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Positive scalar curvature and periodic fundamental groups

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If M is a smooth manifold and g is a riemannian metric on M, then the *scalar curvature* is a smooth real valued function $k_g : M \to \mathbb{R}$ that measures the average sectional curvature at a point of M; more precisely, k_g is formed by double contraction of the riemannian curvature tensor of g (compare [He, pages 74–75]). Geometrically speaking, the scalar curvature function measures the difference between the volumes of the riemannian and euclidean geodesic disks. The existence of a riemannian metric with positive scalar curvature on a smooth manifold turns out to be of interest in many contexts. For example, by results of J. Kazdan and F. Warner [KW] the entire question of realizing a smooth function as a scalar curvature reduces to the existence of such a metric, and results of R. Schoen [Schn] on the Yamabe problem show that a metric with positive scalar curvature can be conformally deformed to one with *constant* positive scalar curvature. Furthermore, the existence of a riemannian metric with positive scalar curvature is directly related to some questions and results from index theory, transformation groups, and the applications of differential geometry to general relativity; discussions of these relationships can be found in papers by H. B. Lawson and M. Gromov [GL1–2], and R. Schoen and S.-T. Yau [SY]. Several results from the past two decades have shown that differential-topological invariants often yield necessary or sufficient conditions for a manifold to admit a positive scalar curvature metric. In most cases the invariants involve characteristic class data and the manifold's fundamental group (compare [GL1–2], [Miy1], [Ros1–3], [SY]). As in surgery theory, there are major differences between the techniques for studying finite and infinite fundamental groups; the latter often require geometric and analytic input related to the Index Theorem, while algebraic and homotopy-theoretic methods are often preferable for finite groups.

For various technical reasons one expects that the most tractable finite fundamental groups are those of odd order. In [Ros3] J. Rosenberg considered the special case of cyclic groups $\mathbb{Z}_p$, where p is an odd prime. The main conclusion of [Ros3] in this direction is that a closed smooth manifold M with fundamental group $\mathbb{Z}_p$ and dimension at least 5 admits a riemannian metric with positive scalar curvature if and only if its universal covering $\tilde{M}$ does. Motivated by this result and by earlier results in [Ros2] in the non-spin case, Rosenberg conjectured that a similar relation holds for arbitrary odd order groups.

In Section 1 of this paper we shall prove Rosenberg's conjecture for manifolds with *periodic* odd order fundamental groups (Theorem 1.8). Recall that a finite group G is periodic if there is a $d > 0$ such that $H^i(G) \cong H^{d+i}(G)$ for all $i > 0$; the least such d is called the *period* of G and is denoted by d_G. One reason for interest in such groups is that a finite group of odd order acts freely and smoothly on some (homotopy) sphere if and only if G is periodic (compare [DM]). The most basic class of such actions is given by *orthogonal* (or *linear*) *spherical spaceforms* in which the action of G on S^n is induced by an orthogonal representation on $\mathbb{R}^{n+1}$ that is free except at the origin. For these examples the quotient manifolds S^n/G admit complete riemannian metrics of constant positive sectional curvature, and in fact an arbitrary manifold M^n admits such a metric if and only if M^n is an orthogonal spherical spaceform.

Although the orbit manifold of a free *nonlinear* differentiable action on a homotopy sphere cannot admit a metric with constant positive sectional curvature, it is still meaningful to ask if there is a metric on M^n with weaker positive curvature properties, and scalar curvature provides a natural starting point. The existence of metrics with reasonable positive curvature properties has attracted particular interest when the group G admits a free differentiable action on some homotopy sphere but does not act freely and orthogonally on any sphere (compare [Schu, Problem 8.13, page 558]); the simplest examples are nonabelian groups of order pq, where p and q are distinct odd primes, and the smallest such group has order 21 (compare [Pe], [Lee]). Our results imply a complete characterization of those quotient manifolds $M = S^n/G$ admitting riemannian metrics with positive scalar curvature. Specifically, M admits such a metric if and only if its universal covering does (see Corollary 1.8). It follows that either M admits a positive scalar curvature metric or the connected sum of M with some homotopy sphere does (see Complement 1.9). Special cases of these results beyond [Ros3] had previously been verified by Rosenberg.

In Section 2 we consider the existence of metrics with positive scalar curvature on the orbit manifolds Σ/G, where G has even order and acts freely and smoothly on the homotopy sphere Σ. As in the odd order case, the group G must be periodic, and the dimension of Σ must be congruent to $-1 \bmod$ the period d_G of G. Furthermore, there are systematic families of such groups that act smoothly and freely on homotopy spheres but never orthogonally; in all cases it is possible to find examples in each dimension congruent to $-1 \bmod 2d_G$, and in most cases it is possible to find examples in each dimension congruent to $-1 \bmod d_G$ (compare [DM]). If the order of G is greater than 2, then there are infinitely many differentiably inequivalent examples in every such dimension, and we prove that there is always an infinite subfamily of manifolds that admit riemannian metrics with positive scalar curvature (see Theorem 2.1). In fact, in each dimension congruent to 4 except 3 itself, one can show that *every* smooth spherical spaceform admits such a metric (see Theorem 2.2).

