**Zeitschrift:** Commentarii Mathematici Helvetici

**Herausgeber:** Schweizerische Mathematische Gesellschaft

**Band:** 65 (1990)

**Artikel:** Witt group of hyperelliptic curves.

**Autor:** Parimala, R. / Sujatha, R.

**DOI:** https://doi.org/10.5169/seals-49743

Comment. Math. Helvetici 65 (1990) 559–580 0010-2571/90/040559-22$1.50 + 0.20/0


# Witt group of hyperelliptic curves

R. PARIMALA and R. SUJATHA

## Introduction

Let $k$ be a perfect field of characteristic $\neq 2$. Let $X$ be a smooth projective curve over $k$. Let $W(k(X), \Omega_{k(X)})$ denote the Witt group of the function field of $X$ with values in the module of differentials $\Omega_{k(X)}$ of $k(X)$. A residue homomorphism

$$\partial : W(k(X), \Omega_{k(X)}) \rightarrow \bigoplus_{x \in X} W(k(x))$$

was defined in [7], $k(x)$ denoting the residue field at points $x \in X$ and a residue theorem was proved; namely the composite

$$W(k(X), \Omega_{k(X)}) \xrightarrow{\partial} \bigoplus_{x \in X} W(k(x)) \xrightarrow{\text{trace}} W(k)$$

is zero. Thus image $\partial$ is contained in the subgroup $(\oplus_{x \in X} W(k(x)))^0$ consisting of tuples $(\mu_x)$ with $\Sigma_{x \in X}$ trace $\mu_x = 0$. The kernel and cokernel of $\partial$ are well understood if $X = \mathbf{P}^1$ [13] or if $X$ is an anisotropic conic over $k$ [14]. To have an intrinsic description of these groups for curves of higher genus is an interesting question posed by Milnor in [13].

In this paper, we study this problem for smooth hyperelliptic curves $X$ with a rational point of ramification over $\mathbf{P}^1$. Let $\pi : X \rightarrow \mathbf{P}^1$ be a covering defined over $\mathbf{A}^1$ by the equation $y^2 = f(T)$. We exhibit an exact sequence (§3)

$$0 \rightarrow \mathring{W}(X) \rightarrow W(k(X)) \xrightarrow{\partial^0} \left(\bigoplus_{x \in X} W(k(x))\right)^0 \rightarrow \frac{W(k[T]_f)}{\langle 1, -f \rangle W(k)} \rightarrow W(X) \rightarrow 0.$$

where $\partial^0$ is simply the residue map $\partial$ through an identification of $W(k(X), \Omega_{k(X)})$ with $W(k(X))$ for a suitable choice of a differential as basis for $\Omega_{k(X)}$. We derive, as a corollary, that if all the ramification points of $\pi$ are $k$-rational, $\mathring{W}(X)$ is generated by one-dimensional forms. This exact sequence may be viewed in two ways: Firstly as characterising coker $\partial^0$ as a subgroup of $\oplus_{x \in S} W(k(x))$, $S$ denoting the set of ramification points of $\pi$ and secondly, as giving the defining relations for expressing $W(X)$ as a quotient of $\oplus_{x \in S} W(k(x))$.

Using the exact sequence above, we give a more precise description of coker $\partial^0$. It contains a subgroup $V_r$ which is a quotient of $\oplus_{x \in S} W(k(x))$, which we call the *ramified part* of coker $\partial^0$. Under the rationality assumption that ${}_4\mathrm{Pic}\, X = {}_4\mathrm{Pic}\, X_{\bar{k}}$, $\bar{k}$ denoting the algebraic closure of $k$, the group $V_r$ is zero. We call $V_{nr} = \operatorname{coker} \partial^0 / V_r$, the *unramified part* of coker $\partial^0$. This group is 2-torsion (§5). It can be computed in terms of certain cohomology groups if ${}_4\mathrm{Pic}\, X = {}_4\mathrm{Pic}\, X_{\bar{k}}$, and supposing further that the curve $Y = X$ or $\mathbf{P}^1$ has the following property: 'Graded Witt group of $Y$ is isomorphic to the cohomology ring'; Curves over local and global fields have this property [16]. We in fact show that under these assumptions on $X$, coker $\partial^0$ is isomorphic to $(\mathrm{Pic}\, X'/2) \oplus NH^3(X')$, where $X' = X \backslash S$, $S$ denoting the set of ramification points of $\pi$ and $NH^n(X')$ denotes the kernel of the map $H^n_{et}(X', \mu_2) \to H^n_{et}(k(X'), \mu_2)$. For a smooth projective hyperelliptic curve over a local field with good reduction, if ${}_4\mathrm{Pic}\, X = {}_4\mathrm{Pic}\, X_{\bar{k}}$, coker $\partial$ is isomorphic to $W(k) \oplus (\mathbf{Z}/2)^{4g}$ where $g$ is the genus of the curve (Theorem 7.1). Further, $W(X)$ is also isomorphic to the group $(\mathbf{Z}/2)^{4g} \oplus W(k)$! (Theorem 7.6).

The computations yield, as a by-product, that for any smooth projective curve $X$ over a local field with good reduction, if ${}_4\mathrm{Pic}\, X = {}_4\mathrm{Pic}\, X_{\bar{k}}$, the classical invariants determine the class of a quadratic space in $W(X)$.

We record here that J. E. Shick [19] has some independent computations of coker $\partial$ for power series fields over $\mathbb{R}$ and of $\mathbb{C}$.

We thank D. S. Nagaraj for carefully going through the manuscript.

## 1. Kernel of the residue homomorphism

Let $k$ be a perfect field of characteristic $\neq 2$. Let $X$ be a smooth projective curve defined over $k$. For a line bundle $\mathscr{L}$ on $X$, let $W(X, \mathscr{L})$ denote the Witt group of quadratic spaces on $X$ with values in $\mathscr{L}$ [9]. Let $W(X) = W(X, \mathcal{O}_X)$.

LEMMA 1.1. *The group $W(X, \mathscr{L})$ depends upto isomorphism, only on the class of $\mathscr{L}$ in* $\mathrm{Pic}\, X/2$. *In particular,* $W(X, \mathscr{L}^2) \xrightarrow{\sim} W(X)$.

*Proof.* Let $\mathscr{M} \in \mathrm{Pic}\, X$ and $(\mathscr{E}, q)$ be a quadratic space with values in $\mathscr{L} \otimes \mathscr{M}^2$, i.e., $q : \mathscr{E} \to \mathscr{E}^* \otimes \mathscr{L} \otimes \mathscr{M}^2$, where for any bundle $\mathscr{F}$, $\mathscr{F}^*$ denotes the dual of $\mathscr{F}$, is an isomorphism such that $q^t \otimes 1_{\mathscr{L} \otimes \mathscr{M}^2} = q$. The assignment

$$(\mathscr{E}, q) \to (\mathscr{E} \otimes \mathscr{M}^*, q \otimes 1_{\mathscr{M}^*})$$

defines an isomorphism

$$W(X, \mathscr{L} \otimes \mathscr{M}^2) \xrightarrow{\sim} W(X, \mathscr{L}).$$

Let $\Omega_X$ denote the sheaf of differentials on $X$. Let

$$\partial : W(k(X), \Omega_{k(X)}) \to \bigoplus_{x \in X} W(k(x))$$

be the residue homomorphism defined in [7], $k(x)$ denoting the residue field at the closed point $x$ of $X$. (Throughout, the notation $x \in X$ stands for the set of all closed points $x$ in $X$).

LEMMA 1.2. *The kernel of the residue map*

$$\partial : W(k(X), \Omega_{k(X)}) \to \bigoplus_{x \in X} W(k(x))$$

is $W(X, \Omega_X)$.

*Proof.* Let $q$ be a quadratic space over $k(X)$ with values in $\Omega_{k(X)}$, whose class belongs to $\ker \partial$. Let $x$ be a closed point of $X$ and $\pi_x$ a local parameter at $x$. Identifying $W(k(X))$ with $W(k(X), \Omega_{k(X)})$ through $d\pi_x$, the residue map $\partial_x$: $W(k(X)) \to W(k(x))$ is simply the second residue homomorphism with respect to $\pi_x$. Thus $q$ which maps to zero under $\partial_x$ (cf. [17], p. 207) is isometric to $q_x \otimes_{\mathcal{O}_{X,x}} k(X)$ for some $q_x \in W(\mathcal{O}_{X,x})$. The spaces $q_x \cdot d\pi_x$ over $\mathcal{O}_{X,x}$ with values in $\Omega_{X,x}$ become isometric to $q$ over $k(X)$. They patch up to yield a quadratic space $q_X$ over $X$ with values in $\Omega_X$ in view of the following

LEMMA 1.3. *Let $\mathcal{L}$ be a line bundle on $X$, $q$ a quadratic space over $k(X)$ with values in $\mathcal{L}_{k(X)}$. Suppose, for every $x \in X$, there exists a quadratic space $q_x$ over $\mathcal{O}_{X,x}$ with value in $\mathcal{L} \otimes \mathcal{O}_{X,x}$ such that $q_x \otimes k(X) \xrightarrow{\sim} q$. Then there exists a quadratic space $q_X$ over $X$ with values in $\mathcal{L}$ such that $q_X \otimes k(X) \xrightarrow{\sim} q$.*

*Proof.* The proof of ([6], Corollary 2.7) in the case $\mathcal{L} = \mathcal{O}_X$ goes through verbatim for any line bundle $\mathcal{L}$.

REMARK. If $X = \mathbf{P}^1$, $\ker \partial \xrightarrow{\sim} W(X) \xrightarrow{\sim} W(k)$. ([13], Proposition 5.3). If $X$ is an anisotropic conic, $\ker \partial = W(X, \Omega_X)$ is computed as $\mathcal{B}_{\mathcal{C}}$ in ([14], Theorem 6.2).

