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Smooth algebras and vanishing of Hochschild homology

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For a given homomorphism of commutative noetherian rings $A \rightarrow B$, we consider the $B \otimes_A B$ -module structure in B induced by the canonical surjective homomorphism $B \otimes_A B \rightarrow B, b \otimes b' \mapsto bb'$. We denote by $fd_{B \otimes_A B}(B)$ the flat dimension of this module.

Let K be a field; then a noetherian K -algebra C is called geometrically regular if for any finite field extension $L | K$, the ring $C \otimes_K L$ is regular. A homomorphism $A \rightarrow B$ of noetherian rings is called regular if it is flat and its fibers are geometrically regular.

In this paper we prove the following result

THEOREM 1. *Let A be a noetherian ring and let B be a flat and noetherian A -algebra. If $fd_{B \otimes_A B}(B) < \infty$, then the homomorphism $A \rightarrow B$ is regular.*

This result has been obtained in [8] with two additional hypotheses: B has characteristic zero and $B \otimes_A B$ is a noetherian ring. Moreover, in [8] has been proved that if $A \rightarrow B$ is regular and $B \otimes_A B$ noetherian, then $fd_{B \otimes_A B}(B) < \infty$. Nevertheless, it is not true for an arbitrary regular homomorphism. Indeed, for a regular homomorphism $A \rightarrow B$, we have isomorphisms [2, Proposition C]

$$\bigwedge^n \Omega_{B|A} \xrightarrow{\sim} \text{Tor}_n^{B \otimes_A B}(B, B),$$

where $\Omega_{B|A}$ is the Kahler differentials module of B over A .

The proof of the version of Theorem 1 given in [8] uses the Hodge decomposition of Hochschild cohomology in characteristic zero, as well as a strong result of L. Avramov [3], which characterizes the local complete intersection rings by the finiteness of flat dimension of its cotangent complex. Here we give a more elementary proof based on the properties of Hochschild homology exposed in [5]. We also use a result of J.-L. Brylinski [4] concerning to the localization of this homology.

Observe that the condition $fd_{B \otimes_A B}(B) < \infty$ is equivalent to the existence of an integer n such that the Hochschild homology

$$H_n(B, M) = \text{Tor}_n^{B \otimes_A B}(B, M) = 0$$

for all $B \otimes_A B$ -modules M .

The results we will be using are the following

R.1. Let K be a commutative ring and let A, L be commutative K -algebras such that A is K -flat. Then for any $(L \otimes_K A) \otimes_L (L \otimes_K A)$ -module M there are isomorphisms $H_n(A, M) \simeq H_n(L \otimes_K A, M)$ (on the left hand side we have homology of A over K , and on the right one, homology of $L \otimes_K A$ over L).

R.2. Let K be a field and let A be a commutative K -algebra. Then for any A -modules M and N we have isomorphisms $H_n(A, M \otimes_K N) \simeq \text{Tor}_n^A(M, N)$.

R.3. Let $A \rightarrow B$ be a flat homomorphism of commutative rings. Let q be a prime ideal of B and let p be its contraction in A . Then for any $B \otimes_A B$ -module M we have isomorphisms

$$\text{Tor}_n^{B \otimes_A B}(B, M)_q \simeq \text{Tor}_n^{B_q \otimes_{A_p} B_q}(B_q, M_q).$$

R.1 and R.2 are in [5]. R.3 is in [4].

Proof of Theorem 1. Using R.3 it is sufficient to show that if $A \rightarrow B$ is a flat homomorphism of noetherian local rings, K the residue field of A , and $fd_{B \otimes_A B}(B) < \infty$, then $B \otimes_A K$ is a geometrically regular K -algebra. Let $C = B \otimes_A K$. It follows from R.1 that $fd_{C \otimes_K C}(C) < \infty$. Since C is noetherian, R.2 implies that the global dimension of C is finite. Then C is a regular local ring (by the classical result of J. P. Serre [10]). Moreover, a new application of R.1 and R.2, shows that C remains regular after a finite extension of K . Therefore C is geometrically regular over K , and Theorem 1 is proved.