Our methods rely on a result of Gromov–Lawson [GL2] and Schoen–Yau [SY]: If M^n has a riemannian metric with positive scalar curvature and N^n is obtained from M^n by surgery on an embedded sphere of dimension $\leq (n-3)$, then N^n also admits such a metric. It follows that the existence of a positive scalar curvature metric essentially depends upon the bordism class of a closed manifold M^n and its 2-connected reference map $M^n \to K(\pi_1(M^n), 1)$. The results of this paper will be proved by a combination of previous results of T. Miyazaki [Miy1–2] and J. Rosenberg [Ros1–3] on such bordism classes, additional homotopy-theoretic techniques from bordism theory, and surgery-theoretic results on the existence of smooth spherical spaceforms as in [DM] or [Ma].

It appears that techniques from surgery theory and homotopy theory can yield much further information on the existence of riemannian metrics with positive scalar curvature on smooth spherical spaceforms. More generally, such methods should also yield quantitative criteria for determining when a positive scalar curvature metric g on a closed spin manifold M with finite fundamental group can be propagated to a second closed spin manifold N that is homotopy equivalent to M. We shall consider these questions in subsequent papers.

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1. Rosenberg's conjecture

We begin with some notation. If π is a finite group and k is a positive integer, let $\Omega_k^{\mathrm{Spin}}(B\pi)$ be the k-th spin bordism group of π. As in [CF] or [St], this group can be described by equivalence classes of pairs $(f: V \to B\pi, \sigma)$, where (V, σ) is a closed k-dimensional spin manifold (with spin structure σ) and f is a continuous map into the classifying space of π, modulo bordism, and the group can also be described algebraically as the stable homotopy group $\pi_k(M\,\mathrm{Spin} \wedge B\pi_+)$, where M Spin is the Thom spectrum associated to B Spin and $B\pi_+$ is the disjoint union of $B\pi$ and a point. Define $\mathrm{Pos}_k(\pi) \subset \Omega_k^{\mathrm{Spin}}(B\pi)$ to be the set of all classes with representatives of the form $(V \to B\pi, \sigma)$, where V has a riemannian metric with positive scalar curvature. It is immediate from [GL2] and [SY] that $\mathrm{Pos}_k(\pi)$ is a subgroup if $k \geq 5$. Furthermore, the results in the latter papers imply an important invariance property for $\mathrm{Pos}_k(\pi)$ (compare [Ros2]).

PROPOSITION 1.1. *Let $f: M \to B\pi$ represent a class in* $\mathrm{Pos}_k(\pi)$, *where* $k \geq 5$. *If f is 2-connected, then M admits a riemannian metric with positive scalar curvature.* □

Following standard practice we define $\tilde{\Omega}_k^{\mathrm{Spin}}(B\pi)$ to be the kernel of the homomorphism $\Omega_k^{\mathrm{Spin}}(B\pi) \to \Omega_k^{\mathrm{Spin}}(\{pt\})$ induced by the constant map. The proof of Rosenberg's conjecture for $\mathbb{Z}_p$ in [Ros3] has two steps. One involves results of T. Miyazaki [Miy2] that yield lower bounds for the set $\mathrm{Pos}_k(\{pt\}) \subset \Omega_k^{\mathrm{Spin}}(\{pt\})$, and the other is a proof that $\mathrm{Pos}_k(\pi)$ contains $\tilde{\Omega}_k^{\mathrm{Spin}}(B\pi)$ for all $k \geq 0$. In fact, the methods of [Ros3] and the equivalence of ordinary and spin bordism away from 2 immediately yield a reduction of Rosenberg's conjecture to the case of spin manifolds:

(1.2) *Let π be a finite group of odd order. Then Rosenberg's conjecture is true for π if and only if* $\mathrm{Pos}_k(\pi)$ *contains* $\tilde{\Omega}_k^{\mathrm{Spin}}(B\pi)$ *for all* $k \geq 0$. □

The subgroups $\mathrm{Pos}_k(\pi)$ have covariant and contravariant naturality properties that are extremely useful in formal manipulations. We begin with covariant naturality.

(1.3) *If $\varphi: \pi \to \pi'$ is a group homomorphism and $B\varphi$ is the associated map of classifying spaces, then*

$$(B\varphi)_*: \Omega_k^{\mathrm{Spin}}(B\pi) \to \Omega_k^{\mathrm{Spin}}(B\pi')$$

sends $\mathrm{Pos}_k(\pi)$ *to* $\mathrm{Pos}_k(\pi')$.

