \subset \partial H$  in place of  $M_1 \subset \partial C$ , we obtain a proper orientable smooth stable minimal surface  $\Gamma_2 \subset H$  with  $\Gamma_2 \cap \partial H = \partial M_2$ . Note  $\Gamma_2$  separates  $H$  into two components, one of which contains  $\Gamma_1$  and the other that contains  $M_2$ .

It follows from Lemma 2 that  $\text{dist}(\Gamma_1, \Gamma_2) > 0$  since these surfaces have finite total curvature and are minimal in  $\mathbb{R}^3$  outside of some compact neighborhood of their boundary curves. On the other hand, since  $\text{dist}(M_1, M_2) = 0$ , there exist points  $p \in M_1$ ,  $q \in M_2$  far from the origin such that  $\text{dist}(p, q) < \text{dist}(\Gamma_1, \Gamma_2)$ . But

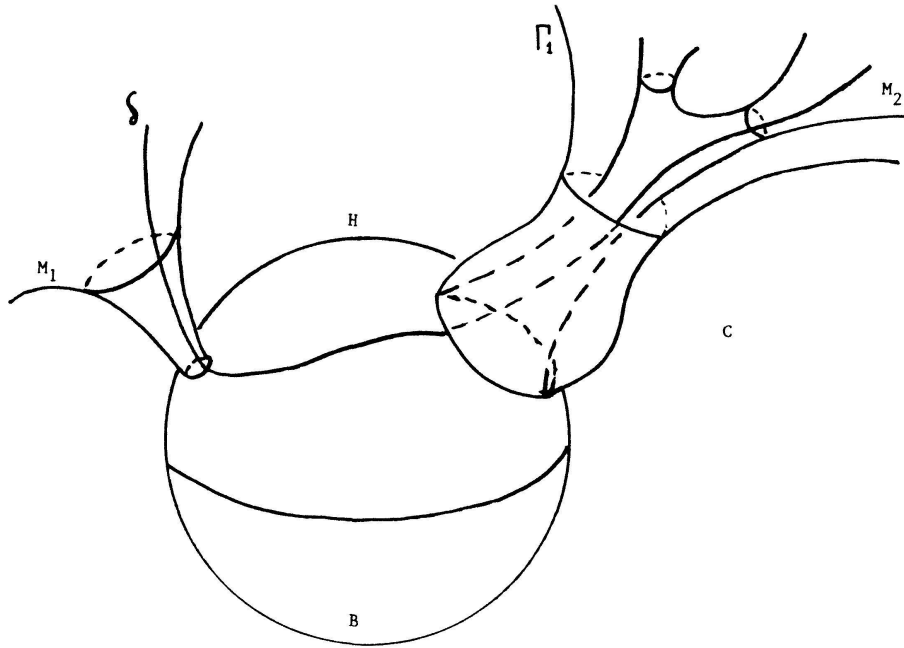


Fig. 1.

any arc joining  $p$  to  $q$  must contain a subarc in  $C$  joining a point of  $\Gamma_1$  to a point of  $\Gamma_2$ . Hence  $\text{dist}(p, q) > \text{dist}(\Gamma_1, \Gamma_2)$ . This contradiction proves Lemma 3.  $\square$

**THEOREM 3.** *Suppose  $M_1$  and  $M_2$  are disjoint properly embedded minimal surfaces in  $\mathbb{R}^3$  with compact boundary and  $M_1$  and  $M_2$  are not parallel planes. Then*

$$\text{dist}(M_1, M_2) = \min \{ \text{dist}(\partial M_1, M_2), \text{dist}(\partial M_2, M_1) \}.$$