PROPOSITION 1.4. *Let $X$ be a smooth hyperelliptic curve with a rational point of ramification over* $\mathbf{P}^1$. *Then* $\ker \partial \xrightarrow{\sim} W(X)$.

*Proof.* By (1.1) and (1.2), it suffices to show that $\Omega_X$ is the square of a line bundle on $X$. Let $\pi : X \to \mathbf{P}^1$ be a covering, defined over $\mathbf{A}^1$ by the equation $y^2 = f(T)$, degree $f = 2g + 1$, $g$ being the genus of $X$. The divisor of the differential $dT/y$ is $(2g - 2)P_\infty$, $P_\infty$ being the point of $X$ lying over $\infty$ in $\mathbf{P}^1$. Let $\mathcal{L}$ be the line bundle corresponding to the divisor $(g - 1)P_\infty$. Then $\Omega_X \xrightarrow{\sim} \mathcal{L}^2$.

REMARK. As observed by M. Rost, one could define more generally, a residue map

$$\partial_{\mathscr{L}} : W(k(X), \mathscr{L}_{k(X)}) \to \bigoplus_{x \in X} (W(k(x)), (\mathscr{L} \otimes \Omega_X)(x)),$$

where $(\mathscr{L} \otimes \Omega_X)(x)$ denotes the fibre of the line bundle $(\mathscr{L} \otimes \Omega_X)$ at $x$. If $X = \mathbf{P}^1$, and $\mathscr{L} = \mathcal{O}_X$, $\partial_{\mathcal{O}_X} = \partial$ is the residue homomorphism discussed above, since $\Omega_X$ is a square. If $\mathscr{L} = \mathcal{O}_X(1)$, $\partial_{\mathcal{O}_X(1)}$ is an isomorphism. In the case of an anisotropic conic, $X$ we have $\ker \partial_{\mathcal{O}_X} \xrightarrow{\sim} W(X) \xrightarrow{\sim} W(k)/\langle 1, -a, -b, ab \rangle W(k)$, (cf. [1]), where $X$ is defined by the equation $aX^2 + bY^2 - Z^2 = 0$. One can identify coker $\partial$ with a subgroup of the Witt group of the residue field at the ramified point of the covering $X \to \mathbf{P}^1$.

## 2. Some auxiliary results on trace, transfer and residue homomorphisms

Let $\pi : X \to \mathbf{P}^1$ be a double covering, defined over $\mathbf{A}^1$ by the equation $y^2 = f(T)$, degree $f = 2g + 1$, $g$ being the genus of $X$. We identify $W(k(X))$ and $W(k(T))$ with $W(k(X), \Omega_{k(X)})$ and $W(k(T), \Omega_{k(T)})$ through the basis $dT/2y$ and $dT$ respectively. For $y \in \mathbf{A}^1$, if $p \in k[T]$ is the monic irreducible polynomial which gives a parameter at $y$, the composite map

$$W(k(T)) \xrightarrow{dT} W(k(T), \Omega_{k(T)}) \xrightarrow{\partial_y} W(k(y))$$

is the second residue homomorphism with respect to the parameter $pp'$, $p'$ denoting the derivative of $p$ with respect to $T$. Similarly, one can verify that if $x \in X$ lies over $y \in \mathbf{P}^1$ corresponding to $p(T)$, and $x$ unramified over $y$, on choosing $p(T)$ again as the parameter at $y$, the composite

$$W(k(X)) \xrightarrow{dT/2y} W(k(X), \Omega_{k(X)}) \xrightarrow{\partial_x} W(k(x))$$

is the second residue homomorphism with respect to the parameter $2pp'y$. We again denote by $\partial$ this residue map.

For any finite separable extension $L/K$, let $tr : W(L) \to W(K)$ be the map induced by the linear map trace : $L \to K$ and $i : W(K) \to W(L)$ the map induced by the inclusion of $K$ in $L$. Let $s : W(k(X)) \to W(k(T))$ be the transfer homomorphism induced by the linear map $s : k(X) \to k(T)$ defined by $s(1) = 0$, $s(y) = 1$ where $\{1, y\}$ is a basis for $k(X)$ over $k(T)$ ([17], p. 47).

LEMMA 2.1. *The diagram*

$$\begin{array}{ccc} W(k(X)) & \xrightarrow{(\partial_x)} & \bigoplus_{x/y} W(k(x)) \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle tr} \\ W(k(T)) & \xrightarrow{\partial_y} & W(k(y)) \end{array}$$

*is commutative.*

*Proof.* Since the diagram

$$\begin{array}{ccc} W(\Omega_{k(X)}) & \xrightarrow{\partial} & \bigoplus_{x/y} W(k(x)) \\ {\scriptstyle tr}\downarrow & & \downarrow{\scriptstyle tr} \\ W(\Omega_{k(T)}) & \xrightarrow{\partial} & W(k(y)) \end{array}$$

is commutative ([7], §1), it suffices to show that the diagram

$$\begin{array}{ccc} W(k(X)) & \xrightarrow{dT/2y} & W(k(X), \Omega_{k(X)}) \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle tr} \\ W(k(T)) & \xrightarrow{dT} & W(k(T), \Omega_{k(T)}) \end{array}$$

is commutative. It is enough to check that

$$tr(\langle h_0 + h_1 y\rangle \cdot dT/2y) = s(\langle h_0 + h_1 y\rangle) \cdot dT,$$

for $h_0, h_1 \in k(T)$. We have,

$$\begin{aligned} tr(\langle h_0 + h_1 y\rangle\, dT/2y) &= \langle tr((h_0 + h_1 y)/2y)\rangle \cdot dT \\ &= \begin{pmatrix} h_1 & h_0 \\ h_0 & h_1 f \end{pmatrix} \cdot dT \\ &= s(\langle h_0 + h_1 y\rangle) \cdot dT. \end{aligned}$$

LEMMA 2.2. *The diagram*

$$\begin{array}{ccc} W(k(T)) & \xrightarrow{\partial_y} & W(k(y)) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i'} \\ W(k(X)) & \xrightarrow{\partial_x} & \bigoplus_{x/y} W(k(x)) \end{array}$$

*is commutative if y is an unramified point for $\pi$ and $i'$ is the composite*

$$W(k(y)) \xrightarrow{i} W(k(x)) \xrightarrow{\overline{2y}^{-1}} W(k(x)).$$

*If x is a ramified point for $\pi$, $\partial_x \circ i$ is zero.*

*Proof.* Let $\langle h \rangle \in W(k(T))$ and $x \in X$ such that $x$ is an unramified point for $\pi$ with $\pi(x) = y$. Let $p \in k[T]$ be the monic polynomial corresponding to $y$. Suppose $v_y(h) = 0$. Then $\partial_y(\langle h \rangle) = 0$ and $\partial_x \circ i(\langle h \rangle) = \partial_x(\langle h \rangle) = 0$, since $v_x(h) = v_y(h) = 0$. Suppose $h = up$ with $v_y(u) = 0$. Since $\partial_x$ is the second residue map with respect to the parameter $2pp'y$ and $\partial_y$ the second residue map with respect to $pp'$, we have

$$i' \circ \partial_y(\langle up \rangle) = i' \circ \partial_y \langle (u/p') \cdot pp' \rangle = i'\overline{\langle u/p' \rangle} = \overline{\langle u/p'2y \rangle}$$

and

$$\partial_x \circ i(\langle h \rangle) = \partial_x(\langle (u/2p'y) \cdot 2pp'y \rangle) = \overline{\langle u/2p'y \rangle}$$

Suppose $x \in X$ is a ramified point, lying over $y \in \mathbf{P}^1$. For $h \in k(T)$, $v_x(ih) \equiv 0 \bmod 2$, since $x$ has ramification index 2, and we have $\partial_x \circ i(\langle h \rangle) = 0$.

LEMMA 2.3. *Let $x/y$ be an unramified point for $\pi$. Then the diagram*

$$\begin{array}{ccc} W(k(T)) & \xrightarrow{\partial_y} & W(k(y)) \\ {\scriptstyle \langle 1, -f \rangle} \downarrow & & \downarrow {\scriptstyle \langle 1, -\bar{f} \rangle} \\ W(k(T)) & \xrightarrow{\partial_y} & W(k(y)) \end{array}$$

*is commutative.*

*Proof.* Clear.

We repeatedly use the following lemma which is a consequence of the Lam's exact triangle ([17], Chapter 2, 5.10).

LEMMA 2.4. *The following triangles are exact.*

$$\begin{array}{ccc} W(k(T)) & \xrightarrow{i} & W(k(X)) \\ {\scriptstyle\langle 1,-f\rangle}\nwarrow & & \swarrow{\scriptstyle s} \\ & W(k(T)) & \end{array}$$

$$\begin{array}{ccc} W(k(y)) & \xrightarrow{i'} & W(k(x)) \\ {\scriptstyle\langle 1,-f\rangle}\nwarrow & & \swarrow{\scriptstyle tr} \\ & W(k(y)) & \end{array}$$

where, in the second triangle, $y$ is unramified for $\pi$ and $\pi(x) = y$; if $y$ splits in $X$, we mean by $W(k(x))$, the direct sum $W(k(x_1)) \oplus W(k(x_2))$ with $\pi(x_i) = y$.