This follows because $B\varphi_*$ sends the class represented by $(f: V \to B\pi, \sigma)$ to the class represented by $((B\varphi)f: V \to B\pi', \sigma)$. □

The statement of the contravariant naturality property involves the *transfer homomorphism*

$$(B\varphi)^! : \Omega_k^{\mathrm{Spin}}(B\pi') \to \Omega_k^{\mathrm{Spin}}(B\pi)$$

associated to the inclusion of one finite group π' into another group π. Geometrically, this map takes a class represented by the pair $(f : V \to B\pi, \sigma)$ into the class represented by $(f : V' \to B\pi', \sigma')$, where f' and V' are obtained from the pullback of the diagram $V \to B\pi \leftarrow B\pi'$ and σ' is obtained by lifting σ through the covering space projection $V' \to V$ (compare [CF]). The map $(B\varphi)^!$ can also be described homotopy-theoretically as the homomorphism induced by a transfer map of CW spectra

$$t_\varphi : S^\infty B\pi \to S^\infty B\pi',$$

where $S^\infty X$ denotes the CW spectrem associated to a CW complex X (compare [BG]).

(1.4) *If* $\varphi : \pi \to \pi'$ *is an inclusion and*

$$(B\varphi)^! : \Omega_k^{\mathrm{Spin}}(B\pi') \to \Omega_k^{\mathrm{Spin}}(B\pi)$$

is the associated transfer homomorphism, then $(B\varphi)^!$ *sends* $\mathrm{Pos}_k(\pi')$ *to* $\mathrm{Pos}_k(\pi)$.

This follows because a metric with positive scalar curvature can always be lifted to a covering space. □

The following consequence of (1.3) and (1.4) will be used repeatedly in this paper.

PROPOSITION 1.5. *Let* π *be a finite group, let* p *be a prime dividing the order of* π, *and let* $j_p : \pi_p \to \pi$ *be inclusion of a Sylow* p*-subgroup. Then a class* $\alpha \in \Omega_k^{\mathrm{Spin}}(B\pi)$ *lies in* $\mathrm{Pos}_k(\pi)$ *if and only if the images* $(Bj_p)^!(\alpha)$ *under the associated transfer homomorphisms lie in* $\mathrm{Pos}_k(\pi_p)$ *for all* p.

Proof. Half of the proposition is a restatement of (1.4), so it suffices to show that $\alpha \in \mathrm{Pos}_k(\pi)$ if for every prime p dividing the order $|\pi|$ of π the transfer $(Bj_p)^!\alpha$ lies in $\mathrm{Pos}_k(\pi_p)$.

For each prime p let T_p be the composite $(Bj_p)_*(Bj_p)^!$. Standard transfer arguments as in [BG] show that $T_p \otimes \mathbb{Z}_{(p)}$ is an isomorphism.

We shall need the following elementary fact:

(1.5A). *Let R be a noetherian ring, let Ω be a finitely generated R-module, and let T be an automorphism of Ω. If P is a submodule of Ω such that $T(P)$ is contained in P, the $T(P)$ is in fact equal to P.*

Proof of (1.5A). The ascending chain of submodules

$$P \subset T^{-1}(P) \subset T^{-2}(P) \cdots$$

must terminate because Ω is noetherian. But if $T^{-k}(P) = T^{-k-1}(P)$, then $T(P) = P$ (apply T^{k+1} to both sides of the equation). □

We now return to the proof of 1.5. Assume now that for every prime p dividing the order $|\pi|$ of π the transfer $(Bj_p)^! \alpha$ lies in $\mathrm{Pos}_k(\pi_p)$. By (1.3) it follows that $T_p(\alpha) \in \mathrm{Pos}_k(\pi)$ for all p. Since $T_p \otimes \mathbb{Z}_{(p)}$ is an automorphism of $\Omega_k^{\mathrm{Spin}}(B\pi)_{(p)}$ and $\mathrm{Pos}_k(\pi)_{(p)}$ is T_p-invariant by (1.3) and (1.4), it follows from (1.5A) that the image of $\alpha_{(p)}$ of α in $\Omega_k^{\mathrm{Spin}}(B\pi)_{(p)}$ lies in $\mathrm{Pos}_k(\pi)_{(p)}$. A similar conclusion holds for all primes p not dividing $|\pi|$, for if j_1 denotes the inclusion of the trivial subgroup in π then T_{p*} is again a bijection by transfer considerations, and the hypotheses imply that $(Bj_1)^! \alpha$ lies in $\mathrm{Pos}_k(\{pt\})$. Since $\Omega_k^{\mathrm{Spin}}(B\pi)$ is finitely generated and the image classes $\alpha_{(p)}$ lie in $\mathrm{Pos}_k(\pi)_{(p)}$ for all primes p, elementary considerations imply that α must lie in $\mathrm{Pos}_k(\pi)$. □

COROLLARY 1.6. *Let π be a finite group of odd order. Then Rosenberg's conjecture is true for π if and only if it is true for each Sylow subgroup π_p of π.* □

Proposition 1.5 reflects well known results on stable splittings of classifying space spectra $S^\infty B\pi$ into p-primary components (where p is a prime dividing the order of π). In fact, the subgroups $\mathrm{Pos}_k(\pi)$ are compatible with all of the splittings of $S^\infty B\pi$ that have been discovered during the past decade (e.g., see the expository article by S. Priddy [Pr]).