*Proof.* By the Strong Halfspace Theorem we may assume that  $\partial M_1$  or  $\partial M_2$  is nonempty. Let  $(p_i, q_i) \in M_1 \times M_2$  be a sequence of points such that  $\lim(\text{dist}(p_i, q_i)) = \text{dist}(M_1, M_2)$ . Then a subsequence of the vectors  $v_i = q_i - p_i$  converges to a point  $v$  on the sphere of radius  $\text{dist}(M_1, M_2)$ . Let  $M_3$  be the surface obtained by translating  $M_1$  by the vector  $v$ . By Lemma 3 we know that  $M_3 \cap M_2 \neq \emptyset$ . There are two cases to consider:

1.  $\partial M_3 \cap M_2 \neq \emptyset$  or  $\partial M_2 \cap M_3 \neq \emptyset$ .
2.  $\text{Int}(M_3) \cap \text{Int}(M_2) \neq \emptyset$ .

Lemma 1 shows possibility 2 occurs only when  $M_2$  and  $M_3$  are contained in a plane. Hence we are in case 1. But case 1 implies

$$\text{dist}(M_1, M_2) = \min \{ \text{dist}(\partial M_1, M_2), \text{dist}(\partial M_2, M_1) \},$$

which completes the proof of the theorem.  $\square$



### 3. Reduction to the case of finite total curvature

In this section we reduce the proof of the weak maximum principle at infinity (Theorem 1 in the Introduction) to the case when the surfaces  $M_1$  and  $M_2$  are embedded stable minimal annuli with finite total curvature. We will call a noncompact surface an annulus if it is homeomorphic to  $S^1 \times [0, 1)$ .

**LEMMA 4.** *Suppose that the weak maximum principle at infinity holds in all flat manifolds of the form  $\mathbb{R}^3/S_\theta$  for the special case of embedded stable minimal annuli of finite total curvature. Then the weak maximum principle at infinity holds in all complete flat three-manifolds.*

*Proof.* By Corollary 1 the weak maximum principle at infinity holds if  $M_1$  or  $M_2$  is compact. We will now assume they are both noncompact. Let  $N$  be an arbitrary flat three-manifold and suppose  $M_1, M_2 \subset N$  are two properly immersed noncompact disjoint minimal surfaces with compact boundary (possibly empty). In particular,  $N$  is noncompact. By the classification of complete flat noncompact three-manifolds [22], we know that  $N$  is finitely covered by  $\mathbb{R}^3$ , by  $\mathbb{R}^3/S_\theta$  or by  $\mathbb{T} \times \mathbb{R}$  where  $\mathbb{T}$  is a flat torus. After taking possibly a finite sheeted covering space of  $N$  and lifting the surfaces  $M_1, M_2$ , we may assume that  $N$  is  $\mathbb{R}^3$ ,  $\mathbb{R}^3/S_\theta$  or  $\mathbb{T} \times \mathbb{R}$ .

Choose a smooth compact analytic subdomain  $D$  of  $N$  such that  $\partial D$  intersects  $M_1 \cup M_2$  transversely,  $\partial M_1 \cup \partial M_2 \subset D$  and  $D$  has nonempty intersection with  $M_1$  and with  $M_2$ . Without loss of generality we will replace  $M_1$  and  $M_2$  by their intersection with  $N - \text{Int}(D)$  and assume they are connected. Let  $C$  be a component of  $N - (M_1 \cup M_2 \cup D)$  that contains points of  $M_1$  and of  $M_2$  in its boundary. We consider  $C$  with its induced metric (the distance between two points is the infimum of the lengths of paths in  $C$  joining the points). The metric completion of  $C$ , denoted  $\tilde{C}$ , is a desingularization of  $\bar{C}$  which is the closure of  $C$ .

Notice that  $\tilde{C}$  is an analytic manifold whose boundary is defined by analytic inequalities, hence by [10] the boundary of  $\tilde{C}$  has an analytic triangulation. We denote by  $M_1(C)$ ,  $M_2(C)$ ,  $D(C)$  the points of  $\tilde{C}$  that project to  $M_1$ ,  $M_2$ ,  $D$ , respectively.

