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Some examples of higher rank manifolds of nonnegative curvature

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1. Introduction

Let M be a complete Riemannian manifold. We recall the notion of rank from [2] (cf. also [3]). It measures the amount of flatness in a manifold.

DEFINITION 1.1. If γ is a (complete) geodesic in M we define the rank of γ, $\operatorname{rk}\gamma$, as the dimension of the space of parallel Jacobi fields along γ. Let the rank of M, $\operatorname{rk} M$, be the minimum of the ranks of all geodesics in M. Also, we call a geodesic γ regular if $\operatorname{rk}\gamma = \operatorname{rk} M$.

Recall that a metric on M is locally irreducible if the universal cover of M does not split isometrically as a product. In nonpositive (sectional) curvature and higher rank, all locally irreducible finite volume manifolds (with bounded curvature) are locally symmetric spaces [1], [8], [12]. This result uses the special properties of nonpositive curvature in an essential way. In fact, Heintze found examples of normally homogeneous nonsymmetric spaces of nonnegative curvature and higher rank [16]. In this note, we will obtain more examples of higher rank and nonnegative curvature with some new features. Indeed, the whole point of this paper is to show that higher rank metrics in nonnegative curvature can be very complicated.

One should compare our situation with the pinching theorems. There there is a duality between positive and negative curvature. In fact, if M is any rank 1 compact locally symmetric space with nonconstant curvature then any other 1/4-pinched metric on M must be symmetric. For positive curvature, this is a consequence of Berger's famous rigidity theorem [10]. For negative curvature, this was proved by Hamenstädt [15]. Notice though that there really is no theorem dual to the sphere theorem in negative curvature, due to the Gromov–Thurston examples of compact manifolds with arbitrarily pinched sectional curvatures which are not homotopy equivalent to a space with constant curvature. Similarly, our examples show that duality fails for the higher-rank rigidity theorems.

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The construction of our examples is based on a simple lower estimate of the rank of certain submersion metrics.

THEOREM 1.2. *Suppose M is a compact Riemannian manifold and H a compact group of isometries of M which acts on M with only principal orbits. Let* $\pi : M \to B \overset{\text{def}}{=} M/H$ *be the associated Riemannian submersion. Then* $\operatorname{rk} B \geq \operatorname{rk} M - \dim F$ *where F is the fiber of the submersion.*

As Remark 2.4 shows, one cannot in general improve the estimate of the rank by $\operatorname{rk} M - \operatorname{rk} F$. Also note that the compactness of M is essential. For a noncompact counterexample see Example 2.2. We do not know whether the theorem holds true for a general submersion with compact total space. If the submersion has totally geodesic fibers it follows quite easily. Also one can always estimate the rank of geodesics in the base space that are covered by a closed horizontal geodesic.

The rank of a manifold really is an infinitesimal measure of the amount of flatness in a manifold. More globally, let us make the

DEFINITION 1.3. A k-flat F in a Riemannian manifold is a totally geodesic isometric immersion of $\mathbf{R}^k$ into M.

One can then ask whether every geodesic lies in a k-flat. Let us call the largest such k the *global rank* of M. Of course, the rank of M is always at least as big as the global rank. Whether a converse holds, that is whether one can integrate the parallel Jacobi fields to flats is only known in nonpositive curvature [2]. In all of our examples however, the two ranks are in fact equal (cf. Corollary 2.5).

In Section 3 we use Theorem 1.2 to determine the rank of various standard submersion metrics. In particular, we see in 3.3.1 that higher-rank metrics of nonnegative curvature are not infinitesimally rigid, even for the standard symmetric spaces.

COROLLARY 1.4. *Let M be a rank k globally symmetric space of the compact type with the standard symmetric metric* g_0. *Then there is a* 1*-parameter variation of metrics* g_t *of* g_0 *of constant (global) rank k and nonnegative curvature such that none of the metrics* g_t *for* $t > 0$ *is symmetric.*

Let M be a manifold of global rank at least 2. One can study the "intersection pattern" of the flats at a point $p \in M$. More precisely, choose a sphere S centered at p of radius less than the injectivity radius of p. Since the global rank is at least 2, the intersections of the k-flats through p with S define a (singular) foliation of S. Call a point $x \in S$ *regular* if the geodesic through p and x is regular. We define *Weyl* *chambers* as the connected components of the regular points of the leaves of this foliation. Note that the Weyl chambers are convex subsets of S (possibly empty). Call this tesselation of the regular points of S by the Weyl chambers the *building germ* Δ_p of the metric at p. Note that Δ_p is independent of S. The building germ is a cell complex where the cells are convex subsets of S.

We do not know how complicated these building germs can be. When M is a symmetric space, the building germ at any point is just a spherical building in the sense of Tits [23]. This follows from the fact that flats in M correspond to flats in the symmetric space of noncompact type dual to M [17]. For the deformations in Corollary 1.4 the building germs are "combinatorically isomorphic" to those of the symmetric space (in the sense that there is a bijection of the flats through a point p in the deformation to the flats through p in the symmetric space that preserves intersections).

Thus not even the full intersection structure of the flats determines the metric. However, we can define a finer invariant, the *Tits metric* d_T on S. If x and y are two points on S, let $d_T(x, y)$ be the length (in the round metric on S) of the shortest path connecting x to y that is piecewise contained in a k-flat through p. If there is no such path, we set $d_T(x, y) = \infty$. It is easy to see that the Tits metric is finite in all of our examples. Note that d_T makes the building germ into a metric space. We will see that the building germs of the deformations in Corollary 1.4 are not isometric to that of the symmetric space. We do not know whether the Tits metric determines the metric in general.

Let us call a manifold *strongly inhomogeneous* if it does not have the homotopy type of a compact homogeneous space. Eschenburg constructed strongly inhomogeneous compact 7-manifolds of positive curvature [13]. In Section 4 we use his examples to show

THEOREM 1.5. *There are strongly inhomogeneous compact* 9-*manifolds with locally irreducible metrics of nonnegative curvature and* (*global*) *rank* 2.