PROPOSITION 1.7. *Suppose that $S^\infty B\pi$ is (stably) equivalent to a wedge of spectra $X_1 \vee \cdots \vee X_r$. For each i such that $1 \leq i \leq r$ let $E_i : S^\infty B\pi \to S^\infty B\pi$ be the homotopy idempotent given by projection onto X_i followed by inclusion, and let E_{i*} be the induced idempotent on $\Omega_k^{\mathrm{Spin}}(B\pi)$. Then a class $\alpha \in \Omega_k^{\mathrm{Spin}}(B\pi)$ lies in* $\mathrm{Pos}_k(\pi)$ *if and only if $E_{i*}\alpha$ lies in* $\mathrm{Pos}_k(\pi)$ *for all i.*

Proof. Since $\alpha = \Sigma_i E_{i*}\alpha$ the if direction is trivial, so it suffices to prove the only if direction. Suppose that $\alpha \in \mathrm{Pos}_k(\pi)$. By Proposition 1.5 and other transfer considerations as in [Pr] it suffices to prove the result when π is a p-group. In this case the truth of the Segal Conjecture implies that the S-maps E_i of $S^\infty B\pi$ are given by sums

$$E_i \simeq \sum_j c_{ij} \cdot a_{ij1} \cdot a_{ij2} \cdots a_{ijq(j)}$$

where each c_{ij} is a p-adic integer and each a_{ijk} is either a map $S^\infty B\varphi$ associated to a group homomorphism φ as in (1.3) or a transfer t_φ associated to an inclusion φ as in (1.4); as noted in [Pr], this follows by combining the results of [Ca] and [LMM]. Since the induced self-map E_{i*} of $\Omega_k^{\mathrm{Spin}}(B\pi)$ is completely determined by the restriction of E_i to a finite subspectrum of $S^\infty B\pi$ with finite stable homotopy groups, for each k it is possible to approximate the p-adic integers c_{ij} by ordinary integers c'_{ij} such that

$$E_{i*} = \sum_j c'_{ij} \cdot a_{ij1*} \cdot a_{ij2*} \cdots a_{ijq(j)*}$$

on the group $\Omega_k^{\mathrm{Spin}}(B\pi)$; of course, one needs increasingly better approximations to the p-adic integers c_{ij} as $k \to \infty$. By (1.3) and (1.4) the map E_{i*} must send $\mathrm{Pos}_k(\pi)$ to itself, and consequently $E_{i*}\alpha$ lies in $\mathrm{Pos}_k(\pi)$ if $\alpha \in \mathrm{Pos}_k(\pi)$. □

We are now ready to prove our result on Rosenberg's conjecture:

THEOREM 1.8. *Let π be a finite periodic group of odd order, and let M be a closed spin manifold with fundamental group π and dimension ≥ 5. Then M admits a riemannian metric with positive scalar curvature if and only if its universal covering $\tilde{M}$ admits such a metric.*

Proof. Recall that a finite group π is periodic if and only if each of its Sylow p-subgroups π_p is periodic, and for p odd this happens if and only if π_p is cyclic (see [CE, Theorem XII.11.6]). Therefore by Corollary 1.6 it suffices to prove the theorem for $\pi = \mathbb{Z}_{p^r}$, where p is an odd prime.

The Atiyah–Hirzebruch spectral sequence for the groups $\Omega_k^{\mathrm{Spin}}(B\mathbb{Z}_{p^r})$ collapses for all $r \geq 1$; the considerations used in [Ros3] to verify this when $r = 1$ extend to all values of r.

Let bo be the stable homotopy spectrum for connective real K-theory (compare [ABP]), and let $D : M\,\mathrm{Spin} \to bo$ be the morphism of ring spectra induced by the Dirac orientation of a spin vector bundle (see [ABP] or [St]). The associated natural transformation of homology theories

$$\Omega_k^{\mathrm{Spin}}(X) \to bo_k(X)$$

will be denoted by $(D_X)_*$ or more simply by D_*. As noted in [Ros3, §1], it is well known that $\mathrm{Pos}_k(\{pt\})$ is contained in the kernel of D_*, and if $k \le 23$ the results of [Ros3, §1] show that equality holds. Furthermore, the results of Miyazaki [Miy2] imply that $2^r\Omega_k^{\mathrm{Spin}}(\{pt\}) \subset \mathrm{Pos}_k(\{pt\})$ for some $r \ge 0$. Therefore the collapsing of the Atiyah–Hirzebruch spectral sequence implies that $\mathrm{Pos}_k(\mathbb{Z}_{p^r})$ contains the kernel of D_* restricted to the reduced group $\tilde{\Omega}_k^{\mathrm{Spin}}(B\mathbb{Z}_{p^r})$. As in [Ros3, §1] this reduces the proof to showing that the image of $\mathrm{Pos}_k(\mathbb{Z}_{p^r})$ contains all of $\tilde{bo}_k(B\mathbb{Z}_{p^r})$; furthermore, the methods of [Ros2, Theorem 2.14] and [Ros3, Theorem 3.1] imply that it suffices to prove containment when $k = 5$. We shall prove this assertion by induction on r.