If  $N = \mathbb{R}^3/S_\theta$  where  $S_\theta$  is a screw motion, then choose the domain  $D$  to be a solid torus which is a regular neighborhood of the image of the axis of  $S_\theta$ . If  $N = \mathbb{T} \times \mathbb{R}$ , then choose  $D$  to be of the form  $\mathbb{T} \times [-t_0, t_0]$  for some  $t_0$  and in the case  $N = \mathbb{R}^3$  choose  $D$  to be a ball. In all cases, the fundamental group of each component  $\Delta$  of  $N - D$  is generated by the fundamental group of the boundary of the component. It then follows from separation theorems that a properly embedded surface  $\Sigma$  in  $\Delta$ , separates  $\Delta$  into two components. This separation property has the useful consequence in our constructions that if  $\Sigma$  is a properly embedded surface in  $\tilde{C}$  with

$\partial\Sigma = \partial M_1(C)$  or  $\partial\Sigma = \partial M_2(C)$  or  $\partial\Sigma = \emptyset$ , then  $\Sigma$  separates  $\tilde{C}$  into two components.

We next check that  $\partial\tilde{C}$  is connected. If  $\partial\tilde{C}$  is not connected, then  $\tilde{C}$  contains a properly embedded connected surface  $\Sigma \subset \text{Int}(\tilde{C})$  that separates one component of  $\partial\tilde{C}$  from another such component. The surface  $\Sigma$  can be considered to lie in  $N$  and  $\Sigma$  is disjoint from  $D$ . By our previous discussion  $\Sigma$  separates  $N$  into a component that contains  $D$  and another component that contains some point of  $M_1$  or  $M_2$ . But since  $\Sigma$  is disjoint from  $M_1 \cup M_2$  it is clear that either  $M_1$  or  $M_2$  is disjoint from  $D$ , which is contrary to our choice of  $D$ . Thus  $\partial\tilde{C}$  is connected.

Change the metric in a compact neighborhood of  $D(C)$  in  $\tilde{C}$  so that the new metric satisfies:

1. The 2-simplices of  $\partial\tilde{C}$  have nonnegative mean curvature and the edges of two adjacent simplices meet at an angle less than or equal to  $\pi$ .
2. If  $\sigma_1$  is 2-simplex in  $D(C)$  and  $\sigma_2$  a 2-simplex in  $M_1(C) \cup M_2(C)$  that are adjacent, then the angle between  $\sigma_1, \sigma_2$  is less than or equal to  $\pi$  and different from  $\pi$  at some point.

We make this change of metric so that the least-area Plateau problem can be solved in  $\tilde{C}$ , i.e. if  $\delta$  is a smooth cycle in  $\tilde{C}$ , that is null homologous in  $\tilde{C} \bmod 2$ , then  $\delta$  bounds a least-area surface  $\Sigma$  and  $\text{Int}(\Sigma)$  is smooth and embedded (see [16] and [21]). Moreover if  $\text{Int}(\Sigma)$  meets  $\partial\tilde{C}$  at a point  $x$ , then the maximum principle implies that the connected component of  $\Sigma$  containing  $x$  is contained in  $\partial\tilde{C}$ .

Since  $\text{dist}(M_1, M_2) = 0$  and  $\partial M_1 \cup \partial M_2$  is compact, we can choose the component  $C$  so that  $\text{dist}(M_1(C), M_2(C)) = 0$  in the metric induced by the Riemannian metric on  $\tilde{C}$ . Notice that the boundary of  $M_1(C) \cup M_2(C)$  is contained in the boundary of  $D(C)$ . Let  $\Sigma_1 \subset \Sigma_2 \subset \dots$  be a compact exhaustion of  $M_1(C)$  by piecewise smooth subdomains where  $\partial M_1(C) \subset \partial\Sigma_1$ . Let  $\tilde{\Sigma}_i$  be a least-area surface in  $\tilde{C}$  with  $\partial\tilde{\Sigma}_i = \partial\Sigma_i$ , and that is  $\mathbb{Z}_2$ -homologous to  $\Sigma_i$  (rel  $(\partial\Sigma_i)$ ). The cycle  $\tilde{\Sigma}_i \cup \Sigma_i$  bounds in  $\tilde{C}$ . In particular  $\tilde{\Sigma}_i$  is orientable. (See Figure 1 where an analogous situation is described.) As in the proof of Lemma 3, a subsequence of the  $\tilde{\Sigma}_i$  converge to a least-area surface  $\Gamma_1$ .