## 3. An exact sequence

Let $\pi : X \to \mathbf{P}^1$ be a hyperelliptic curve defined over $\mathbf{A}^1$ by the equation $y^2 = f(T)$, degree $f = 2g + 1$, $g$ being the genus of $X$. Let

$$\partial^0 : W(k(X)) \to (\bigoplus_{x \in X} W(k(x)))^0$$

be the residue homomorphism as defined in §2, identifying $W(k(X))$ with $W(k(X), \Omega_{k(X)})$ through the basis $dT/2y$, $(\oplus_{x\in X} W(k(x)))^0$ denoting the kernel of the trace map $\oplus_{x \in X} W(k(x)) \xrightarrow{tr} W(k)$. We fix the following notation: $S$ = set of ramification points for $\pi$, $X' = X \setminus S$, $Y = \mathbf{P}^1$, $Y' = Y \setminus \pi(S)$. We have the following commutative diagram with exact rows and columns, in view of (2.1), (2.3) and (2.4) and ([13], Theorem 5.3).

$$\begin{array}{ccccccccc} & & & & W(k(X)) & \xrightarrow{\partial} & \bigoplus_{x\in X} W(k(x)) & & & & \\ & & & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle tr} & & & & \\ 0 & \longrightarrow & W(k) & \longrightarrow & W(k(T)) & \xrightarrow{\partial} & \bigoplus_{y\in Y'} W(k(y)) \;\oplus \bigoplus_{y \in \pi(S)} W(k(y)) & \xrightarrow{\text{trace}} & W(k) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle\langle 1,-f\rangle} & & \downarrow{\scriptstyle\langle 1,-f\rangle} & & & & \\ 0 & \longrightarrow & W(Y') & \longrightarrow & W(k(T)) & \xrightarrow{\partial} & \bigoplus_{y\in Y'} W(k(y)). & & & & \end{array}$$

We define a homomorphism $\alpha : (\oplus_{x\in X} W(k(x)))^0 \to W(A)/\langle 1, -f\rangle \cdot W(k)$, where $A = k[T]_f$ as follows. Let $\theta \in (\oplus_{x\in X} W(k(x)))^0$. Then there exists $q \in W(kT))$ with

$\partial(q) = tr\,\theta$. We have for $y \in Y'$

$$\begin{aligned}\partial_y(\langle 1, -f\rangle \cdot q) &= \langle 1, -\bar{f}\rangle \partial_y(q)\\ &= \langle 1, -\bar{f}\rangle tr(\theta_x)\\ &= 0.\end{aligned}$$

Hence $\langle 1, -f\rangle q \in W(Y') = W(A)$. Let $\alpha(\theta)$ denote its class in $W(A)/\langle 1, -f\rangle \cdot W(k)$. If $q_1, q_2$ and two lifts of $tr\theta$ in $W(k(T))$, $q_1 - q_2 \in W(k)$ and $\langle 1, -f\rangle q_1$ and $\langle 1, -f\rangle q_2$ define the same class in $W(A)/\langle 1, -f\rangle W(k)$. Thus $\alpha$ is well-defined.

LEMMA 3.1. $\ker \alpha = \partial^0(W(k(X)))$.

*Proof.* Since $\partial \circ s = tr \circ \partial$ and $\langle 1, -f\rangle \circ s = 0$, we have

$$\partial(W(k(X))) \subset \ker \alpha.$$

Let $\theta \in \oplus_{x \in X} W(k(x))^0$ with $\alpha(\theta) = 0$. Let $q_1 \in W(k(T))$ be such that $\partial(q_1) = tr\theta$. Then $\langle 1, -f\rangle q_1 \in \langle 1, -f\rangle W(k)$. Replacing $q_1$ by $q_1 - q_0$ for a suitable $q_0 \in W(k)$, we assume that $\langle 1, -f\rangle q_1 = 0$. Thus, by (2.4), there exists $q_2 \in W(k(X))$ such that $s(q_2) = q_1$. We have $tr(\theta - \partial q_2) = tr\theta - \partial s q_2 = tr\theta - \partial q_1 = 0$. The fact that $\theta - \partial q_2 \in \partial W(k(X))$ follows from the following

LEMMA 3.2. *Let $\theta \in (\oplus_{x \in X} W(k(x)))^0$ with $tr\theta = 0$ in $(\oplus_{y \in Y} W(k(y)))^0$. Then $\theta \in \partial \circ i(W(k(T)))$.*

SUBLEMMA 3.3. *Let $(\mu_x) \in \oplus_{x \in X'} W(k(x))$ be such that $tr(\mu_x) = 0$ in $\oplus_{y \in Y'} W(k(y))$. Then there exists $q \in W(k(T))$ such that $\partial_x(i(q)) = \mu_x$, for $x \in X'$.*

*Proof.* By (2.4), there exists $(\nu_y) \in \oplus_{y \in Y'} W(k(y))$ such that $i'(\nu_y) = \mu_x$. Since $Y' \subset \mathbf{A}^1$, the residue map $\partial : W(k(T)) \to \oplus_{y \in Y'} W(k(y))$ is surjective. Let $q \in W(k(T))$ be such that $\partial_y(q) = \nu_y$ for $y \in Y'$. Then, by (2.2), $\partial \circ i(q) = i' \circ \partial_y(q) = \mu_x$ for $x \in X'$.

*Proof of* 3.2. By (3.3), there exists $q \in W(k(T))$ such that $\partial_x \circ i(q) = \theta_x$ for $x \in X'$. Further, by (2.2), $\partial_x \circ i(q) = 0$ for $x \in S = X \backslash X'$. Since for $x \in S$, $\theta_x = tr\,\theta_x = 0$, we have, $\partial(i(q)) = \theta$.

Let $A = k[T]_f$, $B = (k[T, y]/(y^2 - f))_f$ be the co-ordinate rings of $Y'$ and $X'$ respectively. Since for $x \in S$, $q \in W(A)$, $\partial_x \circ i(q) = 0$ by (2.2), the natural map $W(A) \xrightarrow{i} W(B)$ has its image contained in $W(X)$. This map vanishes on $\langle 1, -f\rangle \cdot W(k)$ and induces a map $\beta : W(A)/\langle 1, -f\rangle W(k) \to W(X)$.

THEOREM 3.4. *The sequence*

$$0 \longrightarrow W(X) \xrightarrow{i} W(k(X)) \xrightarrow{\partial^0} \Big(\bigoplus_{x\in X} W(k(x))\Big)^0 \xrightarrow{\alpha} W(A)/\langle 1, -f\rangle W(k) \xrightarrow{\beta} W(X) \longrightarrow 0$$

*is exact.*

*Proof.* Exactness at $W(X)$ (left) and $W(k(X))$ are proved in ([10], p. 277) noting that $\partial^0$ is the second residue homomorphism at all points $x \in X$. The exactness at $(\oplus_{x\in X} W(k(x)))^0$ is proved in (3.1). That $\beta \circ \alpha = 0$ follows from the fact that $i \circ \langle 1, -f\rangle = 0$, (2.4). We now prove the surjectivity of $\beta$. We identify $W(X)$ with the subgroup of $W(k(X))$ which is the kernel of $\partial^0$. Let $q \in W(X)$. Then $\partial \circ s(q) = tr \circ \partial q = 0$ so that $s(q) \in W(k)$. Further $\langle 1, -f\rangle s(q) = 0(2.4)$. This implies that $s(q) = 0$ in view of the fact that for any anisotropic quadratic space $q$ over $k$, $q \not\to g \cdot q$ for any odd degree polynomial $g$. Thus, there exists $q_1 \in W(k(T))$ with $i(q_1) = q$. We have $i' \circ \partial_y(q_1) = \partial_x \circ i(q_1) = 0$ for $y \in Y'$. There exists $\mu_y \in W(k(y))$ such that $\langle 1, -\bar{f}\rangle(\mu_y) = \partial_y(q_1)$. Since $\partial : W(k(T)) \to \oplus_{y\in Y'} W(k(y))$ is surjective, there exists $q_2 \in W(k(T))$ such that $\partial_y(q_2) = \mu_y$ for every $y \in Y'$. We have $\partial_y(q_1 - \langle 1, -f\rangle q_2) = \langle 1, -\bar{f}\rangle \mu_y - \langle 1, -\bar{f}\rangle \partial_y(q_2) = 0$ for $y \in Y'$ so that $q_1 - \langle 1, -f\rangle q_2 \in W(A)$ and maps to $q$ under $\beta$. We now prove exactness at $W(A)/\langle 1, -f\rangle \cdot W(k)$. Let $q \in W(A)$ be such that $\beta(\bar{q}) = 0$ in $W(k(X))$. By (2.4), there exists $q_1 \in W(k(T))$ such that $\langle 1, -f\rangle \cdot q_1 = q$. Since $\langle 1, -\bar{f}\rangle\, \partial_y(q_1) = 0$ for $y \in Y'$, there exists $\mu_x \in W(k(x))$, $x/y$ such that $tr(\mu_x) = \partial_y(q_1)$. For $x \in S$, we set $\mu_x = \partial_y(q_1)$. Clearly $(\mu_x) \in (\oplus_{x\in X} W(k(x)))^0$ and $\alpha((\mu_x)) = q$.

COROLLARY 3.5. *If all ramification points of X are defined over k, then $W(X)$ is generated by discriminants.*

*Proof.* Suppose $f = \Pi_i\,(T - \alpha_i)$, $\alpha_i \in k$. An immediate consequence of the Milnor sequence ([13] Theorem 5.3) is that $W(k[T]_f)$ is generated by $\langle \lambda(T - \alpha_i)\rangle$ and $\langle \mu\rangle$, $\mu \in k^*$, $1 \leqslant i \leqslant 2g + 1$. Since $\beta$ is surjective, their images under $\beta$, which are precisely the discriminants of $W(X)$, generate $W(X)$.

## 4. Some computations for hyperelliptic curves

Let $X$ be a smooth hyperelliptic curve defined over $k$. We assume throughout that $X$ has a rational point of ramification. Let $\pi : X \to \mathbf{P}^1$ be a double covering as in §3. If genus $X > 1$, since any two double coverings $\pi_1, \pi_2 : X \to \mathbf{P}^1$ differ by an automorphism of $X$, the space $X' = X \backslash S$, $S$ denoting the set of ramification points of the covering $\pi : X \to \mathbf{P}^1$ determines and is determined by $X$. Following notations of §3, let $A = k[T]_f$ and $B = (k[T, y]/(y^2 - f))_f$ be the co-ordinate rings of $Y'$ and $X'$ respectively.