In fact, these manifolds are 2-sphere bundles over the Eschenburg examples. It is much easier to find inhomogeneous metrics of higher rank and nonnegative curvature. In fact, one can construct such metrics on $SU(3) \times_{S^1} S^2$, starting from an S^1-invariant metric on S^2.

Finally, in Section 5, we generalize Berger's theorem on the nonexistence of variations positive of first order of the product metric on $S^2 \times S^2$ to metrics of nonnegative curvature with a 2-flat.

We are grateful to T. Farrell for showing us Proposition 4.2. Our proof is a variation on his argument. Also we would like to thank C. H. Sah for several helpful conversations.

2. The rank of submersion metrics

Here we discuss Theorem 1.2 and a variant of it. We will adopt the notations of [5] and [20].

First we prove Theorem 1.2. Recall that M is a compact manifold and H a closed group of isometries of M with only principal orbits. Give $B \overset{\text{def}}{=} M/H$ the submersion metric induced by $\pi : M \to B$. Consider a geodesic $\check{\gamma}$ in B with initial vector $\check{x}$. Since π is a Riemannian submersion we can define diffeomorphisms k^t between the fibres $F_0 = \pi^{-1}(\check{\gamma}(0))$ and $F_t = \pi^{-1}(\check{\gamma}(t))$ in the following way: Let $k^t(p) \overset{\text{def}}{=} \gamma_p(t)$, where γ_p is the (unique) horizontal lift of $\check{\gamma}$ which starts at $p \in F_0$.

Fix $p \in F_0$ and consider a vertical curve c through p with initial vector $v = \dot{c}(0)$. The diffeomorphism k^t gives rise to a geodesic variation α of γ_p:

$$\alpha(s, t) \overset{\text{def}}{=} k^t(c(s)).$$

The corresponding Jacobi field $J_v(t)$ along γ_p with $J_v(0) = v$ is vertical and

$$k^t_* v = J_v(t). \tag{$*$}$$

Set $m := \text{rk } M - \dim F_0$. As rk B is always at least 1, we may assume that $m \geq 2$. Then we can find $(m-1)$ orthonormal parallel Jacobi fields $E_1(t), \ldots, E_{m-1}(t)$ along γ_p which are orthogonal to γ_p and horizontal for $t = 0$. By Lemma 2.1 below, the inner product $\langle E_i, J_v \rangle(t)$ is identically zero for all $v \in T_pF_0$. By $(*)$ we have $T_{\gamma_p(t)}F_t = \{J_v(t) \mid v \in T_pF_0\}$. Thus every parallel field E_i is horizontal for all $t \in \mathbf{R}$. Hence we get $m - 1$ vectorfields $\check{E}_i \overset{\text{def}}{=} \pi_* E_i$ along $\check{\gamma}$ such that

$$0 = \mathscr{H}(E_i') = \check{E}_i' \tag{1}$$

$$0 = v(E_i') = A_{\dot{\gamma}} E_i, \tag{2}$$

where A is the O'Neill tensor of π (cf. [20]). By [5, 9.28f, p. 241] equation (2) implies that $\check{R}(\check{E}_i, \dot{\check{\gamma}})\dot{\check{\gamma}} = 0$. Together with (1) this shows that $\check{E}_i$ is a parallel Jacobi field along $\check{\gamma}$. Therefore we have rk $(B) \geq m$.

To finish the proof of Theorem 1.2 we need the following generalization of the Clairaut integral.

LEMMA 2.1. *Consider the Riemannian submersion $\pi : M \to M/H$, where M is a compact Riemannian manifold and H is a closed subgroup of isometries such that all orbits of the H-action on M are principal orbits. Let $\gamma : \mathbf{R} \to M$ be a horizontal geodesic and E a parallel Jacobi field along γ. If $J_v(t) = k^t_* v$ denotes the vertical*

Jacobi field along γ *defined by* $(*)$ *then*

$$f_v(t) \stackrel{\text{def}}{=} \langle J_v, E\rangle(t)$$

is constant in t.

Proof. As E is parallel we get

$$\begin{aligned} f_v''(t) &= \langle J_v'', E\rangle(t) \\ &= -\langle R(J_v, \dot\gamma)\dot\gamma, E\rangle(t) \\ &= -\langle R(E, \dot\gamma)\dot\gamma, J_v\rangle(t). \end{aligned}$$

Since E is a parallel Jacobi field we deduce that $f_v'' = 0$. Thus $f_v(t) = at + b$ for some $a, b \in \mathbf{R}$. Let $l = \dim F_0 = \dim H - \dim H_p$ where H_p is the isotropy group of p. Choose a basis $v_1, \ldots, v_l$ of T_pF_0 such that $v_i = \mathscr{K}_i(p)$, where $\mathscr{K}_i$ is a Killing field generated by the action of H on M. Since k^t commutes with all elements $h \in H$, we get

$$k^t_* \mathscr{K}_i(p) = \mathscr{K}_i \circ \gamma(t) \qquad (1 \le i \le m-1).$$

Therefore equation $(*)$ yields

$$\mathscr{K}_i \circ \gamma(t) = J_{v_i}(t).$$

Hence J_{v_i} is the restriction of a globally defined vector field on M. Since M is compact we get

$$|J_{v_i}| \le \|\mathscr{K}_i\| < \infty.$$

In particular, f_{v_i} is bounded for all i. By $(*)$, f_v is bounded. Thus $a = 0$ and f_v is constant. □

EXAMPLE 2.2. Theorem 1.2 and Lemma 2.1 do not hold in general if M is not compact. As an example, consider $M = S^1 \times \mathbf{R}^2$ and let S^1 act diagonally on M by rotation. It is not difficult to verify that the rank of M/S^1 is 1.

Consider a Riemannian submersion $\pi : M \to B$. Recall that a k-flat in M is an isometric totally geodesic immersion $F : \mathbf{R}^k \to M$. The next proposition summarizes the relation between k-flats of M and B.

