

**Zeitschrift:** Commentarii Mathematici Helvetici  
**Herausgeber:** Schweizerische Mathematische Gesellschaft  
**Band:** 65 (1990)

**Artikel:** Small eigenvalues on Y-pieces and on Riemann surfaces.  
**Autor:** Schmutz, Paul  
**DOI:** <https://doi.org/10.5169/seals-49745>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. [Siehe Rechtliche Hinweise.](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. [Voir Informations légales.](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. [See Legal notice.](#)

**Download PDF:** 08.02.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

## Small eigenvalues on Y-pieces and on Riemann surfaces

PAUL SCHMUTZ

### I. Introduction

We treat eigenvalues of the Laplacian on Riemann surfaces whose Gauss curvature is identically  $-1$ . We label the eigenvalues in ascending order:

$$0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$$

Each eigenvalue is repeated according to its multiplicity.

We define as *small eigenvalues* those which are less than  $\frac{1}{4}$ . In particular, 0 is taken to be a small eigenvalue. An introduction to the subject is found, for example, in Chapters 1 and 10 of [6].

The question of how many small eigenvalues can exist on closed Riemann surfaces has been treated in two theorems of [3]:

**THEOREM 1.** *Given any  $\varepsilon > 0$  and integer  $g \geq 2$ , there exists a closed Riemann surface of genus  $g$  with  $2g - 2$  eigenvalues smaller than  $\varepsilon$ .*

**THEOREM 2.** *A closed Riemann surface of genus  $g \geq 2$  has at most  $4g - 2$  small eigenvalues.*

In this article we present an improvement of Theorem 2:

**THEOREM 3.** *A closed Riemann surface of genus  $g \geq 2$  has at most  $4g - 4$  small eigenvalues.*

These theorems are proved using the principle of monotonicity. Cut the surface  $M$  into pieces. Then:

- (a) The number of *all* small eigenvalues of all pieces with respect to Neumann boundary conditions is an upper bound for the number of small eigenvalues on  $M$ .
- (b) The number of *all* small eigenvalues of all pieces with respect to Dirichlet boundary conditions is a lower bound for the number of small eigenvalues on  $M$ .

Thus, we must determine the number of small eigenvalues of the pieces.

Considering the fact that a closed Riemann surface of genus  $g$  can be cut into  $2g - 2$   $Y$ -pieces (these are Riemann surfaces of signature  $(0, 3)$  with closed geodesics as boundary components) or also into  $4g - 2$  geodesic triangles, the propositions above follow as corollaries of the following more general theorems:

**THEOREM 1'.** *Given any  $\varepsilon > 0$ , there exists a  $Y$ -piece which has an eigenvalue smaller than  $\varepsilon$  with respect to Dirichlet boundary conditions.*

**THEOREM 2'.** *A geodesic triangle has 0 as its only small eigenvalue with respect to Neumann boundary conditions.*

**THEOREM 3'.** *A  $Y$ -piece has at most two small eigenvalues with respect to Neumann boundary conditions.*

We proceed as follows with the proof of theorem 3', our main theorem. In Section II we provide the necessary base which includes information about the small eigenvalues in the right-angled hexagon (hexagons in the hyperbolic plane  $\mathbb{H}^2$  with six right angles), the Symmetry-Lemma and the Quadrilateral-Lemma. In Section III we prove the main theorem with two different methods. We also prove that a closed Riemann surface of genus  $g$  can be cut into  $4g - 4$  geodesic triangles. In Section IV we classify the  $Y$ -pieces into four types. Finally, in Section V we add some remarks concerning the number of small eigenvalues which can exist on Riemann surfaces.

*Notation:*

- (a) Let  $S$  be a Riemann surface. Then  $S(N)$  (respectively  $S(D)$ ) denotes the eigenvalue problem on  $S$  with respect to Neumann boundary conditions (respectively with respect to Dirichlet boundary conditions). If we have an eigenvalue problem on  $S$  with respect to mixed boundary conditions (on one portion  $D$  of the boundary we have Dirichlet boundary conditions, on the other part we have Neumann boundary conditions), then we write  $S(M; D)$ .
- (b) Let  $H$  be a right-angled hexagon. Then there are three pairs of opposite sides which we denote by  $a/x, b/y, c/z$ , such that among  $a, b, c$  there are no neighbors.