As in the construction of  $\Gamma_1$  at the end of the proof of Lemma 3, we can assume that  $\text{Int}(\Gamma_1) \subset \text{Int}(\tilde{C})$ . The surface  $\Gamma_1$  separates  $\tilde{C}$  into two regions, one of which contains  $M_1(C)$  and the other  $H$  that contains  $M_2(C)$ . Arguing as before with  $M_2(C) \subset \partial H$  in place of  $M_1(C) \subset \partial\tilde{C}$ , we obtain a properly embedded stable minimal surface  $\Gamma_2$  with  $\partial\Gamma_2 = \partial M_2$ . Furthermore,  $\Gamma_2$  separates  $H$  into two components where one of the components has  $\Gamma_1, \Gamma_2$  and part of  $D(C)$  on its boundary.

Recall that  $C$  was chosen so that  $\text{dist}(M_1(C), M_2(C)) = 0$ . Since  $\Gamma_1$  and  $\Gamma_2$  separate  $M_1(C)$  and  $M_2(C)$  in  $\tilde{C}$ , we conclude that  $\text{dist}(\Gamma_1, \Gamma_2) = 0$ . However outside a compact subset of  $\tilde{C}$ , the metric on  $\tilde{C}$  is flat. Theorem 2.1 in [15] states that a stable orientable properly immersed minimal surface with compact boundary in a flat orientable three-manifold has finite total curvature (also see [3]). Thus,  $\Gamma_1$



and  $\Gamma_2$  have a finite number of stable annular ends of finite total curvature. If  $N = \mathbb{R}^3$ , then Lemma 3 shows  $\text{dist}(\Gamma_1, \Gamma_2) > 0$ , a contradiction. If  $N = \mathbb{T} \times \mathbb{R}$ , it was shown in Theorem 3 in [13] that the ends of  $\Gamma_1$  and  $\Gamma_2$  stay a bounded distance from each other, which contradicts  $\text{dist}(\Gamma_1, \Gamma_2) = 0$  (proved by a flux calculation similar to the calculation in the proof of Lemma 2). Thus, if the weak maximum principle at infinity holds in  $\mathbb{R}^3/S_\theta$  for a pair of disjoint embedded stable minimal annuli, then the weak maximum principle at infinity holds in all flat three-manifolds.  $\square$

#### 4. The Proof of the Weak Maximum Principle at Infinity

We now prove Theorem 1 (Weak Maximum Principle at Infinity) stated in the Introduction. By Lemma 4 we need only check the weak maximum principle at infinity for two properly embedded disjoint stable minimal annuli  $A_1, A_2 \subset N = \mathbb{R}^3/S_\theta$  that have finite total curvature and such that  $\text{dist}(A_1, A_2) = 0$ . Let  $\gamma \subset N$  denote the image of the  $x_3$ -axis. After removing compact subdomains from  $A_1$ , and  $A_2$ , we may assume that  $A_1$  and  $A_2$  are disjoint from  $\gamma$ . Let  $D_R$  denote the tubular neighborhood of  $\gamma$  of radius  $R$ .