LEMMA 4.1. *The unit group $U(B)$ is generated by $k^*$, $y$, and divisors of $f$. If $f$ splits into linear factors over $k$, $U(B) \xrightarrow{\sim} k^* \times \mathbf{Z}^{2g+1}$.*

*Proof.* Let $h \in U(B)$. Then $\operatorname{div} h = \Sigma\, n_i x_i$, $x_i \in S$, $\operatorname{div} h$ denoting the divisor of $h$. Let $\sigma$ denote the nontrivial automorphism of $k(X)$ over $k(T)$. Then $\sigma x_i = x_i$, so that $\operatorname{div} \sigma h = \operatorname{div} h$. Thus $h = \lambda \sigma h$, $\lambda \in k^*$. We have, $h^2 = \lambda(h\sigma h) \in U(A)$. Thus $h\sigma h$ is upto a scalar from $k^*$, a power product of divisors of $f$. On the other hand, the only non-square in $k(T)$ which becomes a square in $k(X)$ is $f$. It follows that $h^2 = \mu^2 \Pi_i h_i^{2m_i} f$ or $h^2 = \mu^2 \Pi_i h_i^{2m_i}$, $m_i \in \mathbf{Z}$, $h_i$ divisors of $f$ in $k[T]$. Thus, $h = \pm\mu(\Pi\, h_i^{m_i})y$ or $h = \pm\mu(\Pi\, h_i^{m_i})$. Further, if $f = \Pi_{1 \le i \le 2g+1}(T - \alpha_i)$, $\alpha_i \in k^*$, the homomorphism $k^* \times \mathbf{Z}^{2g+1} \to U(B)$, defined by

$$(\lambda, (n_i)) \to \lambda(T - \alpha_1)^{n_1} \cdots (T - \alpha_{2g})^{n_{2g}} y^{n_{2g+1}}$$

is surjective, by the above remarks. Suppose

$$\lambda(T - \alpha_1)^{n_1} \cdots (T - \alpha_{2g})^{2_{2g}} \cdot y^{n_{2g+1}} = 1$$

is a relation. Then the divisor

$$\sum_{1 \le i \le 2g} 2n_i x_i + n_{2g+1}\left(\sum_{1 \le i \le 2g+1} x_i\right) - \left(\sum 2n_i + n_{2g+1}(2g+1)\right) x_\infty = 0,$$

where $x_i \in S$ lie over $T - \alpha_i$ and $x_\infty$ lies over $\infty$. This implies that $n_i = 0$, $1 \le i \le 2g+1$ and $\lambda = 1$. Thus we have an isomorphism $k^* \times \mathbf{Z}^{2g+1} \xrightarrow{\sim} U(B)$.

LEMMA 4.2. *Suppose $f$ splits into linear factors over $k$. Then the map* $\operatorname{Pic} X' \to \operatorname{Pic} X'_{\bar{k}}$ *is injective.*

*Proof.* Since the divisor classes of degree zero supported on the ramification locus $S$ are precisely the elements of ${}_2\operatorname{Pic} X$, we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & {}_2\operatorname{Pic} X & \longrightarrow & \operatorname{Pic}^0 X & \longrightarrow & \operatorname{Pic} X' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & {}_2\operatorname{Pic} X_{\bar{k}} & \longrightarrow & \operatorname{Pic}^0 X_{\bar{k}} & \longrightarrow & \operatorname{Pic} X'_{\bar{k}} & \longrightarrow & 0.
\end{array}$$

Here $\mathrm{Pic}^0 X$ is the group of divisor classes of degree zero. The first two vertical maps are natural injections. Since $f$ splits into a product of linear factors, ${}_2\mathrm{Pic}\, X = {}_2\mathrm{Pic}\, X_{\bar{k}}$. Since $\mathrm{Pic}^0 X \hookrightarrow \mathrm{Pic}^0 X_{\bar{k}}$ is an injection, it follows that $\mathrm{Pic}\, X' \to \mathrm{Pic}\, X'_{\bar{k}}$ is injective.

LEMMA 4.3. *The group* ${}_2\mathrm{Pic}\, X' \xrightarrow{\sim} (\mathbf{Z}/2)^l$ *where* $l \leq 2g$ *and* $l = 2g$ *if and only if* ${}_4\mathrm{Pic}\, X = {}_4\mathrm{Pic}\, X_{\bar{k}}$.

*Proof.* We have an exact sequence

$$0 \to {}_2\mathrm{Pic}\, X \to {}_4\mathrm{Pic}\, X \to {}_2\mathrm{Pic}\, X' \to 0.$$

Let ${}_2\mathrm{Pic}\, X \xrightarrow{\sim} (\mathbf{Z}/2)^l$, $l \leq 2g$. Let $m$ elements in ${}_2\mathrm{Pic}\, X$ admit a square root over $k$. Then $|{}_4\mathrm{Pic}\, X| = m \cdot 2^l$, with $m \leq 2^l \leq 2^{2g}$. Therefore $|{}_2\mathrm{Pic}\, X'| = m \leq 2^{2g}$ and equality holds if and only if $m = 2^l = 2^{2g}$; i.e., if and only if ${}_4\mathrm{Pic}\, X = {}_4\mathrm{Pic}\, X_{\bar{k}}$.

PROPOSITION 4.4. *Let* Disc *denote the discriminant group of a scheme. Let* $f$ *split into linear factors over* $k$. *Then the composite map* $\mathrm{Disc}\, X' \xrightarrow{N} \mathrm{Disc}\, Y' \to \mathrm{Disc}\, Y'/\mathrm{Disc}\, k$ *is surjective if and only if* ${}_4\mathrm{Pic}\, X = {}_4\mathrm{Pic}\, X_{\bar{k}}$, $N$ *denoting the norm map.*

*Proof.* Since $X'/Y'$ is étale quadratic, we have an exact sequence in étale cohomology groups with $\mu_2$ coefficients ([12], p. 92),

$$0 \longrightarrow H^0(Y') \xrightarrow{\cup \chi_f} H^1(Y') \xrightarrow{i} H^1(X') \xrightarrow{tr} H^1(Y')$$

Here, $H^i(-)$ denotes $H^i_{et}(-, \mu_2)$. The group $H^1(-)$ is simply the discriminant group so that we have an exact sequence

$$1 \longrightarrow \mathrm{Disc}\, Y'/\langle f \rangle \longrightarrow \mathrm{Disc}\, X' \xrightarrow{N} \mathrm{Disc}\, Y'.$$

Since the only square class in $k(T)$ which becomes trivial in $k(X)$ is $\langle f \rangle$, this sequence yields the following exact sequence

$$1 \to \mathrm{Disc}\, Y'/\langle f \rangle \mathrm{Disc}\, k \to \mathrm{Disc}\, X'/\mathrm{Disc}\, k \to \mathrm{Disc}\, Y'/\mathrm{Disc}\, k.$$

We denote $U(B)$ and $U(A)$ by $U(X')$ and $U(Y')$ respectively. By our hypothesis on $f$, $\mathrm{Disc}\, Y' \xrightarrow{\sim} U(Y')/2 \xrightarrow{\sim} (\mathbf{Z}/2)^{2g+1} \times \mathrm{Disc}\, k$ so that $\mathrm{Disc}\, Y'/\langle f \rangle \mathrm{Disc}\, k \xrightarrow{\sim} (\mathbf{Z}/2)^{2g}$ and $\mathrm{Disc}\, Y'/\mathrm{Disc}\, k \xrightarrow{\sim} (\mathbf{Z}/2)^{2g+1}$. Further, the exact sequence

$$1 \to U(X')/2 \to \mathrm{Disc}\, X' \to {}_2\mathrm{Pic}\, X' \to 0$$

gives, by (4.1) and (4.3) that $\mathrm{Disc}\, X'/\mathrm{Disc}\, k \xrightarrow{\sim} (\mathbf{Z}/2)^{2g+1+l}$, where ${}_2(\mathrm{Pic}\, X') \xrightarrow{\sim} (\mathbf{Z}/2)^l$. Clearly, the map $\mathrm{Disc}\, X'/\mathrm{Disc}\, k \to \mathrm{Disc}\, Y'/\mathrm{Disc}\, k$ is surjective if and only if $l = 2g$; i.e., if and only if ${}_4\mathrm{Pic}\, X = {}_4\mathrm{Pic}\, X_{\bar{k}}$.

## 5. Ramified and unramified parts of coker $\partial^0$

Let $(\bigoplus_{x\in S} W(k(x)))^0$ denote the subgroup of $(\bigoplus_{x\in X} W(k(x)))^0$ with non-zero entries only at $x \in S$. Let $V_r$ be the subgroup of coker $\partial^0$, defined by

$$V_r = \left(\bigoplus_{x\in S} W(k(x))\right)^0 \Big/ \left(\partial W(k(X)) \cap \left(\bigoplus_{x\in S} W(k(x))\right)^0\right)$$
$$= \left(\bigoplus_{x\in S} W(k(x))\right)^0 \Big/ \left(\partial W(X')\right)$$

We define $V_{nr} = \mathrm{coker}\, \partial^0 / V_r$. If $p : (\bigoplus_{x\in X} W(k(x)))^0 \to \bigoplus_{x\in X'} W(k(x))$ denotes the restriction of the projection, $p$ is surjective, since $S = X\backslash X'$ contains a rational point. Thus,

$$V_{nr} \xrightarrow{\sim} \bigoplus_{x\in X'} W(k(x))/(p \circ \partial) W(k(X)).$$

LEMMA 5.1. *The map* $\alpha : \mathrm{coker}\, \partial^0 \to W(A)/\langle 1, -\rangle W(k)$ *maps* $V_r$ *onto* $\langle 1, -f\rangle W(A)/\langle 1, -f\rangle W(k)$.

