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A relationship between volume, injectivity radius, and eigenvalues

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Suppose M is a compact Riemannian manifold and C a measurable subset of M having measure A. Expand the indicator function χ of C in a Fourier series in orthonormal eigenfunctions of the Laplace operator to get (in L^2)

$$\chi(y) = \sum_{k=0}^{\infty} a_k \varphi_k(y).$$

By the Parseval theorem,

$$A = \int_M |\chi(y)|^2 \, dy = \sum_{k=0}^{\infty} |a_k|^2,$$

and since $a_0 = A/\sqrt{V}$, where $V = \mathrm{vol}\,(M)$, this implies that

$$1 = \frac{A}{V} + \frac{1}{A} \sum{}' \, |a_k|^2, \tag{1}$$

where the prime on a summation sign means that the term corresponding to index 0 is omitted. This last identity is the core of Siegel's quantitative version of the Minkowski theorem for a convex symmetric body B in R^n, in which the role of C is played by $\frac{1}{2}B$ [7].

Equation (1) becomes more precise if we know something about the Fourier coefficients. We will illustrate this when M is hyperbolic and of dimension n, which we will henceforth assume to be the case. Take C to be a ball about a point x in M of radius equal to the injectivity radius R of M. It then follows from the Selberg pretrace formula (cf. [1], Chapter 11), that the Fourier coefficients are given by $a_k = h(r_k)\varphi_k(x)$, where r_k is either of the two roots of $\delta^2 + r^2 = \lambda_k$. Here $\delta = \frac{1}{2}(n-1)$, λ_k is the kth eigenvalue of the Laplace operator, and the even function h is the Selberg transform of the point-pair invariant which is 1 if its two arguments are within R of each other, and 0 otherwise (cf. [1], Chapter 11).

Equation (1) thus becomes

$$1 = \frac{A}{V} + \frac{1}{A} \sum{}' \, |h(r_k)|^2 |\varphi_k(x)|^2,$$

where the summation is over one of the two r_k's corresponding to each λ_k. For definiteness, we will suppose that the sum is taken over the r_k's which lie on the union of the non-negative reals with the imaginary segment from 0 to δi. Note that the so-called small eigenvalues of M, i.e., those in $(0, \delta^2)$, correspond to r_k's on the open imaginary segment. If $\lambda_k = \delta^2$ is an eigenvalue of multiplicity m, the corresponding $r_k = 0$ is counted m times.

Integrate now over x, to get

$$V = A + \frac{1}{A} \sum{}' \, |h(r_k)|^2,$$

from which we derive

THEOREM 1.

$$1 = \frac{A}{V} + \frac{1}{AV} \sum{}' \, |h(r_k)|^2.$$

In order to apply Theorem 1, we will need to calculate $h(r)$ for our particular point-pair invariant. Now by [1], equation (5), page 275,

$$h(r) = 2\omega_{n-2} \int_0^R \cos ru \, du \int_u^R (z(\rho) - z(u))^{\delta - 1} \sinh \rho \, d\rho,$$

where

$$z(x) = \left(2 \sinh \frac{x}{2}\right)^2 = 2(\cosh x) - 2,$$

and ω_{n-2} is the area of the $(n-2)$-sphere in $R^{n-1} (\omega_0 = 2)$.

I.e.,

$$h(r) = 2^{\delta} \omega_{n-2} \int_0^R \cos ru \, du \int_u^R (\cosh \rho - \cosh u)^{\delta - 1} \sinh \rho \, d\rho,$$

or

$$h(r) = \delta^{-1}2^{\delta}\omega_{n-2}\int_0^R (\cosh R - \cosh u)^{\delta} \cos ru \, du. \tag{2}$$

Note that $h(r)$ is positive and decreasing along the segment from δi to 0, so that the values of $h(r)$ along this segment dominate $h(0)$, which is given by

$$h(0) = \delta^{-1}2^{\delta}\omega_{n-2}\int_0^R (\cosh R - \cosh u)^{\delta} \, du$$

$$= \delta^{-1}2^{\delta}\omega_{n-2}R\cosh^{\delta} R\int_0^1 \left(1 - \frac{\cosh Ru}{\cosh R}\right)^{\delta} du.$$

Now A, the volume of the ball of radius R, is given by

$$\omega_{n-1}\int_0^R \sinh^{n-1} u \, du,$$

which is asymptotic to

$$\frac{\omega_{n-1}}{(n-1)2^{n-1}} e^{(n-1)R}$$

for large R. On the other hand, it follows easily from our last expression for $h(0)$, that $h(0)$ is positive for $R > 0$, and that $|h(0)|^2$ is asymptotic to

$$\delta^{-2}\omega_{n-2}^2 R^2 e^{(n-1)R}$$

for large R. It follows that $A^{-1}|h(0)|^2 \geq c_1(n, R)R^2$, where $c_1(n, R)$ is positive and asymptotic to

$$c_2(n) = \frac{2^{n+1}\omega_{n-2}^2}{(n-1)\omega_{n-1}}$$

for large R.