PROPOSITION 2.3. *Let $\pi : M \to B$ be a Riemannian submersion, γ a horizontal geodesic, $\check{\gamma} \stackrel{\mathrm{def}}{=} \pi \circ \gamma$ and E a parallel Jacobi field along γ.*

1. *If E is horizontal for all t then $\check{E} \stackrel{\mathrm{def}}{=} \pi_* E$ is a parallel Jacobi field along $\check{\gamma}$. Conversely, if $\check{E}$ is a parallel Jacobi field and M has nonnegative curvature then the same holds for the horizontal lift E of $\check{E}$ along γ.*
2. *Let F be a k-flat in M. If F is horizontal then $\check{F} \stackrel{\mathrm{def}}{=} \pi(F)$ is a k-flat in B. Conversely, suppose M has nonnegative curvature. Then, given a k-flat $\check{F}$ in B there is a (uniquely determined) horizontal k-flat F through every point $p \in \pi^{-1}(\check{F})$ with $\pi(F) = \check{F}$.*

Proof. The first claim is a straightforward application of O'Neill's formulas (cf. [20] and the proof of Theorem 1.2). Indeed, we have

$$0 = \langle R(\check{E}, \dot{\check{\gamma}})\dot{\check{\gamma}}, \check{E}\rangle = \langle R(E, \dot{\gamma})\dot{\gamma}, E\rangle + \|A_{\dot{\gamma}}E\|^2.$$

Since M is nonnegatively curved, this shows that $\langle R(E, \dot{\gamma})\dot{\gamma}, E\rangle = 0$, and therefore $R(E, \dot{\gamma})\dot{\gamma} = 0$.

The second claim follows from the fact that the distribution defined by lifting the tangent spaces of $\check{F}$ is integrable. Indeed, as above, the O'Neill tensor vanishes for this distribution. □

REMARK 2.4. The inequality $\operatorname{rk} M/H \geq \operatorname{rk} M - \operatorname{rk} H$ does not hold in general: Let S^3 be the round 3-sphere. Set $M \stackrel{\mathrm{def}}{=} S^3 \times S^3 \times S^3$ and let $H \stackrel{\mathrm{def}}{=} S^3 \cong SU(2)$ act diagonally on M. Then M/H with the submersion metric is diffeomorphic to $S^3 \times S^3$ and has rank 1, as is straightforward to show.

We can apply Theorem 1.2 to the case where M is a compact symmetric space of nonnegative curvature and higher rank. Since a Riemannian submersion is curvature non-decreasing, we obtain manifolds $B = M/H$ of nonnegative curvature and higher rank which are in general neither symmetric nor products. We will study this class in the next section in more detail. The special case of a normal homogeneous space $B = G/H$ is due to E. Heintze [16].

COROLLARY 2.5. *Let M be a compact Riemannian manifold of curvature $K \geq 0$ and H a Lie group acting freely on M by isometries. Then the space of orbits $B \stackrel{\mathrm{def}}{=} M/H$ inherits a metric of non-negative curvature $\check{K}$ and $\operatorname{rk} B \geq \operatorname{rk} M - \dim H$. Furthermore, if M is a symmetric space, $\check{\sigma}$ a 2-plane in TB with $\check{K}(\check{\sigma}) = 0$ then there exists a complete 2-flat $\check{F}$ such that $\check{\sigma}$ is tangent to $\check{F}$.*

Proof. The first part of this corollary follows directly from Theorem 1.2. Let σ be a horizontal lift of $\check{\sigma}$ through a point $p \in M$. Since $\check{K}(\check{\sigma}) = 0$ we obtain by [20] that $K(\sigma) = 0$. Since M is a symmetric space there is a 2-flat F with $\sigma = T_pF$. Let x and y be a basis of σ and let E be the parallel field along γ_x with $E(0) = y$, where γ_x is the geodesic with initial vector x. Then E is tangent to F and horizontal by Lemma 2.1. This shows that F is horizontal. By Proposition 2.3, $\check{F} := \pi(F)$ is the desired flat in B. □

3. Simple applications

3.1. *Normal homogeneous spaces* (*E. Heintze*)

Let $M = G$ be a compact Lie group with a biinvariant metric, H a closed subgroup with $\dim H + 2 \leq \operatorname{rk} G$. By Corollary 2.5, the normal homogeneous space $B \overset{\text{def}}{=} G/H$ carries a metric of nonnegative curvature and $\operatorname{rk} \geq 2$.

3.2. *Biquotients*

(cf. [13, p. 496] and [14]) Let G be a compact Lie group, H a closed subgroup of $G \times G$. We denote the projection from $G \times G$ to the second factor by pr and assume that the metric on G is left-invariant under G and right invariant under $\operatorname{pr}(H)$. Then $h = (h_1, h_2) \in H$ acts on G by an isometry via

$$h \cdot g \overset{\text{def}}{=} h_1 g h_2^{-1}.$$

If the metric on G has non-negative curvature (which of course holds for the biinvariant metric on G) and if H acts freely on G (or, more generally, if all isotropy groups are principal) then $B = G/H$ with the submersion metric has nonnegative curvature and $\operatorname{rk} B \geq \operatorname{rk} M - \dim H$. As an explicit example, take $M = SU(4)$ and $H \subset SU(4) \times SU(4)$. Let H be the circle generated by the tangent vector $(D_1, D_2) \in T_1H$, where the D_i are the diagonal matrices with coefficients $1, 0, -1, 0$ and $2, 2, -4, 0$ respectively. It is easy to check that H acts freely (cf. [13] for the corresponding statement for $SU(3)$).

3.3. *Quotients of products*

(cf. [10, p. 79] and [9]) Let G be a compact Lie group, M_0 a Riemannian manifold and H a subgroup of G which acts on M_0 by isometries. Assume that G

carries a metric right invariant under H. Then H acts isometrically on $M \overset{\text{def}}{=} G \times M_0$ by

$$h \cdot (g, p) \overset{\text{def}}{=} (gh^{-1}, hp).$$

The action is fixed point free and the manifold $B \overset{\text{def}}{=} M/H = G \times_H M_0$ is an example of the type described in Corollary 2.5.