*Proof.* Let $\theta \in (\bigoplus_{x\in S} W(k(x))^0$. Let $q \in W(k(T))$ be such that $\partial(q) = tr\,\theta$.
Since $\partial_y((q) = tr(\theta_y) = 0$ for $y \notin \pi(S)$, $q \in W(A)$ and $\alpha(\bar{\theta}) = \overline{\langle 1, -f\rangle q} \in \langle 1, -f\rangle W(A)/\langle 1, -f\rangle W(k)$.

We now show that $\alpha(V_r) = \langle 1, -f\rangle W(A)/\langle 1, -f)W(k)$. Let $q \in W(A)$. Let $\mu = (\mu_x) \in (\bigoplus_{x\in X} W(k(X)))^0$ be defined by $\mu_x = 0$ for $x \in X'$, $\mu_x = \partial_y(q)$, for $x \in S$, $\pi(x) = y$. Then $\mu \in (\bigoplus_{x\in S} W(k(x)))^0$ and $\alpha(\bar{\mu}) = \overline{\langle 1, -f\rangle q}$ in $\langle 1, -f\rangle W(A)/\langle 1, -f\rangle W(k)$. We thus have an exact sequence

$$0 \to V_{nr} \xrightarrow{\alpha} W(A)/\langle 1, -f\rangle W(A) \xrightarrow{\beta} W(X) \to 0.$$

PROPOSITION 5.2. *The group* $V_{nr}$ is 2-torsion.

*Proof.* Let $\theta \in \bigoplus_{x\in X'} W(k(x))$. Since $\pi(S)$ has a rational point of ramification, there exists $q \in W(k(T))$ such that $\partial(q) = tr\,\theta$. We have, $\langle 1, -f\rangle(\langle 1, f\rangle q) = 0$ so that there exists $q_1 \in W(k(X))$ with $s(q_1) = (\langle 1, f\rangle q)$. Since for $x \in X'$ with $\pi(x) = y$,

$$\partial_y(\langle 1, -f\rangle q) = \langle 1, -\bar{f}\rangle \partial_y(q) = \langle 1, -\bar{f}\rangle tr(\theta_x) = 0,$$

we have $tr(2\theta) = \partial(\langle 1, f\rangle q) = \partial(s(q_1)) = tr(\partial(q_1))$. Thus, by (3.3), there exists $q_2 \in W(k(T))$ such that $\partial_x \circ i(q_2) = 2\theta_x - \partial_x q_1$ for $x \in X'$ and $\partial_x \circ i(q_2) = 0$ for $x \in S$. Thus $2\theta - \partial(q_1 - i(q_2)) \in (\oplus_{x\in S} W(k(x)))^0$ and its image under the projection map $p$ is zero. Thus the class of $2\theta$ in $V_{nr}$ is zero.

Therefore coker $\partial^0$ is an extension of $V_r$ by the 2-torsion group $V_{nr}$. We now show that under the rationality assumption ${}_4\mathrm{Pic}\, X = {}_4\mathrm{Pic}\, X_{\bar{k}}$, $V_r = 0$. We observe that ${}_4\mathrm{Pic}\, X_{\bar{k}}$ being a finite group, there exists a finite separable extension $l/k$ such that $V_r = 0$ for $X_l$.

PROPOSITION 5.3. *Suppose* ${}_2\mathrm{Pic}\, X = {}_2\mathrm{Pic}\, X_{\bar{k}}$. *Then the group* $V_r = 0$ *if and only if* ${}_4\mathrm{Pic}\, X = {}_4\mathrm{Pic}\, X_{\bar{k}}$.

*Proof.* We show that the map $\partial : W(X') \to (\oplus_{x\in S} W(k(x)))^0$ is surjective if and only if ${}_4\mathrm{Pic}\, X = {}_4\mathrm{Pic}\, X_{\bar{k}}$. In view of the commutative diagram

$$\begin{array}{ccccccccc} & & & & W(X') & \longrightarrow & \left(\bigoplus_{x\in S} W(k(x))\right)^0 & & \\ & & & & \downarrow s & & \| & & \\ 0 & \longrightarrow & W(k) & \longrightarrow & W(Y') & \longrightarrow & \left(\bigoplus_{y\in\pi(S)} W(k(y))\right)^0 & \longrightarrow & 0 \end{array} \qquad (**)$$

with $(**)$ exact, we need to show that $s : W(X') \to W(Y')/W(k)$ is surjective if and only if ${}_4\mathrm{Pic}\, X = {}_4\mathrm{Pic}\, X_{\bar{k}}$. By our assumption ${}_2\mathrm{Pic}\, X = {}_2\mathrm{Pic}\, X_{\bar{k}}$, $f$ splits as a product $\Pi_{1\le i\le 2g+1}(T - \alpha_i)$ over $k$. Suppose ${}_4\mathrm{Pic}\, X = {}_4\mathrm{Pic}\, X_{\bar{k}}$. The exact sequence $(**)$ with each $W(k(y)) \xrightarrow{\sim} W(k)$ for $y \in \pi(S)$ implies that $W(Y')$ is generated by Disc $k$ and $\langle\lambda(T-\alpha_i)\rangle$, $\lambda \in k^*$, $1 \le i \le 2g+1$. It is therefore enough to show that given $\langle\lambda(T-\alpha_i)\rangle$, $\lambda \in k^*$, there exists $\mu \in k^*$ such that $\langle\mu, \lambda(T-\alpha_i)\rangle \in s(W(X'))$. By (4.4), there exists $\tilde{z} \in \mathrm{Disc}\, X'$ such that $N(\tilde{z}) = \langle v(T-\alpha_i)\rangle$ for some $v \in k^*$. We have, $s(\tilde{z}) = z_1\langle 1, -v(T-\alpha_i)\rangle$ for some $z_1 \in k(T)$. Thus,

$$s(-z_1^{-1}v^{-1}\cdot\lambda\cdot\tilde{z}) = \langle -v^{-1}\lambda, \lambda(T-\alpha_i)\rangle.$$

Conversely, suppose $W(X') \to W(Y')/W(k)$ is surjective. Then the map restricted to the ideal $I(X')$ of even dimensional forms surjects onto $I(Y')/I(k)$. In view of the commutative diagram

$$\begin{array}{ccc} I(X') & \xrightarrow{s} & I(Y')/I(k) \\ \downarrow & & \downarrow \\ \mathrm{Disc}\, X' & \xrightarrow{N} & \mathrm{Disc}\, Y'/\mathrm{Disc}\, k \end{array}$$

with the vertical maps surjective, it follows that $N : \operatorname{Disc} X' \to \operatorname{Disc} Y'/\operatorname{Disc} k$ is surjective. This implies, by (4.4) that ${}_4\operatorname{Pic} X = {}_4\operatorname{Pic} X_{\bar{k}}$.

In the next section, under certain assumptions on $k$ and $X$, we describe the unramified part $V_{nr}$ of coker $\partial^0$ cohomologically.

## 6. The unramified part of coker $\partial^0$

Let $Y$ be any scheme over $k$. Let the properties $PQ(1)$, $PQ(2)$ for $Y$ be the following.

**PQ(1)**: For every geometric point $y \in Y$, the invariant theorem for quadratic spaces, $I^n(k(y))/I^{n+1}(k(y)) \xrightarrow{\sim} H^n_{et}(k(y), \mu_2)$ holds for all $n \geq 0$.

**PQ(2)**: $Y$ satisfies $PQ(1)$ and the maps $e_n : I_n(Y) \to \Gamma(Y, \mathcal{H}^n)$ defined in ([15, §1) are surjective for $n \geq 0$.

Here, $\mathcal{H}^n$ denotes the Zariski sheaf associated to the presheaf $U \to H^n_{et}(U, \mu_2)$. The class of schemes which satisfy $PQ(2)$ include all smooth quasi projective curves over local fields, in view of [2] and [16]. Conjecturally, all smooth projective curves over any field satisfy $PQ(2)$.

We follow the same notations as in §4 and denote by $\pi : X \to \mathbf{P}^1$ a double cover, $X$ being a smooth hyperelliptic curve with a rational point of ramification. Under the assumptions that $X' = X \backslash S$, $Y' = Y \backslash \pi(S)$ satisfy $PQ(2)$, we shall describe $V_{nr}$ as a certain cohomology group.

LEMMA 6.1. *Let $Y \subseteq \mathbf{P}^1$ be any subscheme. Then $Y$ satisfies $PQ(2)$ if $Y$ satisfies $PQ(1)$.*

*Proof.* We have the following commutative diagram (cf. [5], [10])

$$
\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 & & & I_n(Y) & \xrightarrow{e_n} & \Gamma(Y, \mathcal{H}^n) & \\
 & & & \downarrow & & \downarrow & \\
0 \longrightarrow & I^{n+1}(k(T)) & \longrightarrow & I^n(k(T)) & \longrightarrow & H^n(k(T)) & \longrightarrow 0 \\
 & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & \\
0 \longrightarrow & \Big(\bigoplus_{y \in Y} I^n(k(y))\Big)^0 & \longrightarrow & \Big(\bigoplus_{y \in Y} I^{n-1}(k(y))\Big)^0 & \longrightarrow & \Big(\bigoplus_{y \in Y} H^{n-1}(k(y))\Big)^0. &
\end{array}
$$

Here $(\oplus_{y\in Y} I^m(k(y)))^0$ (resp. $(\oplus_{y\in Y} H^m(k(y)))^0$ denotes the subgroup consisting of trace zero elements. The two vertical columns are exact, by ([10], p. 277) and [5]. By the assumption on $Y$, the two rows are exact. The surjectivity of $e_n : I_n(Y) \to \Gamma(Y, \mathscr{H}^n)$ follows from the surjectivity of the residue map $\partial : I^{n+1}(k(T)) \to (\oplus_{y\in y} I^n(k(y)))^0$ [13], Theorem 5.3).