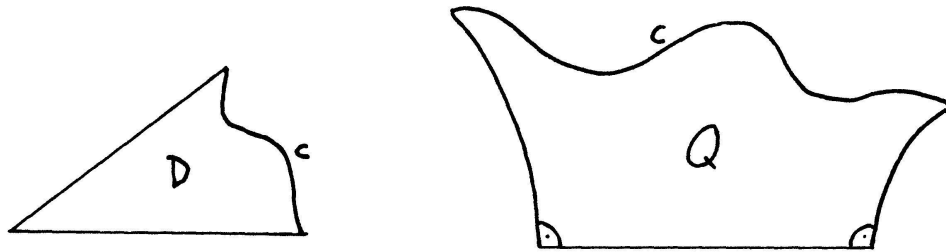
## II. Basic Lemmas

All domains are supposed to be in the hyperbolic plane  $\mathbb{H}^2$ . We refer the reader to [1] or [5] for results concerning hyperbolic trigonometry.

(a) *Right-angled hexagons*

We need two Lemmas from [4] and the Cheeger inequality. Proofs are found in [4] or [9].

LEMMA a. *Let  $D$  be a “triangle” of the following kind: two sides of  $D$  are geodesic segments, the third one a piecewise smooth curve  $c$ . Then  $L(c) > Ar(D)$ . ( $L = \text{length}$ ,  $Ar = \text{area}$ )*



LEMMA b. *Let  $Q$  be a “quadrilateral” of the following kind: three sides of  $Q$  are geodesic segments, which enclose right angles. The fourth side is a piecewise smooth curve  $c$ . Then*

$$L(c) > Ar(Q)$$

This Lemma has the following generalization.

LEMMA b'. *The claim of Lemma b holds if one replaces the two right angles of  $Q$  by angles  $\alpha$  and  $\delta$  with  $\alpha + \delta = \pi$ .*

*Proof.* This change of  $Q$  affects neither  $L(c)$  nor  $Ar(Q)$ .

THEOREM (Cheeger inequality). *Let  $M$  be a Riemann surface and let  $\lambda$  be the smallest nonzero eigenvalue of  $M$ . Then  $\lambda \geq \frac{1}{4}h^2$ , where  $h$  is the isoperimetric constant of Cheeger.*

REMARK. With respect to Neumann boundary conditions,  $h(M)$  is defined as follows:

$$h(M) = \inf \frac{L(\Omega)}{\min \{Ar(M_1), Ar(M_2)\}},$$

where the infimum is with respect to all piecewise smooth curves  $\Omega$  which divide  $M$

into two disjoint subsurfaces  $M_1$  and  $M_2$  with  $\Omega$  as common boundary. With respect to Dirichlet boundary conditions,  $h(M)$  is defined as follows:

$$h(M) = \inf \frac{L(\Omega)}{Ar(M_1)}$$

where  $\Omega$  is as above with  $\partial M_1 \cap \partial M = \phi$ . With respect to Neumann boundary conditions, these results of [9] follow:

LEMMA c. *A geodesic triangle has no nonzero small eigenvalue.*

*Proof.* The Cheeger constant  $h$  is greater than 1, by Lemma a.

LEMMA d. *A geodesic quadrilateral has at most two small eigenvalues.*

*Proof.* Lemma c and principle of monotonicity.

LEMMA e. *A right-angled pentagon has no nonzero small eigenvalue.*

*Proof.* The Cheeger constant  $h$  is greater than 1, by Lemmas a and b.

LEMMA f. *A right-angled hexagon  $H$  has at most two small eigenvalues. Moreover, if  $H$  has two small eigenvalues, then the nodal line of an eigenfunction of  $\lambda_2$  connects two opposite sides of  $H$ .*

*Proof.* Lemma e and principle of monotonicity.