The case $r = 1$ is essentially contained in [Ros3, Theorem 1.3]. Assume that $r \ge 2$ and $D_*(\mathrm{Pos}_5(\pi))$ contains $\tilde{bo}_5(B\pi)$ for $\pi = \mathbb{Z}_{p^{r-1}}$. An Atiyah–Hirzebruch spectral sequence argument implies that the sequence

$$0 \to \tilde{bo}_5(B\mathbb{Z}_{p^{r-1}}) \to \tilde{bo}_5(B\mathbb{Z}_{p^r}) \to \tilde{bo}_5(B\mathbb{Z}_p) \to 0$$

is exact, where the monomorphism is induced by inclusion and the epimorphism is the transfer. By the induction hypothesis and (1.3) we know that the image of $\tilde{bo}_5(B\mathbb{Z}_{p^{r-1}})$ is contained in $D_*(\mathrm{Pos}_5(\mathbb{Z}_{p^r}))$. On the other hand, by [Ros3, Thm. 1.3] we also know that every element in $\tilde{bo}_5(B\mathbb{Z}_p)$ has the form $D_*(\Sigma a_i x_i)$, where the a_i are integers and each x_i is represented by a $\mathbb{Z}_p$ lens space. But every free linear action of $\mathbb{Z}_p$ on a sphere extends to a free linear action of $\mathbb{Z}_{p^r}$, and therefore each x_i lifts to an element $y_i \in \mathrm{Pos}_5(\mathbb{Z}_{p^r})$ represented by an appropriate lens space. Therefore every element $u \in \tilde{bo}_5(B\mathbb{Z}_{p^r})$ can be written as a sum $u_0 + u_1$, where u_0 lies in the image of $\mathrm{Pos}_5(\mathbb{Z}_{p^r})$ and $u_1 \in \tilde{bo}_5(B\mathbb{Z}_{p^{r-1}})$. By the induction hypothesis we know that $u_1 = D_* v$ for some $v \in \mathrm{Pos}_5(\mathbb{Z}_{p^{r-1}})$, and it follows that u lies in the image of $\mathrm{Pos}_5(\mathbb{Z}_{p^r})$. □

REMARK. Recently S. Stolz has announced that $\mathrm{Pos}_k(1)$ is the kernel of the Dirac orientation $D_* : \Omega_k^{\mathrm{Spin}}(X) \to bo_k(X) \cong \pi_k(\mathbb{Z} \times BO)$. This is a strengthening of the result from [Miy2] used in the proof of Proposition 1.5.

Theorem 1.8 has immediate consequences for smooth spherical spaceforms $M^n = \Sigma^n/G$, where $n \ge 5$ and G is an odd order group that acts freely and differentiably on the homotopy sphere Σ^n.

COROLLARY 1.9. *Let G be a finite group of odd order, and assume we are given a free differentiable G-action on the homotopy n-sphere Σ^n, where $n \ge 5$. Then Σ/G admits a riemannian metric with positive scalar curvature if and only if Σ bounds a spin manifold.*

Proof. If M^n as above admits a riemannian metric with positive scalar curvature, then by [Hi, Remark (3), page 46] the universal covering Σ^n bounds a spin manifold. Conversely, if Σ^n bounds a spin manifold, then the invariance principle in Proposition 1.1 implies that Σ admits a riemannian metric with positive scalar curvature (since Σ^n is spin cobordant to S^n and the latter has such a metric). □

COMPLEMENT 1.10. *Let M^n and Σ^n be as in* 1.8, *and assume that $G \neq \{1\}$. If n is not congruent to* $1 \bmod 8$ *then every smooth spherical spaceform M^n as in* 1.8 *admits a riemannian metric with positive scalar curvature. If n is congruent to* $1 \bmod 8$ *then either M^n admits such a metric or else there is a homotopy sphere Σ_0 such that $M \# \Sigma_0$ admits such a metric.*

In particular, Corollary 1.9 and Complement 1.10 answer a question posed by I. Madsen in [Schu, Problem 8.13, page 558]: If G is the nonabelian group of order 21 and Φ is an arbitrary free differentiable action of G on S^5, then S^5/Φ admits a riemannian metric with positive scalar curvature.

Proof. First of all, if G acts freely on a homotopy n-sphere, then it is well-known that n must be odd.

Every homotopy n-sphere bounds a spin manifold if n is not congruent to 1 or $2 \bmod 8$, and in these cases there is a homotopy sphere Σ_n that does not bound a spin manifold such that $\Sigma_n \# \Sigma_n$ is diffeomorphic to S^n, and for every homotopy n-sphere T either T or $T \# \Sigma_n$ bounds a spin manifold (compare [Bru, Theorem 1.1 and Section 2]).