3.3.1. *Deforming symmetric metrics*

Here we prove Corollary 1.4. By varying the metric on factors, we can restrict ourselves to the irreducible case. Also, it clearly suffices to consider the case when the rank is bigger than 1. Thus we let M be an irreducible globally symmetric space of the compact type with rank $k > 1$ with the standard symmetric metric g_0. In the construction above, let $G = H = S^1$ be a subgroup of the isometries of M. Then $B = S^1 \times_{S^1} M$ is diffeomorphic to M and the rank of the submersion is at least k (we will see below that it is actually k). If we multiply the given metric on S^1 by μ^2 then the metric on $(B, g_\mu) = ((\mu^2 S^1) \times_{S^1} M)$ converges to the initial metric on M as $\mu \to \infty$. Setting $t = 1/\mu^2$, we obtain a deformation $t \to g_t$ of the symmetric metric g_0 on M in the category of higher rank manifolds of non-negative curvature. If the S^1-action on M is fixed point free, then this deformation is exactly of the type described in [5, p. 252]. Fix some $t = 1/\mu^2 > 0$ and suppose from now on that $H = S^1$ acts on M by translations. We will now study the structure of the flats in (M, g_t). We refer to the Introduction for the definition of building germs and the Tits metrics on them.

Let $((d/dx), k)$ be the infinitesimal generator of the action of S^1 on $S^1 \times M$. By a simple calculation (cf. also Proposition 2.3) we obtain the

LEMMA 3.1. *Let $p \in M$, and let F be a k-flat through p in the standard metric. Then the image $\hat{F}$ of T_pF in $S^1 \times M$ under the map*

$$f \mapsto \exp_{(s,p)}\left(-f \cdot \kappa(p)\frac{d}{dx}, f\right)$$

is horizontal for any $s \in S^1$. Conversely, every horizontal k-flat is of this form.

COROLLARY 3.2. *The building germ Δ' of $S^1 \times_{S^1} M$ at any point is (combinatorially) isomorphic with the Tits building Δ of M.*

Proof. The map from the lemma gives the desired isomorphism. □

COROLLARY 3.3. *The Tits metrics on Δ and Δ' are not isometric.*

Proof. Fix a point $p \in M$. Since M is a symmetric space, the building germ Δ has some additional structure, namely Δ is simplicial. We call the unit vectors corresponding to vertices in Δ the *maximally singular* vectors. The idea of the proof is that some $(k-1)$-simplex in Δ becomes "bigger" in Δ' with respect to the Tits metrics.

To find this $(k-1)$-simplex we first claim that there is a maximally singular unit vector w based at p with $w \perp \kappa(p)$. In fact, the vertices of Δ decompose into finitely many connected sets (in the Hausdorff topology on S) such that each such set contains exactly one vertex from each simplex. This follows from the description of the building of the dual symmetric space of the noncompact type by parabolic subgroups [23]. The set a vertex belongs to is called the *type* of the vertex. Now our claim follows easily. In fact, let w_1 and w_2 be vertices of the same type in $(k-1)$-simplices that $\kappa(p)$ and $-\kappa(p)$ belong to. Since the diameter of any $(k-1)$-simplex in Δ is at most $\pi/2$, w_1 and w_2 lie in the northern and southern hemisphere defined by $\kappa(p)$ respectively. Connect w_1 to w_2 by a path in the vertices of the same type as w_1. The intersection point of this path with the equator defines a maximally singular vector w perpendicular to $\kappa(p)$.

We may also assume that w is tangent to a flat F which is not perpendicular to $\kappa(p)$. Indeed, let w' be a unit vector not perpendicular to $\kappa(p)$. Then w and w' are connected by a finite chain of flats. The first flat (starting from w) that is not orthogonal to $\kappa(p)$ contains a maximally singular vector as desired.

For a unit vector v tangent to F, let $\hat{v}$ denote the unit vector tangent to $S^1 \times M$ in direction of $(-v \cdot \kappa(p)(d/dx), v)$. An easy calculation shows that

$$\hat{w} \cdot \hat{v} = \frac{w \cdot v}{\sqrt{1 + (v \cdot \kappa(p))^2}}. \tag{$*$}$$

Let $\mathscr{C}$ be a $(k-1)$-simplex containing w. Since F is not perpendicular to $\kappa(p)$, $\mathscr{C}$ is not perpendicular to $\kappa(p)$. Let $\{v_1, \ldots, v_l\}$ be the vertices of $\mathscr{C}$ which are not orthogonal to $\kappa(p)$, and let $\{w_0 = w, w_1, \ldots, w_{k-l-1}\}$ be the remaining vertices. Since M and therefore its Tits building are irreducible, not all v_i can be orthogonal to all w_j. Thus there are vertices v and w' of $\mathscr{C}$ such that w' is orthogonal to $\kappa(p)$ and v is neither orthogonal to $\kappa(p)$ nor to w'.

By the above calculation ($*$), the distances between w' and the other vertices of $\mathscr{C}$ orthogonal to $\kappa(p)$ are not decreased and the distances between w' and the vertices not orthogonal to $\kappa(p)$ definitely become bigger. This shows that there cannot be a type preserving isometry between the building germs of M and $S^1 \times_{S^1} M$. Also note that this finishes the proof in rank 2.

Finally, let ϕ be any isometry between the building germs. Identifying the building germs combinatorially as in Corollary 3.2, ϕ defines a combinatorial isomorphism. In fact, it suffices to see that ϕ maps regular points to regular points. Regular points in the building germ of M are characterized by the property that they have neighborhoods that are balls. Since the combinatorial isomorphism from the last corollary is continuous the same characterization holds for Δ'. An isometry, of course, preserves this property.