### (b) Symmetry-Lemma

SYMMETRY-LEMMA. *Let  $M$  be a compact Riemann surface with a (nontrivial) involution  $\Psi$  and a symmetrical axis  $t$  (composed by geodesic segments) which divides  $M$  into two isometric parts  $A$  and  $B$  and which is composed by fixed points with respect to  $\Psi$ . The eigenvalues on  $M(N)$  we denote by  $\lambda_i$ . The eigenvalues on  $A(N)$  and the eigenvalues on  $A(M; t)$  we order in a list and label them  $\mu_i$ . Then  $\lambda_i = \mu_i$ , for every  $i = 1, 2, 3, \dots$  Moreover, every eigenfunction on  $A(N)$  or on  $A(M; t)$  is a restriction of an eigenfunction on  $M(N)$ .*

*Proof.* It is easy to show ([9]) that every eigenspace on  $M(N)$  has an orthogonal basis of eigenfunctions which are either symmetric or antisymmetric with respect to  $\Psi$ . In the following, we suppose that we have on  $M(N)$  such an orthogonal basis of eigenfunctions of this kind.

(i) Let  $\phi$  be a symmetric eigenfunction on  $M(N)$ . Then  $\phi|_A$  is an eigenfunction on  $A(N)$ . If  $\psi$  is another symmetric eigenfunction on  $M(N)$ , then

$(\phi | A, \psi | A) = 0$ . Similarly, antisymmetric eigenfunctions  $\phi^*$  and  $\psi^*$  on  $M(N)$ , restricted to  $A$ , are eigenfunctions on  $A(M; t)$  and  $(\phi^* | A, \psi^* | A) = 0$ .

(ii) Now let  $\phi_1, \dots, \phi_n$  be an orthogonal basis of the eigenspace of an eigenvalue  $\lambda$  on  $A(N)$ ,  $n \geq 1$ . Let  $\phi'_1, \dots, \phi'_n$  be the corresponding symmetric functions on  $M$  which are produced by reflection with respect to  $t$  of the  $\phi_j$ . The  $\phi'_j$  are pairwise orthogonal and are also orthogonal to all antisymmetric eigenfunctions on  $M(N)$ . Thus there are symmetric eigenfunctions  $\psi'_1, \dots, \psi'_n$  on  $M(N)$ , for which  $(\phi'_j, \psi'_j) \neq 0, j = 1, \dots, n$ . We define  $\psi_j := \psi'_j | A$ . Then the  $\psi_j$  are eigenfunctions on  $A(N)$ . Moreover, they are eigenfunctions of the eigenvalue  $\lambda$ , since otherwise  $(\phi_j, \psi_j) = (\phi'_j, \psi'_j) = 0, j = 1, \dots, n$ . Thus, the  $\psi_j$  form an orthogonal basis of the eigenspace of the eigenvalue  $\lambda$  on  $A(N)$  and the  $\phi_j$  can be represented in this basis. It follows that the  $\phi'_j$  can be represented in the  $\psi'_j$  and are therefore eigenfunctions on  $M(N)$ .

The proof is analogous for eigenfunctions on  $A(M; t)$ . □

**COROLLARY.** *Let  $H$  be a right-angled hexagon and let  $H(N)$  have two small eigenvalues. Let the nodal line  $t$  of an eigenfunction  $\phi$  of  $\lambda_2$  connect the two opposite sides  $c$  and  $z$  of  $H$ . Reflect  $H$  with respect to one of the other four sides of  $H$ , producing an octagon  $A$ . Then  $A(N)$  has three small eigenvalues.*