Using the Weierstrass representation when  $\theta = 0$  and a related analytic representation when  $\theta \neq 0$ , we derived analytic formulas for a minimal annulus  $A$  of finite total curvature in  $N$  [12]. When  $A$  is embedded, we proved that it is asymptotic to one of the following standard ends (see [12] for precise definitions):

1. A plane or catenoid in  $N$ ;
2. A flat vertical annulus in  $N$ ;
3. Helicoid-type ends.

We will now derive a contradiction if  $\text{dist}(A_1, A_2) = 0$ . It follows immediately from the description of standard ends in [12] that if  $A_1$  is asymptotic to one of these standard ends  $S$ , then  $A_2$  is also asymptotic to the same end  $S$ . In particular  $A_1$  is asymptotic to  $A_2$  and, after removing compact subdomains of  $A_1$  and  $A_2$ , we may assume that  $A_1$  is a small graph over  $A_2$ .

Suppose that the limiting unit normal vector to  $S$  is  $v$ . Note  $v$  is vertical when  $N = \mathbb{R}^3/S_\theta$  and  $\theta \neq 0$ . It follows that  $N$  has a parallel Killing vector field  $V$  that is generated by translation in the direction  $v$  in  $\mathbb{R}^3$ . Without loss of generality, we may assume  $A_1$  and  $A_2$  are chosen so that the normals to  $A_1$  and  $A_2$  make a small angle with  $v$ . Thus, after a small translation of  $A_1$  along the direction  $v$ , we obtain a new annulus  $A_3$  whose boundary is above  $A_2$  and that eventually lies below  $A_2$ . Standard ends do not intersect themselves after a small translation in the  $v$  or  $-v$  directions. Thus, as in the proof of Lemma 2,  $A_3$  intersects  $A_2$  transversely in a simple closed curve  $\alpha$  that is homotopically nontrivial on both  $A_2$  and  $A_3$ .

Let  $E_2, E_3$  denote the ends of  $A_2, A_3$ , respectively, with boundary curve  $\alpha$ . Let  $V_2, V_3$  denote the orthogonal projection of  $V$  onto  $E_2$  and  $E_3$ . Let  $\eta_2, \eta_3$  denote the conormals to  $E_2, E_3$ , respectively. Since  $V_2$  and  $V_3$  are divergence free, the fluxes

$$F_2 = \int_{\alpha} V_2 \cdot \eta_2, \quad F_3 = \int_{\alpha} V_3 \cdot \eta_3$$

are geometric invariants of  $E_2$  and  $E_3$ . However, as shown in [12],  $F_2$  and  $F_3$  only depend on the corresponding flux of  $S$ . We conclude that  $F_2 = F_3$ . However, since  $\tilde{E}_3$  lies below  $\tilde{E}_2$  along  $\alpha$ ,  $V_2 \cdot \eta_2 < V_3 \cdot \eta_3$  along  $\alpha$  and so  $F_2 < F_3$ . This contradiction completes the proof of the weak maximum principle at infinity.  $\square$

### 5. The Proof of the Strong Maximum Principle at Infinity

We are now in a position to prove Theorem 2 (Strong Maximum Principle at Infinity) stated in the Introduction. After possibly taking a finite sheeted cover of  $N$  and lifting the surfaces to this cover, we may assume that  $N$  is  $\mathbb{R}^3, S^1 \times \mathbb{R}^2, \mathbb{T} \times \mathbb{R}$  or  $\mathbb{R}^3/S_{\theta}$  where  $\theta$  is not a rational multiple of  $\pi$ . First suppose that  $N \neq \mathbb{R}^3/S_{\theta}$ .