Identifying Δ and Δ' combinatorially as in Corollary 3.2, ϕ defines a combinatorial automorphism of Δ. Then a finite power ϕ^l of ϕ is type-preserving [23, Corollary 5.10]. Now interpret ϕ^{l-1} as an isometry of Δ. Then $\phi \circ \phi^{l-1}$ defines a type-preserving isometry between the building germs which is impossible. □

Proof of Corollary 1.4. The argument for the last corollary can also be used to show that the metrics g_t are not symmetric. Let f be a maximally singular vector orthogonal to $\kappa(p)$. Since the building is irreducible, there is a maximally singular vector f' in the star of f that is not orthogonal to f. Let T be the set of maximally singular vectors of the same type as f' in the star of f. Again, T is connected, and as above, we may assume that not every vector in T is perpendicular to $\kappa(p)$. Using the geodesic symmetry in p we see that there are vectors in T strictly to either side of the equator defined by $\kappa(p)$. Since T is connected, there are also vectors in T perpendicular to $\kappa(p)$. Using formula $(*)$ from the proof of Corollary 3.3, we see that the distance between vertices of fixed type is not constant. Thus g_t is not symmetric. □

This finally proves all our claims about the deformations of the symmetric metrics made in the Introduction.

4. A strongly inhomogeneous manifold of nonnegative curvature and higher rank

In this section we will construct a compact Riemannian manifold of nonnegative curvature and higher rank which topologically is not a product and which is not homotopy equivalent to any compact Riemannian homogeneous space. This will prove Theorem 1.5.

We combine the constructions from 3.2 and 3.3: Consider $G = SU(3)$ and let $H = H_{klpq}$ be a closed one-parameter subgroup of $G \times G$ as in [13]. The numbers k, l, p and q describe how $H = S^1$ is embedded into $G \times G$. Choose k, l, p and q such that the action of H on G does not have fixed points and such that (cf. [13])

$$r := |(k^2 + l^2 + kl) - (p^2 + q^2 + pq)| \equiv 2 \bmod 3.$$

Let H also act on the standard sphere S^2 by rotation. Then the space $X \stackrel{\text{def}}{=} SU(3) \times_H S^2$ has nonnegative curvature and rank at least 2. Topologically, X is a 2-sphere bundle over Eschenburg's strongly inhomogeneous 7-manifold $Y \stackrel{\text{def}}{=} SU(3)/H$. Metrically however we just endow $SU(3)$ with the biinvariant metric unlike Eschenburg who strives for positive curvature on the biquotient Y.

Proposition 2.5 implies that every flat 2-plane σ (i.e. $K(\sigma) = 0$) tangent to X is tangent to a complete 2-flat $F : \mathbf{R}^2 \hookrightarrow X$. Thus the structure of the flat 2-planes in X is similar to that of symmetric spaces or normal homogeneous spaces. However, X is strongly inhomogeneous, simply connected and irreducible (topologically, i.e. X is not a product). We will show this in the next two sections.

4.1. *Homotopy and Homology of X*

Using standard techniques from algebraic topology, we calculate the homotopy and integral cohomology groups of X. For simplicity, we write $H^q(\cdot)$ for $H^q(\cdot, \mathbf{Z})$. Also we denote the cyclic group of order p by $\mathbf{Z}_p$.

PROPOSITION 4.1. *Let r be defined as above. Then*
(a) *X is connected and simply connected and*

$$\pi_2(X) = \pi_3(X) = \mathbf{Z}^2$$
$$\pi_4(X) = \mathbf{Z}_2$$

(b) *$H^q(X)$ is isomorphic to $H^q(Y) \oplus H^{q-2}(Y)$. In particular, we obtain*

$$H^1(X) = 0 \qquad H^2(X) = \mathbf{Z}^2$$
$$H^3(X) = 0 \qquad H^4(X) = \mathbf{Z} \oplus \mathbf{Z}_r$$
$$H^5(X) = \mathbf{Z} \qquad H^6(X) = \mathbf{Z}_r$$
$$H^7(X) = \mathbf{Z}^2 \qquad H^8(X) = 0.$$

Proof. The homotopy groups can be calculated easily from the exact homotopy sequence of the fibration $S^1 \to SU(3) \times S^2 \to SU(3) \times_{S^1} S^2$.

As for the cohomology groups, the 2-sphere bundle $S^2 \to X \xrightarrow{\pi} Y$ gives rise to the Gysin sequence

$$\cdots \to H^{p-3}(Y) \xrightarrow{\mu} H^p(Y) \xrightarrow{\pi^*} H^p(X) \xrightarrow{\psi} H^{p-2}(Y) \to \cdots$$

where μ is multiplication with the Euler class $e \in H^3(Y)$. In our case, e is 0 since the fixed points of the S^1-action on S^2 generate cross sections $s: Y \to X$. Thus the Gysin sequence breaks up into pieces of length 3. Moreover for every $\alpha \in H^p(X)$ we have a unique decomposition:

$$\alpha = \pi^*(\alpha_1) + a\pi^*(\alpha_2)$$

where $\alpha_1 \in H^p(Y)$, $\alpha_2 \in H^{p-2}(Y)$ and a is an element in $H^2(X)$ such that $\psi(a) = 1$ in $H^0(Y)$ (cf. [19, p. 273]). This shows that $H^q(X)$ is isomorphic to $H^q(Y) \oplus H^{q-2}(Y)$. The precise formulas for the cohomology groups now follow from Proposition 36 of [13]. □

4.2. *Irreducibility*

In this section we will show that X is irreducible. This was shown to us by T. Farrell. Our proof is a variation of his argument.

PROPOSITION 4.2. (*T. Farrell*) *The manifold X is topologically irreducible, more precisely, X is not homotopy equivalent to any product of closed manifolds.*

We begin by reducing to a special case.

LEMMA 4.3. *Let V be a closed simply connected product manifold,* $V = M^m \times N^n$ *with* $1 \le m = \dim M \le n = \dim N$. *Suppose V has the same integral cohomology groups as X. Then M is homeomorphic to* S^2 *and N is a closed 7-manifold.*

Proof. Since V is simply connected and closed, we only need to show that $m = 2$. For the same reason, we see that $m \ne 1$. Suppose that $m > 2$. Künneth's exact sequence

$$0 \to (H^*(M) \otimes H^*(N))^k \to H^k(M \times N) \to \bigoplus_{p+q=k+1} \operatorname{Tor}(H^p(M), H^q(N)) \to 0$$

implies $H^3(M) = H^3(N) = 0$ since $H^3(X) = 0$. Therefore, we see that $m = 4$ and $n = 5$.