If Σ bounds a spin manifold, then M admits a riemannian metric with positive scalar curvature by 1.8. If Σ does not bound a spin manifold, then by the preceding observations the homotopy sphere $\Sigma \# \Sigma_n$ bounds a spin manifold and is the universal covering of $M \# \Sigma_n$ (the universal covering of the latter is the connected sum of M and $|G| = \text{order}\ (G)$ copies of Σ_n, and since $|G|$ is odd the connected sum of $|G|$ copies of Σ_n is diffeomorphic to Σ_n). It follows that $M \# \Sigma_n$ admits a riemannian metric with positive scalar curvature by 1.8. □

REMARK. As noted in the introduction, if the odd order group G acts freely and smoothly on some sphere but never orthogonally, one can also ask if there is *some* free action for which the orbit manifold has a riemannian metric with positive curvature properties that are stronger than positive scalar curvature but (necessarily) weaker than constant positive sectional curvature. As noted in the first paragraph of this paper, if a metric with positive scalar curvature exists, then there is a metric with constant positive scalar curvature by Schoen's results on the Yamabe problem [Schn]. Two natural strengthenings of positive scalar curvature are (variable) positive sectional curvature and positive Ricci curvature (definitions may be found in many references – for example, see [He, pages 74–75]; in particular, positive sectional curvature implies positive Ricci curvature). Examples of nonlinear smooth spherical spaceforms with cyclic fundamental groups and positive Ricci curvature metrics can be obtained from results of J. Cheeger [Ch] and the work of Hernández-Andrade [He]. If π is a nonabelian group of order pq, where $p > q$ are odd primes, then there are smooth spherical spaceforms for G that are closely related to certain Brieskorn manifolds with positive Ricci curvature. Specifically, if V is the Brieskorn manifold defined by the intersection of the zero set of

$$z_1^p + \cdots + z_{kq}^p + z_{kq+1}^q \qquad (k > 0)$$

with the unit sphere in $\mathbb{C}^{kq+1}$, then V is $(kq-2)$-connected by general results on Brieskorn manifolds, the group π acts freely and differentiably on V (compare [Pe] for the case $k = 1$), and for all sufficiently large positive integers k the methods and results of H. Hernández-Andrade [He] imply that V admits a π-invariant riemannian metric with positive Ricci curvature. Furthermore, one can combine the methods of T. Petrie [Pe] with subsequent results of A. Bak [Bak] and C. T. C. Wall [Wa2–3] to perform π-equivariant surgery on embedded $(kq-1)$-spheres in V to obtain homotopy spheres with free differentiable π-actions. It seems natural to ask whether the orbit manifolds of these free π-actions also admit riemannian metrics with positive Ricci curvature.

2. Spaceforms with even order fundamental groups

The results of Section 1 completely describe the smooth spherical spaceforms admitting riemannian metrics with positive scalar curvature when the fundamental group has odd order. If the fundamental group G of the smooth spherical spaceform M has even order, then the study of the scalar curvature problem for M is considerably more difficult. However, for each group π that arises and for at least half of the possible dimensions there are infinite families of spaceforms that admit riemannian metrics with positive scalar curvature.

A precise statement of the result for even order groups requires some additional notation. If the nontrivial finite group π acts freely on some sphere, then the results of surgery theory yield free differentiable actions on homotopy spheres in all dimensions ≥ 5 and congruent to $-1 \bmod 2d_\pi$ (twice the period of π) in all cases, and for most π the results also yield actions in all dimensions ≥ 5 congruent to -1 mod the period d_π. The hypotheses of [DM, corollary 5.11(a), page 275] describe sufficient conditions for the stronger conclusion to hold (in this connection also see [Ma2]). Set $a = a_\pi$ equal to 1 if these conditions are satisfied, and set $a = a_\pi$ equal to 2 if they do not.

THEOREM 2.1. *Let $\pi \neq 1, \mathbb{Z}_2$ be a finite group that acts freely and smoothly on some sphere, let $d = d_\pi$ be the period of π, and let $a = a_\pi$ be defined as above. Then for each positive integer k with $kad > 5$ there exist infinitely many differentiably inequivalent smooth spherical spaceforms M^{kad-1} of dimension $kad - 1$ such that M admits a riemannian metric with positive scalar curvature.*

Proof of Theorem 2.1. Let π be a finite group that acts freely and smoothly on some sphere. Then for each $k > 0$ the methods of [Ma], Section 4, imply that π acts freely on some homotopy $(kad - 1)$-sphere such that for each Sylow p-subgroup π_p the manifold Σ/π_p is normally cobordant to an orthogonal spaceform. In fact, the results of [Ma] and results on Wall groups from [Wa2] imply that there are infinitely many differentiably inequivalent free π-actions on the standard sphere S^{kad-1} that are equivariantly normally cobordant to a given free action of this type. Let $\{T_j^{kad-1}\}$ denote such an infinite family of smooth π-actions. For many choices of π it is even possible to construct an infinite family of distinct free π-actions that are equivariantly h-cobordant to a specific example (compare [Mlnr]).

Since π has order ≥ 3, its period d is even. The balance of the proof splits into two cases depending upon the residue class of kad mod 4.

Case 1. Assume that kad is divisible by 4.