Suppose  $\text{dist}(\partial M_1, M_2) \leq \text{dist}(\partial M_2, M_1)$  and that  $\text{dist}(\partial M_1, M_2) > \text{dist}(M_1, M_2) > 0$ . Consider a sequence of points  $(p_i, q_i) \in M_1 \times M_2$  such that  $\lim(\text{dist}(p_i, q_i)) = \text{dist}(M_1, M_2)$ . Consider the isometry  $I_i$  of  $N$  taking  $p_i$  to  $q_i$  that lifts to a translation in  $\mathbb{R}^3$ . We may assume after picking a subsequence that  $I_i$  converges to an isometry  $I : N \rightarrow N$ . If  $I(M_1) \cap M_2 \neq \emptyset$ , then there exist interior points  $p \in M_1$ , and  $q \in M_2$  with  $\text{dist}(p, q) = \text{dist}(M_1, M_2)$ , which is impossible by Lemma 1. On the other hand,  $\text{dist}(I(M_1), M_2) = 0$  so the weak maximum principle at infinity shows  $I(M_1) \cap M_2 \neq \emptyset$ . This proves the strong maximum principle at infinity in the case  $N \neq \mathbb{R}^3/S_{\theta}$ . Assume now that  $N = \mathbb{R}^3/S_{\theta}$ ,  $\theta$  an irrational multiple of  $\pi$ .

The proof of the strong maximum principle at infinity that we just gave for  $N \neq \mathbb{R}^3/S_{\theta}$ ,  $\theta$  an irrational multiple of  $\pi$ , fails to work when  $N = \mathbb{R}^3/S_{\theta}$  because for  $p \in M_1$  and  $q \in M_2$  there does not always exist an isometry of  $N$  taking  $p$  to  $q$ . Let  $(p_i, q_i) \in M_1 \times M_2$  with  $\lim(\text{dist}(p_i, q_i)) = \text{dist}(M_1, M_2)$  and consider lifts  $M_1(i)$  and  $M_2(i)$  to  $\mathbb{R}^3$  so the lifted points  $\tilde{p}_i, \tilde{q}_i$  have the same distance in  $\mathbb{R}^3$ . If the vectors  $(\tilde{q}_i - \tilde{p}_i)$  converge to a vertical vector  $v$ , then translation in  $\mathbb{R}^3$  by  $v$  induces an isometry  $I : N \rightarrow N$  that moves points a distance  $\text{dist}(M_1, M_2)$  and such that  $\text{dist}(I(M_1), M_2) = 0$ . In this case the argument in the previous paragraph shows that the strong maximum principle at infinity holds for  $M_1$  and  $M_2$ .

When  $M_1$  and  $M_2$  are embedded in  $N$  with finite total curvature, then the vector  $v$  is always vertical. To see this first note that the ends of  $M_1$  and  $M_2$  are asymptotic

to standard ends and hence have vertical normal vectors at infinity (see Proposition 5.1 in [12]). Since the Gaussian curvature of  $M_1$  and  $M_2$  is asymptotic to zero, and the surfaces are a positive distance apart, it is clear that the sequence of points  $(\tilde{q}_i - \tilde{p}_i)$  converges to a vertical vector. We will now reduce the proof of the general case to the case of embedded surfaces of finite total curvature (where the principle is true by the previous discussion).

For the moment assume that  $M_1$  and  $M_2$  are *embedded* in  $N$ . Also assume that  $\text{dist}(M_1, M_2) < \min\{\text{dist}(\partial M_1, M_2), \text{dist}(\partial M_2, M_1)\}$ . Let  $\gamma$  denote the image of the  $x_3$ -axis in  $\mathbb{R}^3$  and let  $D_R$  denote the tubular neighborhood of  $\gamma$  of radius  $R$ . The failure of the strong maximum principle at infinity to hold for  $M_1$  and  $M_2$  means that distance between  $M_1$  and  $M_2$  is never obtained by points on the surfaces. This property also holds if we remove a bounded subset from each of the surfaces. There exists a  $T > 0$  such that after removing  $M_i \cap D_T$  from  $M_i$ ,  $i = 1, 2$ , the new surfaces (which we also call  $M_1$  and  $M_2$ ) have their boundary in  $\partial D_T$ .  $M_1 \cup M_2$  separates  $N - D_T$  into a finite number of components where 1 or 2 of these components have both  $M_1$  and  $M_2$  on their boundary. Let  $C$  be one of these components where the distance from  $M_1$  to  $M_2$  in  $C$  equals the distance from  $M_1$  to  $M_2$  in  $N$ .