LEMMA 6.2. *Suppose* $\mathbf{P}^1$ and $X$ satisfy $PQ(1)$. Then the sequence

$$I_n(A) \xrightarrow{i} I_n(B) \xrightarrow{s} I_n(A)$$

is exact for $n \geq 0$.

*Proof.* Since $B/A$ is unramified, by (2.1), (2.2) and (2.3), we have the following commutative diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & I_n(A) & \longrightarrow & I_n(B) & \longrightarrow & I_n(A) \\
 & & \downarrow & & \downarrow & & \downarrow \\
I^{n-1}(k(T)) & \xrightarrow{\langle 1, -f\rangle} & I^n(k(T)) & \longrightarrow & I^n(k(X)) & \longrightarrow & I^n(k(T)) \\
\partial\downarrow & & \partial\downarrow & & \partial\downarrow & & \partial\downarrow \\
\bigoplus_{y\in Y'} I^{n-2}(k(y)) & \xrightarrow{\langle 1, -\bar{f}\rangle} & \bigoplus_{y\in Y'} I^{n-1}(k(y)) & \longrightarrow & \bigoplus_{x\in X'} I^{n-1}(k(x)) & \longrightarrow & \bigoplus_{y\in Y'} I^{n-1}(k(y)).
\end{array}$$

The vertical columns are exact by ([10], p. 277). Exactness of the rows is a consequence of the assumption $PQ(1)$ for $X$ and $\mathbf{P}^1$ [3]. Exactness of the top row follows from the surjectivity of $\partial : I^{n-1}(k(T)) \to \oplus_{y\in Y'} I^{n-2}(k(y))$, $Y'$ being contained in $\mathbf{A}^1$.

LEMMA 6.3. *Suppose* $X'$, *and* $Y'$ *satisfy* $PQ(2)$. *Then*

$$(\langle 1, -f\rangle W(A)) \cap I_n(A) \xrightarrow{\sim} \langle 1, -f\rangle I_{n-1}(A).$$

*Proof.* We assume, by induction, that

$$(\langle 1, -f\rangle W(A)) \cap I_m(A) = \langle 1, -f\rangle I_{m-1}(A)$$

for $m \leq n-1$. Let $q \in (\langle 1, -f\rangle W(A)) \cap I_n(A)$. By induction, we may write $q = \langle 1, -f\rangle q_1$, $q_1 \in I_{n-2}(A)$. Since $X'$, $Y'$ satisfy $PQ(1)$, and $B/A$ is étale quadratic, we have the following commutative diagram

$$\begin{array}{ccccc} I_{n-2}(B) & \xrightarrow{s} & I_{n-2}(A) & \xrightarrow{\langle 1, -f\rangle} & I_{n-1}(A) \\ & & \downarrow{\scriptstyle e_{n-2}} & & \downarrow{\scriptstyle e_{n-1}} \\ H^{n-2}(B) & \xrightarrow{tr} & H^{n-2}(A) & \xrightarrow{\cup\chi_f} & H^{n-1}(A) \end{array}$$

with the bottom row exact. Since

$$\langle 1, -f\rangle q_1 = q \in I_n(A),\ e_{n-1}(q) = 0; \quad \text{i.e., } \chi_f \cup e_{n-2}(q_1) = 0.$$

Therefore, there exists $\theta \in H^{n-2}(B)$ such that $tr\theta = e_{n-2}(q_1)$. Let $\tilde{\theta} \in \Gamma(B, \mathscr{H}^{n-2})$ be the image of $\theta$ in $H^{n-2}(k(X))$. By the assumption that $X'$ satisfies $PQ(2)$, there exists $q_2 \in I_{n-2}(B)$ such that $e_{n-2}(q_2) = \tilde{\theta}$. The diagram

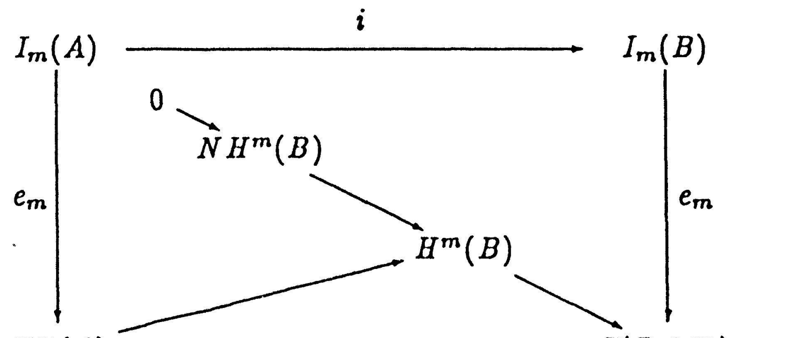
$$\begin{array}{ccc} I_{n-2}(B) & \xrightarrow{s} & I_{n-2}(A) \\ \downarrow{\scriptstyle e_{n-2}} & & \downarrow{\scriptstyle e_{n-2}} \\ \Gamma(B, \mathscr{H}^{n-2}) & \xrightarrow{tr} & \Gamma(A, \mathscr{H}^{n-2}) = H^{n-2}(A) \end{array}$$

can be verified to be commutative, so that $e_{n-2}(q_1 - sq_2) = 0$. Thus $q_1 - sq_2 \in I_{n-1}(A)$ and $\langle 1, -f\rangle(q_1 - sq_2) = \langle 1, -f\rangle q_1 \in \langle 1, -f\rangle I_{n-1}(A)$. This proves the lemma.

We now assume that $X'$ and $Y'$ satisfy $PQ(2)$. The group $V_{nr} = \ker(W(A)/\langle 1, -f\rangle W(A) \xrightarrow{i} W(X))$ has a filtration induced by the filtration $\{I_m(A)\}$ on $W(A)$. Since the map $W(X) \to W(B)$ is injective and since $i$ preserves filtration, by (6.3), we have,

$$\begin{aligned}(V_{nr})_m &= \ker(I_m(A)/(\langle 1, -f\rangle W(A) \cap I_m(A)) \xrightarrow{i} I_m(B)) \\ &= \ker(I_m(A)/\langle 1, -f\rangle I_{m-1}(A) \xrightarrow{i} I_m(B)).\end{aligned}$$

We now define a map $\eta_m : (V_{nr})_m \to NH^m(B) = \ker(H^m(B) \to \Gamma(B, \mathscr{H}^m))$ as follows. Consider the following commutative diagram:



Let $x \in I_m(A)$ be such that $i(x) = 0$. Then the element $i(e_m(x)) \in H^m(B)$ maps to zero in $\Gamma(B, \mathscr{H}^m)$, by the commutativity of the above diagram. Hence $i(e_m(x)) \in NH^m(B)$. We define $\eta_m(\bar{x}) = i \circ e_m(x)$. To show that $\eta_m$ is well-defined, we need to check that for $x \in \langle 1, -f\rangle I_{m-1}(A)$, $\eta_m(\bar{x}) = 0$. Let $x = \langle 1, -f\rangle x'$, $x' \in I_{m-1}(A)$. We have, $i(e_m(x)) = i(\chi_f \cup e_{m-1}(x')) = \chi_{i(f)} \cup i \circ e_{m-1}(x') = 0$ since $f$ is a square in $B$. Thus we have a well-defined homomorphism

$$\eta_m : (V_{nr})_m \to NH^m(B).$$

LEMMA 6.4. $\operatorname{Ker} \eta_m = (V_{nr})_{m+1}$.

*Proof.* Let $\eta_m(\bar{x}) = 0$ with $x \in I_m(A)$. Then $ie_m(x) = 0$ and the exactness of the sequence

$$H^{m-1}(A) \xrightarrow{\cup \chi_f} H^m(A) \xrightarrow{i} H_m(B) \xrightarrow{tr} H^m(A) \qquad (***)$$

implies that there exists $y \in H^{m-1}(A)$ such that $\chi_f \cup y = e_m(x)$. By (6.1), there exists $z \in I_{m-1}(A)$ such that $e_{m-1}(z) = y$. We have, $e_m(x - \langle 1, -f\rangle \cdot z) = 0$ so that $x - \langle 1, -f\rangle \cdot z \in I_{m+1}(A)$ and its class in $(V_{nr})_{m+1}$ is simply the class of $x$.

We thus have a filtration $\{(V_{nr})_m\}$ on $V_{nr}$ with successive quotients $(V_{nr})_m/(V_{nr})_{m+1}$ injecting into $NH^m(B)$.

THEOREM 6.5. *Under the assumption that $X'$ and $Y'$ have $PQ(2)$, $V_{nr} \xrightarrow{\sim} \oplus_{m \geq 2} NH^m(B)$.*

*Proof.* Since by (5.2), $V_{nr}$ is a 2-torsion group, it is enough to show that $\eta_m$ maps $(V_{nr})_m$ onto $NH^m(B)$. Let $x \in NH^m(B)$. Since $NH^n(A) = 0\ \forall n$, $trx = 0$, and the exact sequence $(***)$ implies that there exists $y \in H^m(A)$ with $i(y) = x$. By (6.1), there exists $z \in I_m(A)$ with $e_m(z) = y$. Then $e_m \circ i(z) =$ class of $x$ in $\Gamma(B, \mathscr{H}^m)$ which is zero since $x \in NH^m(B)$. Thus $i(z) \in I_{m+1}(B)$ and $s \circ i(z) = 0$. By (6.2), there exists $z' \in I_{m+1}(A)$ with $i(z') = i(z)$. Replacing $z$ by $z - z'$ which again maps to $y$ under $e_m$, we have $i(z) = 0$; i.e., $\bar{z} \in (V_{nr})_m$ with $\eta_m(\bar{z}) = x$.