Note that $H^2(N)$ is torsion by Poincarè duality. Hence the Künneth sequence also shows that $H^2(N) = 0$ and $H^2(M) = \mathbf{Z}^2$. Since $H^p(M)$ is torsion-free for all p, all the torsion groups in the Künneth sequence vanish and we get

$$\mathbf{Z} \oplus H^4(N) \cong H^4(X) = \mathbf{Z} \oplus \mathbf{Z}_r.$$

Therefore we have $H^4(N) = \mathbf{Z}_r$. This gives a contradiction to the exactness of the Künneth sequence

$$0 \to H^2(M) \otimes H^4(N) = \mathbf{Z}^2 \otimes \mathbf{Z}_r \to H^6(X) = \mathbf{Z}_r. \qquad \square$$

View X as the sphere bundle $S(\eta \oplus \epsilon^1)$, where η is the complex line bundle associated with the principal fibration $\xi : S^1 \to SU(3) \to Y$ of Eschenburg's example and ϵ^1 is the trivial $\mathbf{R}$-bundle over Y.

LEMMA 4.4. *The second Stiefel–Whitney class $w_2(\eta \oplus \epsilon^1)$ is not* 0.

Proof. Since $\pi_i(SU(3)) = 0$ for $i < 3$, ξ is 3-universal [21, Theorem 19.4]. Therefore, there exists a map $f: S^2 \to Y$ such that the pullback of ξ is the Hopf-bundle $\zeta : S^1 \to S^3 \to S^2$ [21]. Let v be the complex line bundle associated to ζ. By the functoriality of the Stiefel–Whitney classes we get $f^*w_2(\eta) = w_2(f^*\eta) = w_2(v)$. Since $w_2(v)$ generates $H^2(\mathbf{CP}^1, \mathbf{Z}_2)$, we see that $w_2(\eta)$ is a generator of $H^2(Y, \mathbf{Z}_2) = \mathbf{Z}_2$. Thus $w_2(\eta \oplus \epsilon^1) = w_2(v) \neq 0$. $\square$

LEMMA 4.5. *The space X is not homotopy equivalent to $S^2 \times N$ for any closed manifold N.*

Proof. First recall from the proof of Proposition 4.1(b) that every element $\alpha \in H^p(X)$ has a unique decomposition as

$$\alpha = \pi^*(\alpha_1) + a\pi^*(\alpha_2)$$

where $\alpha_1 \in H^p(Y)$, $\alpha_2 \in H^{p-2}(Y)$ and a is an element in $H^2(X)$ such that $\psi(a) = 1$ in $H^0(Y)$. In particular, choose $\alpha \in H^4(Y)$ and $\beta \in H^2(Y)$ such that

$$a^2 = \pi^*(\alpha) + a\pi^*(\beta).$$

The elements α and β determine the multiplicative structure of $H^*(X)$ completely. By Theorem III of [19] we have $\beta \equiv w_2(\eta \oplus \epsilon^1) \bmod 2$. Lemma 4.4 then shows that $\beta \equiv 1 \bmod 2$.

Now suppose that X is homotopy equivalent to $S^2 \times N$. Let σ_0 be the generator of $H^2(S^2) \hookrightarrow H^2(X)$ and w_0 the generator of $H^2(N) \hookrightarrow H^2(X)$. Note here that $H^2(N) = \mathbf{Z}$ by Künneth. We have the decompositions

$$\sigma_0 = \pi^*\sigma_1 + a\pi^*\sigma_2$$

$$w_0 = \pi^*w_1 + a\pi^*w_2.$$

Therefore we get

$$0 = \sigma_0^2 = \pi^*\sigma_1^2 + 2a\pi^*(\sigma_1\sigma_2) + \pi^*(\alpha\sigma_2^2) + a\pi^*(\sigma_2^2\beta).$$

By Künneth we know that $H^2(N) = \mathbf{Z}$ and therefore $H^4(N) = \mathbf{Z}_r$. Thus we get

$$0 = rw_0^2 = r[\pi^*w_1^2 + \pi^*(\alpha w_2^2) + a(\pi^*(2w_1w_2) + \pi^*(w_2^2\beta))].$$

Since $H^2N = \mathbf{Z}$ is torsion-free, we have

$$0 = 2w_1w_2 + w_2^2\beta = 2\sigma_1\sigma_2 + \sigma_2^2\beta.$$

In particular, we get $w_2^2\beta \equiv \sigma_2^2\beta \equiv 0 \bmod 2$. As $\beta \equiv 1 \bmod 2$, we see that

$$w_2 \equiv \sigma_2 \equiv 0 \bmod 2. \qquad (*)$$

Notice that $H^2(X)$ splits in two different ways as $\mathbf{Z} \oplus \mathbf{Z}$ using Künneth on the one hand and Proposition 4.1 on the other hand. Viewing σ_1 and w_1 as integers, the matrix which transforms one splitting to the other is given by the unimodular matrix

$$U \stackrel{\text{def}}{=} \begin{pmatrix} \sigma_1 & \sigma_2 \\ w_1 & w_2 \end{pmatrix}.$$

On the other hand, det $U \equiv 0 \bmod 2$ by $(*)$ which yields the final contradiction. □

4.3. *Strong Inhomogeneity*

The proof of the next claim is a fairly routine matter. We should say however that our efforts were facilitated by several lucky accidents.

PROPOSITION 4.6. *The manifold X is strongly inhomogeneous.*

Proof. Step 1: Assume to the contrary that X is homotopy equivalent to some compact homogeneous space $\bar{X} = G/H$ where G is a transitive subgroup of the isometry group of $\bar{X}$ and H is the isotropy group of some point $x \in \bar{X}$. In this first step, we will restrict the possibilities for G by fairly general arguments.