Change the metric in a compact neighborhood of  $\partial C$  so that  $\partial C$  is a good barrier (see the proof of Lemma 4) for solving Plateau problems in  $C$ . Suppose  $M_1$  does not have finite total curvature. By the argument in Lemma 4,  $\partial M_1$  is the boundary of an embedded stable minimal surface  $\Gamma$  of finite total curvature and such that  $\Gamma \subset \text{Int}(C)$  and  $\Gamma$  separates  $C$  into a component containing  $M_1$  and a component containing  $M_2$ . Furthermore, the ends of  $\Gamma$  consist of a finite number of annuli. These annuli are asymptotic to either a finite number of parallel flat planes or catenoids in  $N$  or they are asymptotic to a finite number of parallel helicoid-type ends in  $N$ .

In the case the ends of  $\Gamma$  are asymptotic to parallel planes or catenoids, then outside of some large  $D_R$ ,  $\Gamma$  disconnects  $N - D_R$  into regions in which  $M_i \cap (N - D_R)$  lift with compact boundary to  $\mathbb{R}^3$ . Replace  $M_1$  and  $M_2$  by components of  $M_i \cap (N - D_R)$ ,  $i = 1, 2$ , respectively, such that the new  $M_1$  and  $M_2$  are also closer at infinity than along their boundaries.

Let  $\tilde{M}_1$  be a lift of  $M_1$  to  $\mathbb{R}^3$ . First note that there are only a finite number of lifts  $N_1, N_2, \dots, N_k$  of  $M_2$  to  $\mathbb{R}^3$  such that the distance of the lift from  $\tilde{M}_1$  is less than  $2 \cdot \text{dist}(M_1, M_2)$ . This is because the lifts of  $M_2$  to  $\mathbb{R}^3$  are separated by parallel catenoid or planar type ends all essentially a constant distance apart. Clearly one of the surfaces  $N_i$  in  $\{N_1, \dots, N_k\}$  has distance  $\text{dist}(M_1, M_2)$  from  $\tilde{M}_1$ . However  $\min\{\text{dist}(\partial \tilde{M}_1, N_i), \text{dist}(\partial N_i, \tilde{M}_1)\} > \text{dist}(\tilde{M}_1, N_i)$ . This contradicts the strong maximum principle at infinity in  $\mathbb{R}^3$  (Theorem 3). We are left with the possibility that the ends of  $\Gamma$  are asymptotic to parallel helicoid-type ends.

Now choose  $R$  much larger than  $T$ . In particular we choose  $R$  large enough so that the ends of  $\Gamma$  intersect  $\partial D_{R'}$  almost orthogonally in almost helices for  $R' \geq R$ . Let  $\beta = M_1 \cap \partial D_R$  and note that  $\beta$  is homologous to  $\partial M_1$  in the component  $H$  of  $C - \Gamma$  that contains  $M_1$ . Applying the argument in the proof of Lemma 3 to  $\beta$  in  $H$ , we see that  $\beta$  is the boundary of a least-area orientable surface  $\Gamma_2$  of finite total curvature and  $\text{Int}(\Gamma_2) \subset \text{Int}(H)$ .