In this case we claim that the quotient space Σ/π is a spin manifold. Transfer considerations reduce the verification of this to the case where π is a 2-group, and thus to cases where π is either a cyclic or generalized quaternionic 2-group. In the first subcase the stable tangent bundle is given by a balanced product $\Sigma \times_\pi V$ where V is a free complex π-representation with an even number of summands; it is an elementary exercise to check that the first two Stiefel–Whitney classes vanish for such bundles. In the second subcase the stable tangent bundle is also given by a balanced product $\Sigma \times_\pi V$, but in this case V is a quaternionic π-representation; since the first two Stiefel–Whitney classes vanish for quaternionic vector bundles, the claim is also true in this subcase.

Now choose a spin structure σ on Σ/π, and let

$$\alpha \in \Omega_{kad-1}^{\mathrm{Spin}}(B\pi)$$

be the cobordism class associated to $(\Sigma/\pi \to B\pi, \sigma)$. Since the manifolds T_j/π are all normally cobordant to Σ/π and the normal cobordism between them is automatically a spin cobordism, by Proposition 1.1 the manifolds Σ/π and T_j/π will all admit riemannian metrics with positive scalar curvature if $\alpha \in \mathrm{Pos}_{kad-1}(\pi)$.

Let p be a prime dividing the order of π, let π_p be the Sylow p-subgroup of π, and let $j_p : \pi_p \to \pi$ be the inclusion. Since Σ/π_p is normally cobordant to an orthogonal spaceform, by the results of [Wa1] it is obtained from the latter by surgeries on embedded spheres of codimension ≥ 3, and the relevant cobordism is a spin cobordism. Therefore if $(Bj_p)^!$ denotes the transfer associated to the Sylow p-subgroup, it follows that $(Bj_p)^!(\alpha)$ lies in $\mathrm{Pos}_{kad-1}(\pi_p)$ (because the class in question can be represented by an orthogonal spaceform and hence by a manifold with a metric of constant positive sectional curvature). Therefore by Proposition 1.5 it follows that $\alpha \in \mathrm{Pos}_{kad-1}(\pi)$. This proves Theorem 2.1 when kad is divisible by 4.

Case 2. Assume that $kad \geq 6$ is congruent to 2 mod 4.

In this case the Sylow 2-subgroup of π must be cyclic, the period d must be congruent to 2 mod 4, and a must be 1. More important, the quotient Σ/π has a nontrivial second Stiefel–Whitney class (essentially because the same is true for $\mathbb{R}P^{4s+1}$). Therefore it is necessary to modify the preceding arguments in order to handle nonspin manifolds. Similar modifications may be found in [Ros2, Theorem 2.14] and [Miy1, Theorem 5.1].

If n is a positive integer or $n = \infty$, π is a finite group, and $\beta \in H^2(\pi; \mathbb{Z}_2)$ is a cohomology class, define $Y_n(\pi, \beta)$ to be the homotopy pullback of $\beta : K(\pi, 1) \to K(\mathbb{Z}_2, 2)$ and the second Stiefel–Whitney class $w_2 : BSO_n \to K(\mathbb{Z}_2, 2)$. Following standard conventions, if no subscript appears it is understood that $n = \infty$. If $n < m$ the canonical maps classifying spaces $BSO_n \to BSO_m$ yield corresponding morphisms $Y_n(\pi, \beta) \to Y_m(\pi, \beta)$, and if

$$\gamma_n \qquad (n < \infty)$$

denotes the pullback of the universal oriented n-plane bundle with associated Thom space $\mathrm{Th}_n(\pi, \beta)$, then one obtains a sequence of maps

$$S\,\mathrm{Th}_n(\pi, \beta) \to \mathrm{Th}_{n+1}(\pi, \beta)$$

that yield a Thom spectrum $\mathrm{Th}(\pi, \beta)$. The homotopy groups of $\mathrm{Th}(\pi, \beta)$ have the usual sort of interpretation as the bordism groups of manifolds with appropriate normal structure (compare [St]).

NOTE. If $\beta = 0$ then $Y_n(\pi, \beta)$ reduces to $B\,\mathrm{Spin}_n \times K(\pi, 1)$.

If $\varphi : \pi \to \pi'$ is a group homomorphism and $\beta' \in H^2(\pi'; \mathbb{Z}_2)$, then the universality properties of pullbacks yield mappings of spaces $C_n(\varphi) : Y_n(\pi, \varphi^*\beta') \to Y_n(\pi', \beta')$ and an associated morphism of Thom spectra $A\varphi : \mathrm{Th}\,(\pi, \varphi^*\beta') \to \mathrm{Th}\,(\pi', \beta')$. The induced map of stable homotopy groups will be denoted by $(A\varphi)_*$. Similarly, if φ is an inclusion there is an associated transfer morphism of spectra from $\mathrm{Th}\,(\pi', \beta')$ to $\mathrm{Th}\,(\pi, \varphi^*\beta')$; the induced homomorphism of stable homotopy groups will be denoted by $(A\varphi)^!$.