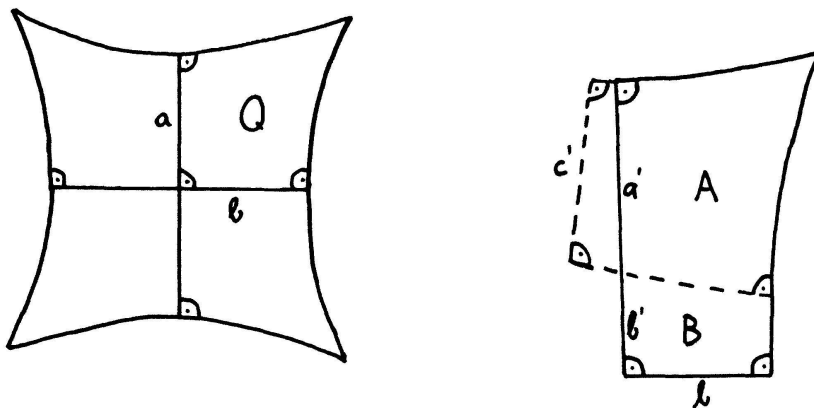
*Proof.*  $A$  is composed of two isometric hexagons  $H$  and  $H'$ . Define the function  $\phi'$  on  $H'$  as the reflection of  $\phi$ . Define the function  $\psi$  on  $A$  as follows:  $\psi | H = \phi, \psi | H' = \phi'$ . Then  $\psi$  is an eigenfunction on  $A$  with three nodal domains. The corollary then follows by Courant's Nodal Domain Theorem. □

(c) *Quadrilateral-Lemma*

**QUADRILATERAL-LEMMA.** *Let  $Q$  be a geodesic quadrilateral with three right angles. Let  $a$  and  $b$  be neighbouring sides, each between two right angles. Let  $L(a) \geq L(b)$ . Then  $Q(M; a)$  has no small eigenvalue.*

*Proof.* Let  $Q(M; a)$  have a small eigenvalue  $\lambda$ .

(i) Suppose that  $L(a) = L(b)$ . We reflect  $Q$  with respect to the side  $a$ , defining a new quadrilateral  $Q'$  which we reflect with respect to the prolonged side  $b$ , defining a quadrilateral  $A$ .  $A(N)$  has two small eigenvalues (because we have also reflected the eigenfunctions). Then, since  $A$  has different axes of symmetry,  $A(N)$  has three small eigenvalues, contradicting Lemma d in IIa.



(ii) Now suppose that  $L(a) > L(b)$ . We symmetrize  $Q$  into a quadrilateral  $Q'$  as in the figure:  $Q'$  has two sides  $c$  and  $c'$  with  $L(c) = L(c')$ .  $Q$  is divided by  $Q'$  into two parts  $A$  and  $B$ . Side  $a$  is divided by  $Q'$  into two parts  $a' \subset A$  and  $b' \subset B$ . Either  $A(M; a')$  or  $B(M; b')$  must have a small eigenvalue. This is impossible for  $B(M; b')$  because of Lemma b of IIa:  $B(M; b')$  has Cheeger constant  $h > 1$ . Thus  $A(M; a')$  has a small eigenvalue with eigenfunction  $\phi$ .

Define a function  $\phi'$  on  $Q'$  by continuing  $\phi$  on  $Q' \setminus Q$  by 0. The Rayleigh-Quotient of  $\phi'$  is less than  $\frac{1}{4}$  and thus there is a small eigenvalue on  $Q'(M; c')$ , contradicting part (i) of this proof.  $\square$

REMARK. The Rayleigh-Quotient of  $f$  (on a surface  $M$ ) is defined as

$$\frac{(\text{grad } f, \text{grad } f)}{(f, f)},$$

where  $(,)$  denotes the inner product on the Hilbert space  $L^2(M)$ .

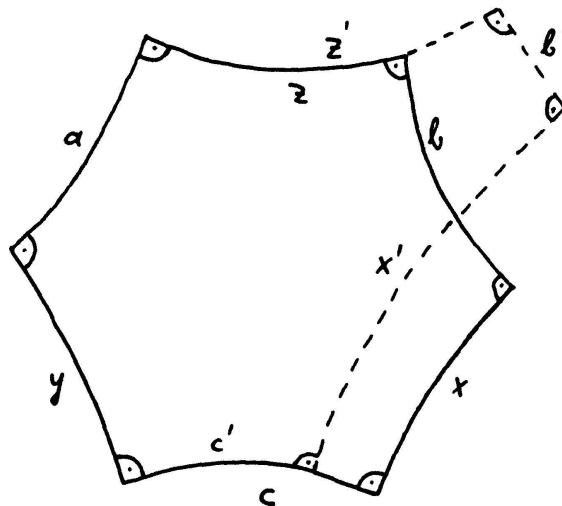
REMARK. The Quadrilateral-Lemma has the following generalization. Its claim holds if the right angle between the sides  $a$  and  $b$  is replaced by another angle. The proof is similar.