Recall that the metric in  $H$  agrees with the induced metric as a subset of  $N$  except in some compact neighborhood  $\Delta$  of  $\partial D_T \cap C$ . We claim that by choosing  $R$  sufficiently large, the surface  $\Gamma_2$  will be disjoint from  $\Delta$  and, hence, can be considered to be a minimal surface in  $N$ . First suppose that  $R$  is large enough so that  $\Delta \subset D_{(1/10)R}$  and  $\Gamma$  intersects  $\partial D_{R'}$  almost orthogonally in almost helices for  $R' \geq \frac{1}{10}R$ . In particular, the components of  $\Gamma \cap (N - D_{\frac{1}{10}R})$  are very flat multisheeted graphs over their projection onto the  $(x_1, x_2)$ -plane. Consider a surface component  $E_R$  of  $\Gamma_2 \cap (D_R - D_{\frac{1}{10}R})$ . Since  $\Gamma_2$  is a stable orientable minimal surface in a flat three-manifold, the curvature estimates of Schoen [19] imply that the Gaussian curvature of  $x \in E_R$  is at most  $\kappa/d^2$  where  $\kappa$  is a universal constant and  $d$  is the minimum of the distances of  $x$  to the boundary of  $D_R$  or  $D_{\frac{1}{10}R}$ . Hence, when  $R$  is large, the surface  $E_R$  is very flat near points in  $\partial D_{\frac{1}{2}R} \cap E_R$ . Since  $E_R$  is caught between the flat helicoid-type ends  $\Gamma \cap (N - D_{\frac{1}{10}R})$ , these curvature estimates imply the existence of an  $\varepsilon$ ,  $0 < \varepsilon < \frac{1}{10}R$ , such that the projection of  $E_R \cap (D_{\frac{1}{2}R} - D_{(\frac{1}{2}-\varepsilon)R})$  onto the  $(x_1, x_2)$ -plane is a submersion. It follows that if  $E_R \cap D_{\frac{1}{10}R} \neq \emptyset$  for  $R$  large, then  $\text{Area}(E_R)$  grows quadratically in  $R$ . Assume that the translational part of  $S_\theta$  is  $(0, 0, 1)$ . Since  $E_R$  is disjoint from  $D_T$ ,  $\partial E_R$  bounds a surface in  $H \cap \partial D_R$  of area less than  $\pi R$ . Since  $\Gamma_2$  is a surface of least area, the area of  $E_R$  grows linearly in  $R$ , a contradiction. This proves that  $\text{Int}(\Gamma_2) \cap \Delta = \emptyset$  for  $R$  large, and hence,  $\Gamma_2$  is a minimal surface in  $N$ .

Since  $\Gamma_2$  separates  $M_1 - D_R$  from  $M_2$  and the surfaces  $M_1$  and  $M_2$  are asymptotically closer at infinity,  $\text{dist}(\Gamma_2, M_2) \leq \text{dist}(M_1, M_2)$ . Since  $M_1$  and  $M_2$  are asymptotically closer at infinity, it is clear that we can choose some large value  $R$  so that  $\text{dist}(\partial \Gamma_2, M_2) < 2 \cdot \text{dist}(M_1, M_2)$  and  $\text{dist}(\partial M_2, \Gamma_2) > \frac{1}{2}R$ . Since  $\partial \Gamma_2 \subset M_1$  is compact,  $\text{dist}(\partial \Gamma_2, M_2) > \text{dist}(M_1, M_2)$ . Hence,  $\Gamma_2$  and  $M_2$  violate the strong maximum principle at infinity in  $N$ . If  $M_2$  also has infinite total curvature, then repeating the above argument with  $\Gamma_2$  and  $M_2$ , we can replace  $M_2$  by a properly embedded minimal surface  $\Gamma_3$  of finite total curvature such that  $\Gamma_2$  and  $\Gamma_3$  violate the strong maximum principle at infinity in  $N$ . As remarked earlier, the strong maximum principle at infinity holds for embedded surfaces of finite total curvature in  $N$ . This contradiction completes the proof in the case  $M_1$  and  $M_2$  are embedded.

If  $M_1$  and  $M_2$  are not embedded, the modification given in the proof of Lemma 4 by metrically completing components of  $N - (M_1 \cup M_2 \cup D_T)$ , reduces the argument to the embedded case. This completes the proof of Theorem 2.

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