Suppose now that π and kad satisfy the conditions for Case 2. Then $H^2(\pi; \mathbb{Z}_2) \cong \mathbb{Z}_2$ and the nonzero class β corresponds to the second Stiefel–Whitney class of smooth spherical spaceforms M^{kad-1} with fundamental group π. Consequently the manifold M^{kad-1} together with an orientation and a reference map to $B\pi$ determine a bordism class associated to an element α of the group $\pi_{kad-1}(\mathrm{Th}\,(\pi, \beta))$.

Since $(kad - 1)$ is odd, the Atiyah–Hirzebruch spectral sequence implies that the stable homotopy group $\pi_{kad-1}(\mathrm{Th}\,(\pi, \beta))$ is finite, and therefore α may be decomposed as a finite sum $\Sigma\, \alpha_p$, where p ranges over all the primes dividing the order of $\pi_{kad-1}(\mathrm{Th}\,(\pi, \beta))$ and the order of α_p is a power of p.

Let $\mathrm{Pos}_{kad-1}\,(\pi, \beta)$ be the set of all classes in $\pi_{kad-1}(\mathrm{Th}\,(\pi, \beta))$ representable by manifolds V^{kad-1} supporting riemannian metrics with positive scalar curvature. Exactly as in Proposition 1.1 (the special case $\beta = 0$), if $k \geq 5$ the results of [GL2] and [SY] imply that $\mathrm{Pos}_{kad-1}\,(\pi, \beta)$ is a subgroup of $\mathrm{Th}\,(\pi, \beta)$ and that all representatives for which the associated reference map $V^{kad-1} \to B\pi$ is 2-connected admit riemannian metrics with positive scalar curvature. Therefore it suffices to prove that each α_p lies in $\mathrm{Pos}_{kad-1}\,(\pi, \beta)_{(p)} \subset \mathrm{Pos}_{kad-1}\,(\pi, \beta)$ (the latter is finite because $\pi_{kad-1}(\mathrm{Th}\,(\pi, \beta))$ is).

As in Case 1, if p is a prime dividing the order of π, let π_p be the Sylow p-subgroup of π, and let $j_p : \pi_p \to \pi$ be the inclusion. By construction the covering spaces Σ/π_p are normally cobordant to orthogonal spaceforms, and the normal cobordisms are automatically (π_p, β)-bordisms. It follows that $(Aj_p)^!\alpha$ lies in $\mathrm{Pos}_{kad-1}\,(\pi_p, \beta)_{(p)}$. If we project onto the p-primary component, we obtain the relations

$$(Aj_p)^!\alpha_p \in \mathrm{Pos}_{kad-1}\,(\pi_p, \beta)_{(p)},$$
$$(Aj_p)_*(Aj_p)^!\alpha_p \in \mathrm{Pos}_{kad-1}\,(\pi, \beta)_{(p)}.$$

As in Case 1, the second of these implies that α_p lies in $\mathrm{Pos}_{kad-1}\,(\pi, \beta)_{(p)}$. By our previous remarks, this completes the proof of Theorem 2.1 when kad is congruent to 2 mod 4. □

Refinements of Theorem 2.1

Roughly speaking, Corollary 1.9 and Theorem 2.1 show that there are basically no obstructions to the existence of positive scalar curvature metric on smooth spherical spaceforms that arise from the fundamental groups or dimensions of these manifolds. The obvious next question is to determine exactly which spherical spaceforms admit such metrics; for fundamental groups or odd order the results of Section 1 answer this question completely. In a subsequent paper we shall prove a result that disposes of half the remaining cases:

THEOREM 2.2. *Let $n = 4k + 3 \geq 7$. Then every n-dimensional smooth spherical spaceform admits a riemannian metric with positive scalar curvature.*

The results of this paper reduce the proof of this result to the case of spaceforms whose fundamental groups are 2-groups. For such groups all spherical spaceforms are homotopy equivalent to linear models, and therefore the existence question can be viewed as a spacial case of the following:

PROPAGATION QUESTION. *Let M^n and N^n be closed smooth manifolds that are homotopy equivalent, and suppose that N has a positive scalar curvature metric. Does M also have such a metric?*

If $N = S^n$ and $n \geq 5$ then this question has an affirmative answer if and only if n is not congruent to 1 or 2 mod 8 (compare [GL1 – 2]). Theorem 2.2 is essentially an affirmative answer to the propagation question for linear spaceforms N^{4k+3} whose fundamental groups are 2-groups.

Our study of this question involves the surgery exact sequence of [Wa1]. It is fairly straightforward to show that the existence of a riemannian metric with positive scalar curvature on a spherical spaceform M homotopy equivalent to the linear spaceform Σ/π (where π is a finite 2-group) depends only upon the 2-local normal invariant of the homotopy equivalence from M to Σ/π. Furthermore, a case by case analysis shows that each normal invariant can be realized by a degree 1 normal map $\varphi : M^* \to \Sigma/\pi$, where φ is 2-connected and M^* has a metric with positive scalar curvature; this analysis uses a variety of techniques from algebraic topology and the representation theory of compact Lie groups. Theorem 2.2 follows directly from these considerations and the invariance property described in Proposition 1.1.

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