COROLLARY 1. Let  $Q$  be an “infinite” quadrilateral, that is, a quadrilateral with four vertices on  $\partial \mathbb{H}^2$ . Let  $a$  and  $b$  be the common orthogonals between opposite sides of  $Q$ . Let  $L(a) > L(b)$ . Let  $Q(N)$  have two small eigenvalues. Then the nodal line  $t$  of an eigenfunction of  $\lambda_2$  lies on  $b$ . Moreover  $L(b) < 2 \sinh^{-1}(1)$ .

*Proof.* It follows from hyperbolic trigonometry that  $a$  and  $b$  are orthogonal and are symmetrical axes of  $Q$ ; moreover  $L(b) < 2 \sinh^{-1}(1)$ . The Symmetry-Lemma asserts that  $t$  lies either on  $a$  or on  $b$ . The Quadrilateral-Lemma now proves the claim.  $\square$

**COROLLARY 2.** *Let  $Q$  be a quadrilateral with two right angles, with a side  $c$  between these two angles and with two vertices on  $\partial\mathbb{H}^2$ . Let  $L(c) \leq 2 \sinh^{-1}(1)$ . Then  $Q(N)$  has no nonzero small eigenvalue.*

**COROLLARY 3.** *Let  $H$  be a right-angled hexagon. Let  $H(M; a, b, c)$  have a small eigenvalue  $\lambda$ . Let  $H'$  be another right-angled hexagon with sides  $a', b', c', x', y', z'$ . Let  $a = a', b > b', c > c', y' = y$ . Then  $H'(M; a', b', c')$  has a small eigenvalue  $\lambda' < \lambda$ .*



*Proof.* Superimpose the two hexagons as shown in the figure. The proof is now the same as the proof of the Quadrilateral-Lemma. □

**PENTAGON-LEMMA.** *Let  $P$  be a right-angled pentagon. Let  $a$  be a side of  $P$ . Let  $P(M; a)$  have a small eigenvalue. Then  $L(a) < \sinh^{-1}(1)$ . (Proof [9].)*

### III. Proof of the main theorem

Every  $Y$ -piece  $M$  is composed of two isometric right-angled hexagons  $H_M$ . The symmetrical axis (composed by three geodesic segments  $a, b, c$  which are each a common orthogonal between two boundary components of  $M$ ) induces an involution  $\Psi$  on  $M$ .

*Proof of the main theorem.* Let  $M$  be a  $Y$ -piece and assume that  $M(N)$  have three small eigenvalues.



Let  $H := H_M$ . Let  $\phi$  and  $\psi$  be (mutually orthogonal) eigenfunctions of the two nonzero small eigenvalues of  $M$  and suppose that  $\phi$  and  $\psi$  are symmetric or antisymmetric with respect to the involution  $\Psi$ .

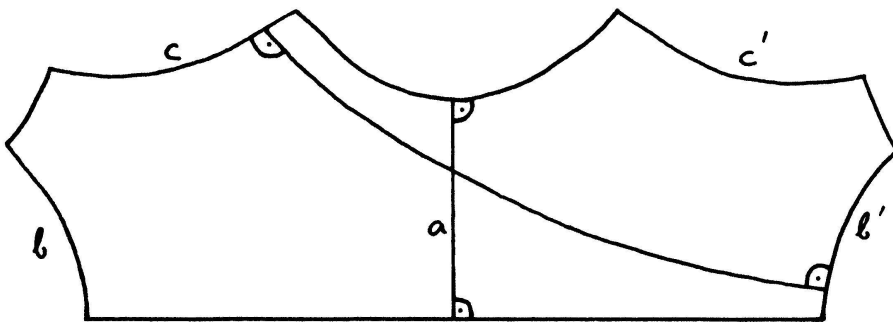
(i)  $\phi$  and  $\psi$  cannot both be symmetric with respect to  $\Psi$ . Otherwise, by the Symmetry-Lemma, the hexagon  $H$  would have three small eigenvalues (with respect to Neumann boundary conditions), contradicting Lemma f of IIa.