As a particular case of Theorem 1 we obtain that a field extension $L | K$ is separable if $fd_{L \otimes_K L}(L) < \infty$. It is a generalization of Theorem 9 in [8].

COROLLARY 2. *Let A be a noetherian ring and let B be an A -algebra of finite type. The following conditions are equivalent*

- (i) B is a smooth A -algebra
- (ii) B is a flat A -module and $fd_{B \otimes_A B}(B) < \infty$.

Proof. Let us recall the characterizations of regularity and smoothness using the André-Quillen 1-dimensional (co) homology functors $H(A, B, -)$ (see [1]). This

functors can be described as follows. Let $B \simeq R/J$ be a presentation of B where R is a polynomial A -algebra, and consider the canonical homomorphism $J/J^2 \rightarrow \Omega_{R|A} \otimes_R B$. Then, for a B -module M we have

$$H_1(A, B, M) \simeq \text{Ker} (J/J^2 \otimes_B M \rightarrow \Omega_{R|A} \otimes_R M)$$

$$H^1(A, B, M) \simeq \text{Coker} (\text{Hom}_B (\Omega_{R|A} \otimes_R B, M) \rightarrow \text{Hom}_B (J/J^2, M)).$$

The above mentioned characterizations are the following:

- (1) $A \rightarrow B$ is regular if and only if $H_1(A, B, -) = 0$, i.e., $H_1(A, B, B) = 0$ and $\Omega_{B|A}$ is a flat B -module;
- (2) B is a smooth A -algebra if and only if $H^1(A, B, -) = 0$, i.e., $H_1(A, B, B) = 0$ and $\Omega_{B|A}$ is a projective B -module. Since B is noetherian and $\Omega_{B|A}$ is a finitely generated B -module, these two conditions on $\Omega_{B|A}$ are equivalent. Then the result is a consequence of [8] and Theorem 1.

REMARK 3. The proof of $H_1(A, B, -) = 0$ for a regular homomorphism is easy if B is an A -algebra of finite type. In the general case, the proof is difficult since it involves the homology functors of higher dimensions (see [1], pp. 330–331).

The homomorphism of rings $B \otimes_A B \rightarrow B$ enable us to consider a B -module M as a $B \otimes_A B$ -module. If $B \otimes_A B$ is a noetherian ring, then the vanishing of $H_n(B, M)$ for an integer n and all B -modules M , is sufficient for $fd_{B \otimes_A B}(B) < \infty$. It is an easy consequence of the following well known result: if R is a local noetherian ring with residue field K , and I is an ideal of R , then $fd_R(R/I) < \infty$ if and only if there exists an integer n such that $\text{Tor}_n^R(R/I, K) = 0$. On the other hand, the calculation of Hochschild homology of complete intersections in [11] shows that there exist locally complete intersection algebras A , of finite type over a field of characteristic zero such that $H_n(A, A) \neq 0$ for infinitely many n . We consider this as an evidence for the following conjecture:

CONJECTURE. *Let K be a field of characteristic zero and let A be a K -algebra of finite type. If $H_n(A, A) = 0$ for n sufficiently large, then A is a smooth K -algebra.*

A conjecture of J. Herzog implies that such an algebra must be at least a locally complete intersection. In fact, since K has characteristic zero, $H_n(K, A, A)$ is a summand of $H_{n+1}(A, A)$ (see [6], where $H_n(K, A, A)$ is called $H_{1,n}(A, A)$). Then $H_n(K, A, A) = 0$ for n sufficiently large. Let R be a polynomial K -algebra of finite type such that there exists a surjective K -homomorphism $R \rightarrow A$. Since the André–Quillen homology of R vanishes in dimensions > 0 , the Jacobi–Zriski sequence

associated to $K \rightarrow R \rightarrow A$ yields $H_n(R, A, A) = 0$ for n sufficiently large. In this situation, Herzog conjectures [7] (p. 62) that A must be a locally complete intersection.

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