## 7. An example

THEOREM 7.1. *Let $X$ be a smooth projective hyperelliptic curve defined over a local field $k$ with residue field characteristic $\neq 2$. Suppose $X$ has a rational point of ramification, $X$ has good reduction and ${}_4\mathrm{Pic}\, X = {}_4\mathrm{Pic}\, X_{\bar{k}}$. Then*

$$\operatorname{coker} \partial \xrightarrow{\sim} W(k) \oplus (\mathbf{Z}/2)^{4g},$$

*$g$ being the genus of $X$.*

In view of results of [2], any curve over a local field satisfies $PQ(1)$. It is shown in [16] that any such curve also satisfies $PQ(2)$. Therefore by our assumption ${}_4\mathrm{Pic}\, X = {}_4\mathrm{Pic}\, X_{\bar{k}}$, we have, $\operatorname{coker} \partial \xrightarrow{\sim} W(k) \oplus (\oplus_{m \geq 2} NH^m(X'))$. Let $G = G(\bar{k}/k)$, $\bar{k}$ denoting the algebraic closure of $k$. Then $cd_2 k \leq 2$ [18] and $cd_2 X'_{\bar{k}} \leq 1$, $X'_{\bar{k}}$ being affine. The spectral sequence ([12], p. 105)

$$H^i(G, H^j(X'_{\bar{k}})) \Rightarrow H^n(X')$$

yields $H^n(X') = 0$ for $n \geq 4$. Thus $\operatorname{coker} \partial^0 \xrightarrow{\sim} NH^2(X') \oplus NH^3(X')$. We shall now compute these groups.

LEMMA 7.2. *Let $X$ be any smooth projective curve of genus $g$ (not necessarily hyperelliptic) over a local field $k$ with residue field characteristic $\neq 2$ and such that $X(k) \neq \varnothing$ and ${}_2\mathrm{Pic}\, X = {}_2\mathrm{Pic}\, X_{\bar{k}}$. Then $H^3(X) \xrightarrow{\sim} (\mathbf{Z}/2)^{2g+2}$ and $\Gamma(X, \mathscr{H}^3) = 0$.*

*Proof.* The only two non-zero terms in the above spectral sequence contributing to $H^3(X)$ are $H^1(G, H^2(X_{\bar{k}}))$ and $H^2(G, H^1(X_{\bar{k}}))$. The only possible non-zero differential $H^0(G, H^2(X_{\bar{k}})) \to H^2(G, H^1(X_{\bar{k}}))$ is zero, $X(k)$ being non-empty, since $H^2(X) \to H^0(G, H^2(X_{\bar{k}}))$ is surjective. Therefore

$$\begin{aligned} H^3(X) &\xrightarrow{\sim} H^2(G, H^1(X_{\bar{k}})) \oplus H^1(G, H^2(X_{\bar{k}})) \\ &\xrightarrow{\sim} (\mathbf{Z}/2)^{2g} \oplus (\mathbf{Z}/2)^2. \end{aligned}$$

In fact the action of $G$ on $H^1(X_{\bar{k}}) \xrightarrow{\sim} {}_2\mathrm{Pic}\, X_{\bar{k}} \xrightarrow{\sim} (\mathbf{Z}/2)^{2g}$ is trivial by our assumption and $H^2(X_{\bar{k}}) \xrightarrow{\sim} \mathrm{Pic}\, X_{\bar{k}}/2 \xrightarrow{\sim} \mathbf{Z}/2$ with trivial action again. Further, $k$ being a local field, $H^2(G, \mathbf{Z}/2) \xrightarrow{\sim} {}_2Br(k) \xrightarrow{\sim} \mathbf{Z}/2$ and $H^1(G, \mathbf{Z}/2) \xrightarrow{\sim} k^*/k^{*2} \xrightarrow{\sim} \mathbf{Z}/2 \times \mathbf{Z}/2$. In view of [4], $NH^3(X) \xrightarrow{\sim} k^*/k^{*2} \times J(k)/2J(k)$. Since $k$ is a local field, by [11], $J(k)$ contains a subgroup $\mathscr{M}$ isomorphic to copies of the valuation ring such that $J(k)/\mathscr{M}$ is finite. The 2-primary part of $J(k)/\mathscr{M}$ is isomorphic to $\Pi_{1 \leq j \leq l}(Z/2^{i_j})$, where $l = \dim_{\mathbf{Z}/2}({}_2\mathrm{Pic}\, X) = 2g$ by our assumption. Therefore $J(k)/2J(k) \xrightarrow{\sim} (\mathbf{Z}/2)^{2g}$, so that $NH^2(X) \xrightarrow{\sim} (\mathbf{Z}/2)^{2g+2}$. Thus $NH^3(X) = H^3(X)$ and $\Gamma(X, \mathscr{H}^3) = 0$.

COROLLARY 7.3. *Let $X$ be a smooth projective curve over a local field $k$ with residue field characteristic $\neq 2$. Suppose $X$ has good reduction and ${}_2\mathrm{Pic}\, X = {}_2\mathrm{Pic}\, X_{\bar{k}}$. Then the classical invariants uniquely determine the class of a quadratic space in $W(X)$.*

*Proof.* In view of ([15], §1), we have injections $rk : W(X)/I(X) \hookrightarrow \mathbf{Z}/2$, $\mathrm{disc} : I(X)/I_2(X) \hookrightarrow H^1(X)$, $c : I_2(X)/I_3(X) \hookrightarrow {}_2Br(X) = \Gamma(X, \mathscr{H}^2)$, where $rk$, disc and $c$ stand for rank, discriminant and Hasse–Witt invariant maps. Since $I_4(X) \hookrightarrow I^4(k(X)) = 0$ [2] and $I_3(X)$ injects into $\Gamma(X, \mathscr{H}^3) = 0$ by (7.2), we have, $rk$, disc and $c$ uniquely determine an element in $W(X)$.

LEMMA 7.4. *Let $X$ be a hyperelliptic curve. Then* $NH^2(X') \xrightarrow{\sim} (\mathbf{Z}/2)^{2g}$, under the assumptions of (7.1) on $X$.

*Proof.* We have $NH^2(X') \xrightarrow{\sim} \operatorname{Pic} X'/2$. The exact sequence

$$0 \to {}_2\operatorname{Pic} X \to \operatorname{Pic}^0 X \to \operatorname{Pic} X' \to 0$$

yields the following long exact sequence

$$0 \to {}_2\operatorname{Pic} X \to {}_2\operatorname{Pic}^0 X \to {}_2\operatorname{Pic} X' \to {}_2\operatorname{Pic} X/2 \to \operatorname{Pic}^0 X/2 \to \operatorname{Pic} X'/2 \to 0.$$

We have ${}_2\operatorname{Pic} X = {}_2\operatorname{Pic}^0 X$, ${}_2\operatorname{Pic} X' \xrightarrow{\sim} (\mathbf{Z}/2)^{2g}$ (4.3), ${}_2\operatorname{Pic} X/2 \xrightarrow{\sim} (\mathbf{Z}/2)^{2g}$ and $\operatorname{Pic}^0 X/2 = J(k)/2J(k) \xrightarrow{\sim} (\mathbf{Z}/2)^{2g}$, in view of (7.2). We therefore have $\operatorname{Pic} X'/2 \xrightarrow{\sim} (\mathbf{Z}/2)^{2g}$.

LEMMA 7.5. *Let $X$ be a hyperelliptic curve. Then $NH^3(X') \xrightarrow{\sim} (\mathbf{Z}/2)^{2g}$, under the assumptions of* (7.1) *on $X$.*

*Proof.* We have an exact sequence

$$0 \to U(X'_{\bar{k}})/2 \to H^1(X'_{\bar{k}}) \to {}_2\operatorname{Pic} X'_{\bar{k}} \to 0.$$

By (4.1) and (4.3), $U(X'_{\bar{k}})/2 \xrightarrow{\sim} (\mathbf{Z}/2)^{2g+1}$ and ${}_2\operatorname{Pic} X'_{\bar{k}} \xrightarrow{\sim} (\mathbf{Z}/2)^{2g}$. Therefore $H^1(X'_{\bar{k}}) \xrightarrow{\sim} (\mathbf{Z}/2)^{4g+1}$. Further, since $U(X'_{\bar{k}})/2$ is generated by $\{y, T - \alpha_i\}$, $1 \le i \le 2g$, which are defined over $k$, and ${}_2\operatorname{Pic} X'_{\bar{k}}$ is also defined over $k$ under the assumption ${}_4\operatorname{Pic} X' = {}_4\operatorname{Pic} X'_{\bar{k}}$, the action of $G$ on $H^1(X'_{\bar{k}})$ is trivial. The only non-zero terms in the spectral sequence

$$H^i(G, H^j(X'_{\bar{k}})) \Rightarrow H^n(X')$$

contributing to $H^3(X')$ is $H^2(G, H^1(X'_{\bar{k}}))$ with all the differentials vanishing, as before. We therefore have

$$H^3(X') \xrightarrow{\sim} H^2(G, H^1(X'_{\bar{k}})) \xrightarrow{\sim} (\mathbf{Z}/2)^{4g+1}.$$

We shall now compute $\Gamma(X', \mathscr{H}^3)$. The sequence

$$H^3(k(X)) \xrightarrow{tr} H^3(k(T)) \xrightarrow{\cup \chi_f} H^4(k(T))$$

is exact and since $cd_2(k) \leq 2$, $cd_2(k(T)) \leq 3$, $H^4(k(T)) = 0$. Thus

$$tr : H^3(k(X)) \to H^3(k(T))$$

is surjective. It induces a map

$$tr : \Gamma(X', \mathscr{H}^3) \to \Gamma(Y', \mathscr{H}^3) \xrightarrow{\sim} H^3(Y').$$