This has an interesting consequence, since it follows from Theorem 1 that

$$1 > \frac{A}{V} + \frac{1}{AV}\sum{}'' |h(r_k)|^2, \tag{3}$$

where the sum is over the small eigenvalues. On the other hand, we have seen that for such an eigenvalue, $|h(r_k)|^2 > |h(0)|^2$, so if we denote by $N(M)$ the number of small eigenvalues for M, we conclude that

$$1 > \frac{A}{V} + c_1(n, R)\frac{N(M)R^2}{V},$$

which implies the following theorem, which is of interest for large R:

THEOREM 2.

$$N(M) < \alpha(n, R)\frac{V-A}{R^2},$$

where $\alpha(n, R)$ *is positive and asymptotic to* $1/c_2(n)$ *for large* R.

We conclude with another application of Theorem 1. Recall that a_0, the zeroth Fourier coefficient of χ, is equal to $A/\sqrt{V}$, and that $\varphi_0(y) \equiv 1/\sqrt{V}$. Since $a_0 = h(r_0)\varphi_0(y)$, it follows immediately that $h(r_0) = h(\delta i) = A$. Thus, if λ_k is close to 0, or equivalently, if r_k is close to δi, it will be the case that $h(r_k) \sim A$. In more detail, suppose $\epsilon \in (0, 1)$, and that $r_k = \delta' i$, where $|\delta - \delta'| < \epsilon/R$.

By (2),

$$A - h(\delta' i) = \delta^{-1}2^\delta\omega_{n-2}\int_0^R (\cosh R - \cosh u)^\delta(\cosh \delta u - \cosh \delta' u)\,du,$$

and by the mean value theorem this last expression is equal to

$$\delta^{-1}2^\delta\omega_{n-2}\int_0^R (\cosh R - \cosh u)^\delta(\sinh w(u))(\delta - \delta')u\,du,$$

where $w(u)$ is between $\delta' u$ and δu.

This is dominated by

$$\epsilon\delta^{-1}2^\delta\omega_{n-2}\int_0^R (\cosh R - \cosh u)^\delta(\cosh \delta u)\,du = \epsilon A.$$

I.e.,

$$|A - h(\delta' i)| < \epsilon A,$$

or

$$\left|1 - \frac{h(\delta' i)}{A}\right| < \epsilon,$$

from which we easily deduce that

$$\frac{|h(\delta' i)|^2}{A} > A(1-\epsilon)^2.$$

Suppose now that M has $s_\epsilon(M)$ very small eigenvalues in the above sense, i.e., eigenvalues for which $|r_k - \delta i| < \epsilon/R$.

By Theorem 1,

$$1 > \frac{A}{V} + \frac{1}{AV}\sum\nolimits''' |h(r_k)|^2,$$

where the sum is taken over the $s_\epsilon(M)$ very small eigenvalues of M.

We conclude from this that

$$1 + (1-\epsilon)^2 s_\epsilon(M) < \frac{V}{A},$$

which implies the following theorem which is of interest for large R:

THEOREM 3.

$$\frac{1}{(1-\epsilon)^2} + s_\epsilon(M) < \frac{1}{(1-\epsilon)^2}\frac{V}{A}.$$

COROLLARY.

$$1 + s_\epsilon(M) < \frac{1}{(1-\epsilon)^2}\frac{V}{A}.$$

The corollary has an interesting consequence in two dimensions. Suppose M is of genus g, and $s_\epsilon(M) = 2g - 3$. (We remark that such examples can be produced, and that for a given genus, if ϵ is small enough this value of $s_\epsilon(M)$ is maximal, since it is known [2, 3, 6] that there exists $\epsilon(g) > 0$ such that for $\epsilon < \epsilon(g)$, $s_\epsilon(m) \leq 2g - 3$. Additionally, for a given genus, $s_\epsilon(M) = 2g - 3$ for ϵ sufficiently small implies that there are no other eigenvalues λ_k in $(0, \delta^2)$ [5].)

By the corollary, bearing in mind that $V = 4\pi(g-1)$,

$$2g - 2 < \frac{1}{(1-\epsilon)^2} \frac{4\pi(g-1)}{A},$$

or

$$A < \frac{2\pi}{(1-\epsilon)^2}.$$

Since $A = 2\pi(\cosh R - 1)$, we conclude that if $2g-3$ very small eigenvalues are present, the injectivity radius R of M must be less than a quantity which for small ϵ is near $\cosh^{-1} 2 \approx 1.317$. Note that for fixed ϵ, this estimate on the injectivity radius is uniform in the genus. In general, in view of the corollary, in any dimension an inequality of the form $s_\epsilon(M) \geq cV$ imposes a computable upper bound on R. Similarly, in any dimension an inequality of the form $V/A \leq c$ imposes a computable upper bound on $s_\epsilon(M)$. Finally, since it is known that for fixed genus in two dimensions, λ_{2g-3} cannot tend to zero unless R tends to zero [2, 3, 6], results like the last one derive interest for small ϵ from the fact that they are uniform in the genus.

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