Since dim $\bar{X}$ = dim $X = 9$, H is a subgroup of $O(9)$. By Proposition 4.1 and the exact homotopy sequence for $H \to G \to \bar{X}$, we obtain:

1. $\pi_0(G) = \pi_0(H)$
2. the sequence $0 \to \mathbf{Z}^2 \to \pi_1(H) \to \pi_1(G) \to 0$ is exact
3. the sequence $0 \to \pi_3(H) \to \pi_3(G) \to \mathbf{Z}^2 \to 0$ is exact (since $\pi_4(\bar{X}) = \pi_4(X) = \mathbf{Z}_2$).

As in [13, 4.2 and 4.3] we see that we may assume without loss of generality (possibly replacing G by a finite cover) that

- H and G are connected
- G is compact, semisimple and simply connected
- $H = H' \times T^2$ where H' is semisimple and simply connected, and T^2 is the 2-torus.

Notice that by (3), G has $p + 2$ simple factors if H' has p such factors.

Fortunately, there are further restrictions on the Lie group G. For any compact Lie group G define

$$m(G) = \min \{\dim M \mid M \text{ is a manifold on which } G \text{ acts almost effectively}\}$$

(cf. [18, Chapter 4]). Assume that G is simply connected. Decompose $G = G_1 \times \cdots \times G_s$ such that each G_i is either simple or Spin (4) and there is at most one $SU(2)$. Thus each pair of $SU(2)$'s has been combined into a Spin (4). A theorem due to L. N. Mann says that

$$m(G) = \sum m(G_i)$$

(cf. [18, p. 68]).

In our case, G is a subgroup of the isometry group of $\bar{X}$ (up to a finite cover). Therefore G acts almost effectively on $\bar{X}$. In particular, we see that $m(G) \le \dim \bar{X} = 9$. As the number of simple factors of G is at least 2, we obtain the following list of possible factors G_0 of G (cf. [18, p. 68]):

G_0	rank	$\dim G_0$	$m(G_0)$
$SU(2)$	1	3	2
$SU(3)$	2	8	4
$SU(4)$	3	15	6
Spin (5)	2	10	4
Spin (7)	3	21	6
Spin (8)	4	28	7
$G_{(2)}$	2	14	6

Step 2: Here we complete the proof of the proposition by checking all possible candidates for G_0 from the table above. We argue using the number p of simple factors of H'. We will denote the Lie algebra of a Lie group G by $\underline{G}$.

Case I: $p = 0$

In this case $H = T^2$ and G has two simple factors: $G = G_1 \times G_2$. Moreover, we get $\dim G = \dim \bar{X} + \dim H = 11$. Thus (up to permutation) we see that $G_1 = SU(2)$ and $G_2 = SU(3)$. Let p_1 and p_2 denote the projections onto $SU(2)$ and $SU(3)$ respectively. If $\dim p_1(T^2) = 0$ or $\dim p_2(T^2) = 0$ then $\bar{X}$ is a product manifold, in contradiction to Proposition 4.2. Thus $\dim p_1(T^2) = 1$. If $\dim p_2(T^2) = 1$ then again $\bar{X}$ would be a product manifold.

Finally, we get to the most critical case of all, namely that $\dim p_2(T^2) = 2$. We may arrange the projection in such a way that one of the S^1's projects to 0 in $SU(2)$. Then

$$\bar{X} = (SU(2) \times (SU(3)/S^1))/S^1$$

is a fiber bundle $\pi : \bar{X} \to S^2$ whose fiber is the homogeneous space $W \overset{\text{def}}{=} SU(3)/S^1$, the so-called *Wallach example*. Let D_+ and D_- be the closed northern and southern hemisphere in S^2 respectively. Then the triad

$$(\bar{X}, \bar{X}_+, \bar{X}_-) \overset{\text{def}}{=} (\bar{X}, \pi^{-1}(D_+), \pi^{-1}(D_-))$$

is exact. Set $A = \bar{X}_+ \cap \bar{X}_-$. Then the Mayer–Vietoris sequence

$$\cdots \to H_4\bar{X}_+ \oplus H_4\bar{X}_- \to H_4\bar{X} \to H_3A \xrightarrow{\psi} H_3\bar{X}_+ \oplus H_3\bar{X}_- \to H_3\bar{X} \to H_2A \to \cdots$$

is exact. By [13], the Wallach examples have $H_3W \cong H^4W \cong \mathbf{Z}_s$ for some integer s. Also note that $\bar{X}_+$, $\bar{X}_-$ and A are trivial bundles and that $H_2A = \mathbf{Z}$. Thus the Mayer–Vietoris sequence above gives the exact sequence

$$0 \to \mathbf{Z} \to \mathbf{Z} \oplus \mathbf{Z}_s \xrightarrow{\psi} \mathbf{Z}_s \oplus \mathbf{Z}_s \to \mathbf{Z}_r \to 0.$$

Note that the map $\psi : H_3A = H_3W \oplus H_2W \otimes \mathbf{Z} \to H_3\bar{X}_+ \oplus H_3\bar{X}_-$ is 0 on $H_2W \otimes \mathbf{Z}$ and consists of inclusions on H_3W. Therefore

$$0 \to \mathbf{Z}_s \to \mathbf{Z}_s \oplus \mathbf{Z}_s \to \mathbf{Z}_r \to 0$$

is exact. This implies that $s = r \equiv 2 \bmod 3$ by our choice of r. However, $s \equiv 0$ or $1 \bmod 3$ since W is a Wallach example (cf. [13]).

Case II: $p = 1$

In this case, $H = H_1 \times T^2$ where H_1 is simple and $G = G_1 \times G_2 \times G_3$ with simple factors G_i. Since $m(G) \le 9$, we conclude from the table that at least one G_i, say G_1, must be isomorphic to $SU(2)$. Up to a covering, H is a subgroup of $O(9)$ so that $\operatorname{rk} H \le 4$ and therefore $\operatorname{rk} H_1 \le 2$. Thus we have the following possibilities:

$$H_1 = \begin{cases} G_{(2)} & (1) \\ \mathrm{Spin}\,(5) & (2) \\ SU(3) & (3) \\ SU(2) & (4) \end{cases}$$

Let us first make the

OBSERVATION 4.7. Suppose that (in addition to G_1) G_2 is isomorphic with $SU(2)$ and that $\underline{H_1}$ projects trivially into $\underline{G_1} + \underline{G_2}$. Then $\pi_4(\bar{X})$ contains $\pi_4 SU(2) \oplus \pi_4 SU(2) = \mathbf{Z}_2 \oplus \mathbf{Z}_2$. In particular, $\bar{X}$ is not homotopy equivalent with X as $\pi_4 X = \mathbf{Z}_2$.