We show that this map is surjective. Let $\lambda \in H^3(Y')$ and $\mu \in H^3(k(X))$ be such that $tr\, \mu = \lambda$, identifying $H^3(Y')$ with a subgroup of $H^3(k(T))$. In view of the commutative diagram

$$\begin{array}{ccccc} H^3(k(T)) & \xrightarrow{i} & H^3(k(X)) & \xrightarrow{tr} & H^3(k(T)) \\ \partial \downarrow & & \partial \downarrow & & \partial \downarrow \\ \bigoplus_{y \in Y'} H^2(k(y)) & \xrightarrow{i} & \bigoplus_{x \in X'} H^2(k(x)) & \xrightarrow{tr} & \bigoplus_{y \in Y'} H^2(k(y)) \end{array}$$

with exact rows, $tr \circ \partial\mu = \partial \circ tr\mu = \partial(\lambda) = 0$ and hence there exists $\nu \in \oplus_{y \in Y'} H^2(k(y))$ with $i(\nu) = \partial(\mu)$. Since $Y' \subset \mathbf{A}^1$, $\partial : H^3(k(T)) \to \oplus_{y \in Y'} H^2(k(y))$ is surjective and hence there exists $\tilde{\nu} \in H^3(k(T))$ with $\partial(\tilde{\nu}) = \nu$. We have $\partial(\mu - i\tilde{\nu}) = 0$ so that $(\mu - i\tilde{\nu}) \in \Gamma(X', \mathscr{H}^3)$ and maps to $\lambda \in \Gamma(Y', \mathscr{H}^3) = H^3(A)$. We thus have a surjection $tr : \Gamma(X', \mathscr{H}^3) \to \Gamma(Y', \mathscr{H}^3)$. We now compute its kernel. Since $H^3(k) = 0$, the map $\partial : H^3(A) \to (\oplus_{y \in \pi(S)} H^2(k(y)))^0$ is an isomorphism. Since the square

$$\begin{array}{ccc} \Gamma(X', \mathscr{H}^3) & \xrightarrow{\partial} & \left(\bigoplus_{x \in S} H^2(k(x))\right)^0 \\ tr \downarrow & & \| \\ H^3(A) & \xrightarrow{\sim} & \left(\bigoplus_{y \in \pi(S)} H^2(k(y))\right)^0 \end{array}$$

is commutative, we have, $\ker tr = \ker \partial = \Gamma(X, \mathscr{H}^3) = 0$, by [5] and (7.2). Thus, $\Gamma(X', \mathscr{H}^3) \xrightarrow{\sim} H^3(A) \xrightarrow{\sim} (\mathbf{Z}/2)^{2g+1}$. Therefore $NH^3(X') \xrightarrow{\sim} (\mathbf{Z}/2)^{2g}$.

This completes the proof of Theorem 7.1. Finally, we use the exact sequence (§3) to compute the defining relations for $W(X)$ as a quotient of $\oplus_{x \in S} W(k(x))$. More precisely, we have the following

THEOREM 7.6. *Under the same hypothesis as in* (7.1),

$$W(X) \xrightarrow{\sim} (\mathbf{Z}/2)^{4g} \oplus W(k).$$

*Proof.* In view of (3.4) and (7.1), we have an exact sequence

$$0 \to (\mathbf{Z}/2)^{4g} \to W(A)/(\langle 1, -f\rangle W(k)) \to W(X) \to 0 \qquad (*)$$

The residue map $\partial : W(A) \to \oplus_{1 \le i \le 2g+1} W(k)$ is surjective, with kernel $W(k)$. We have, in $W(k(T))$, $(W(k) \cap \langle 1, -f\rangle \cdot W(k)) = 0$. In fact, for $q \in W(k) \cap \langle 1, -f\rangle \cdot W(k)$, $q$ extends to zero in $W(X)$. Since $X(k)$ is non-empty, specialising at a rational point yields $q = 0$ in $W(k)$. We thus have an exact sequence

$$0 \to W(k) \to W(A)/(\langle 1, -f\rangle \cdot W(k)) \to \bigoplus_{2g+1} W(k)/\partial(\langle 1, -f\rangle W(k)) \to 0.$$

The image of the map $\eta : W(k) \to \oplus_{2g+1} W(k)$ defined by

$$\eta(q) = (-f'(\alpha_1)q, -f'(\alpha_2)q, \ldots, -f'(\alpha_{2g+1})q)$$

is precisely $\partial(\langle 1, -f\rangle \cdot W(k))$. The map $\eta$ is injective, since for $q \in W(k)$, $\eta(q) = 0$ implies that $\partial(\langle 1, -f\rangle q) = 0$; i.e., $\langle 1, -f\rangle q \in W(k) \cap \langle 1, -f\rangle W(k) = 0$ and $q \xrightarrow{\sim} fq$. Since degree $f$ is odd, $q = 0$. Clearly $\eta$ is a split injection, a section $t$ being given by $t(q_1, q_2, \ldots, q_{2g+1}) = -f'(\alpha_1) \cdot q_1$. We thus have an isomorphism

$$\tilde{\eta} : W(A)/(\langle 1, -f\rangle W(k)) \to W(k) \oplus \left( \bigoplus_{2g} W(k) \right)$$

given by $\tilde{\eta}(\bar{q}) = (\tilde{q}, (\partial_{x_i} q))$, $2 \le i \le 2g+1$, $x_i \in S$, $\tilde{q}$ denoting specialisation at $\infty$. If $\bar{q} \in W(A)/ < 1, -f\rangle W(k)$, maps to zero in $W(X)$, specialising at $\infty$, we see that $\tilde{q} = 0$, so that in the sequence $(*)$, $(\mathbf{Z}/2)^{4g}$ injects into the factor $\oplus_{2g} W(k) \xrightarrow{\sim} \oplus_{4g} W(F)$ where $F$ denotes the residue field of $k$. If $-1$ is a square in $F$, $W(F) \xrightarrow{\sim} (\mathbf{Z}/2)^2$ and if $-1$ is not a square in $F$, $W(F) \xrightarrow{\sim} \mathbf{Z}/4$. Therefore,

$$W(X) \xrightarrow{\sim} W(k) \oplus W(F)^{4g}/(\mathbf{Z}/2)^{4g}$$
$$\xrightarrow{\sim} W(k) \oplus (\mathbf{Z}/2)^{4g}.$$

The above theorem leads one to the following natural questions.

QUESTION 1. *For a smooth hyperelliptic curve $X$ over an arbitrary ground field $k$, (with ${}_4\mathrm{Pic}\, X = {}_4\mathrm{Pic}\, X_{\bar{k}}$), is $W(X)$ isomorphic to $W(k) \oplus (\mathbf{Z}/2)^{4g}$?*

A positive answer to this question will also provide evidence to an affirmative answer to the following more general

QUESTION. (*Scharlau*) *Let $X$ be a smooth projective curve over a field $k$. If $W(k)$ is finitely generated, is $W(X)$ finitely generated?*

QUESTION 2. *For a smooth projective curve $X$ over $k$ with $X(k) \neq \emptyset$, is* coker $\partial \xrightarrow{\sim} W(X)$?

## REFERENCES


[1] J. K. ARASON, *Appendix II to the paper, Amenable Fields and Pfister Extensions by R. Elman, T. Y. Lam and A. R. Wadsworth.* In Conf. on Quadratic forms, Kingston, p. 492, (1976).

[2] J. K. ARASON, R. ELMAN and B. JACOB, *Fields of cohomological 2-dimension three.* Math. Ann. *274*, 649–657 (1986).

[3] J. K. ARASON, R. ELMAN and B. JACOB, *The graded Witt Ring and Galois Cohomology.* CMS Conference Proceedings, *4*, 1973.

[4] S. BLOCH, *Algebraic K-Theory and Classifield Theory for arithmetic surfaces.* Ann. of Math. *114*, 229–265 (1981).

[5] S. BLOCK, A. OGUS, *Gersten's Conjecture and the homology of schemes.* Ann. Sci. Ecole Norm. Sup. *7*, 181–202 (1974).

[6] J.-L. COLLIOT-THÉLÈNE, J. J. SANSUC, *Fibrés quadratiques et composantes connexes réelles.* Math. Ann. *224*, 105–134 (1979).

[7] W. D. GEYER, G. HARDER, M. KNEBUSCH and W. SCHARLAU, *Ein Residuensatz für symmetrische Bilinearformen.* Inv. Math. *11*, 319–328 (1970).

[8] M. KNEBUSCH, *On Algebraic curves over real closed fields II*, Math. Z. *151*, 189–205 (1976).

[9] M. KNEBUSCH, W. SCHARLAU, *Quadratische Formen und Quadratische Reziprozitätsgesetze.* Math. Z. *121*, 346–368 (1971).

[10] M. KNEBUSCH, *Symmetric bilinear forms over algebraic varieties.* In Conf. on Quadratic forms, Kingston, 103–283 (1976).

[11] A. MATTUCK, *Abelian varieties over p-adic ground fields.* Ann. of Math. *62*, 92–119 (1955).

[12] J. S. MILNE, *Etale cohomology.* Princeton Math. Series *33* (1980).

[13] J. MILNOR, *Algebraic K-theory and Quadratic forms*, Invent Math. *9*, 318–344 (1970).

[14] R. PARIMALA, *Witt groups of conics, elliptic and hyperelliptic curves.* Journal of Number Theory *28*, 69–93 (1988).

[15] R. PARIMALA, *Witt group of affine three folds.* Duke Math. Journal *57*, 947–954 (1988).

[16] R. PARIMALA and R. SRIDHARAN, *Graded Witt-rings and unramified cohomology rings of curves.* Preprint, Bombay.

[17] W. SCHARLAU, *Quadratic and Hermitian forms.* In Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, New York, Berlin (1985).

[18] J. P. SERRE, *Cohomologie Galoisienne.* Lecture Notes in Mathematics *5*, Springer-Verlag, Berlin, Heidelberg, New York (1986).

[19] J. SHICK, *Abstract of Ph.D. Thesis.* Univ. of Calif., San Diego (1988).



*School of Mathematics*
*Tata Institute of Fundamental Research*
*Homi Bhabha Road*
*Bombay 400 005*
*India*


Received November 20, 1989