Now we will exclude all the possibilities for H_1:

$H_1 = G_{(2)}$: From the table we find that $G_2 = SU(3)$ and $G_3 = G_{(2)}$ (up to permutation). Thus $\bar{X} \cong SU(2) \times SU(3)/T^2$ since $H_1 = G_{(2)}$ must project trivially onto $SU(2) \times SU(3)$. This however is our Case I.

$H_1 = \mathrm{Spin}\,(5)$: By the observation above we may assume that only $G_1 = SU(2)$. From the table we see that $G_2 = SU(3)$ and $G_3 = \mathrm{Spin}\,(5)$. Again $H_1 = \mathrm{Spin}\,(5)$ projects trivially onto $SU(2) \times SU(3)$.

$H_1 = SU(3)$: Then we see that $G_2 = G_3 = SU(3)$. Therefore we have again that $\bar{X} \cong SU(2) \times SU(3)/T^2$.

$H_1 = SU(2)$: Then $\dim G = \dim \bar{X} + \dim H = 14$. Therefore we get $G_2 = SU(2)$, $G_3 = SU(3)$ and $\bar{X} = SU(2) \times SU(2) \times SU(3)/SU(2) \times T^2$. If $\underline{H_1}$ projects trivially into $\underline{G_1} + \underline{G_2}$ then we are done by our observation. Otherwise, $\bar{X}$ is again homeomorphic to $SU(2) \times SU(3)/T^2$.

Case III: $p = 2$

In this case, $H = H_1 \times H_2 \times T^2$ and $G = G_1 \times G_2 \times G_3 \times G_4$ with simple factors H_i and G_i. Since $\operatorname{rk} H \le 4$, H_1 and H_2 must have rank 1. Thus $H_1 = H_2 = SU(2)$. Since $m(G) \le 9$, at least 3 factors, say G_1, G_2 and G_3 equal $SU(2)$. Therefore $G_4 = SU(3)$ since $\dim G = 17$. By the observation, $\underline{H_1}$ or $\underline{H_2}$ must project nontrivially into $\underline{G_1} + \underline{G_2}$. Therefore $\bar{X} \cong SU(2) \times SU(3) \times SU(3)/SU(2) \times T^2$ and we are back to the previous case.

Clearly $p \le 2$ as $\operatorname{rk} H \le 4$, and we have checked all the possibilities. □

5. First order rigidity

A famous problem of Hopf asks whether there is a metric of strictly positive curvature on $S^2 \times S^2$. More generally, one can ask whether there are metrics of positive curvature on any manifold M admitting a metric g of nonnegative curvature and rank at least 2. Let us consider a differentiable variation $t \mapsto g_t$ of the metric $g = g_0$. We call g_t *positive* if (M, g_t) is complete and has strictly positive curvature K_t for all $t > 0$. We call g_t *positive of first order* if the derivative

$$K'(\sigma) \overset{\text{def}}{=} \frac{d}{dt}\bigg|_0 K_t(\sigma)$$

is strictly positive for all 2-planes $\sigma \in K_0^{-1}(0)$ [4].

If a metric variation is positive of first order and M is compact then the variation is positive. If M_1 and M_2 are compact Riemannian manifolds without Killing fields, then the Riemannian product does not admit any positive variations which depend analytically on t [6]. Much less is known if there are Killing fields [7]. Notice Remark 2.4 where one can deform the product metric on $S^3 \times S^3$ to a metric of rank 1.

Let us now consider variations g_t positive of first order. Riemannian products and symmetric spaces of higher rank do not admit such variations [4], [22]. The obstruction to their existence are the embedded flat k-tori $i : T^k \hookrightarrow M$. More precisely, if g_t is a variation positive of first order, then the pulled back metric $\bar{g}_t \overset{\text{def}}{=} i^* g_t$ on T^k is also positive of first order. Thus $\bar{g}_t$ is positive. This is impossible since T^k does not admit a metric of positive curvature. By a similar argument we have

PROPOSITION 5.1. *Let M be a compact manifold of nonnegative curvature. Suppose there exists an immersed totally geodesic k-flat $i : \mathbf{R}^k \hookrightarrow M$ with $k > 1$. Then M does not admit a variation positive of first order.*

Proof. Let g_0 be the metric of M. Suppose g_t is a variation positive of first order of g_0. Then the pulled back variation $\bar{g}_t \overset{\text{def}}{=} i^* g_t$ is also positive of first order. Now we can estimate the curvature functions $\bar{K}_t$ of $\bar{g}_t$ from below. Let $\bar{\sigma}$ be a 2-plane in $T\mathbf{R}^k$. Using the formulas in [6, Section 3] and the compactness of M it is straightforward to check that the coefficients of the Taylor expansion around $t = 0$ of $t \mapsto \bar{K}_t(\bar{\sigma})$ are bounded from above by a constant independent of $\bar{\sigma}$. Since $\bar{K}_t'(\bar{\sigma}) = K'(i_* \bar{\sigma})$ by [22, Lemma 4.1], we obtain

$$\bar{K}_t(\bar{\sigma}) = t(K'(i_* \bar{\sigma}) + tD(\bar{\sigma})) \geq t(\delta + t(-C))$$

where D is the remainder term in the Taylor expansion and δ and C are positive constants independent of $\bar{\sigma}$. Since M is compact, we have $\frac{1}{2}i^*g_0 \leq i^*g_t \leq 2i^*g_0$ for all small t. Therefore $\bar{g}_t$ is a complete metric on $\mathbf{R}^k$ with positive curvature bounded from below. This is clearly a contradiction to the theorem of Myers [10]. □

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