(ii)  $\phi$  and  $\psi$  cannot both be antisymmetric. Otherwise,  $\phi$  and  $\psi$  would have an even number of nodal domains, by antisymmetry, and hence two nodal domains, by Courant's Nodal Domain Theorem. Then the nodal lines of  $\phi$  and  $\psi$  would be identically the symmetrical axis of  $M$  and  $\phi$  and  $\psi$  could not be orthogonal.

It follows that we may assume that  $\phi$  is symmetric and  $\psi$  antisymmetric.

(iii) Claim. We can assume without loss of generality that two sides of  $S$  are arbitrary small.

*Proof.* The Symmetry-Lemma says that  $H(N)$  has two small eigenvalues and that  $H(M; a, b, c)$  has one small eigenvalue. These two conditions we denote by condition  $N$  and condition  $M$  for  $H$ . Let the nodal line of  $\phi$  on  $M$  connect the sides  $c$  and  $z$  of  $H$ . We now reflect  $H$  with respect to the side  $a$ , the result being an octagon  $A$  (figure). This we cut along the common orthogonal between the sides  $c$  and  $b'$  (the reflected  $b$ ) and the result is two right-angled hexagons,  $H_1$  and  $H_2$ . By Corollary 3 of IIc, condition  $M$  holds for these two hexagons. By the corollary of IIb,  $A(N)$  has three small eigenvalues. Thus, condition  $N$  holds for one of the two hexagons by the principle of monotonicity. We now select that hexagon for which the conditions  $M$  and  $N$  both hold and repeat the process. Thereby, two of the three sides  $a, b, c$  are reduced each time. It is easy to show ([9]) that in this way one can make two of the three sides arbitrarily small.



(iv) Thus, supposing the sides  $a$  and  $b$  of  $H$  to be very small, we reflect  $H$  with respect to the side  $c$ , defining an octagon  $Q$ . By the Symmetry-Lemma  $Q(N)$  has three small eigenvalues.  $Q$  has four very small sides  $a, b, a', b'$  where  $a', b'$  are reflected sides  $a, b$ . We cut  $Q$  along the common orthogonal between  $a$  and  $b'$ ,

defining two right-angled hexagons. Both have three very small sides, so that their Cheeger constant  $h$  satisfies  $h > 1$ , by IIa. Thus the hexagons have no nonzero small eigenvalue with respect to Neumann boundary conditions. It follows by the principle of monotonicity that  $Q(N)$  has *at most two small eigenvalues*, contradicting the conclusion of the Symmetry-Lemma of above. So the  $Y$ -piece  $M$  has at most two small eigenvalues.  $\square$

**COROLLARY 1.** *Let  $H$  be a right-angled hexagon and let  $H(N)$  have two small eigenvalues. Then  $H$  has a pair of opposite sides which are both strictly longer than  $2 \sinh^{-1}(1)$ .*

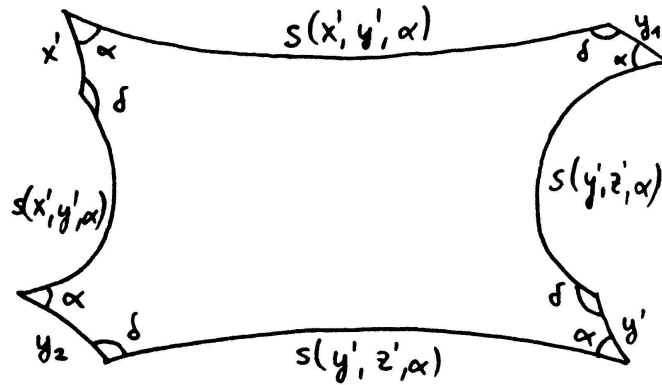
*Proof.* We iterate the process of the above proof. During this, the three sides  $a, b, c$  of  $H$  do not get longer. Repeating the process arbitrarily often, the hexagon  $H$  converges into a quadrilateral  $Q$  with two vertices on  $\partial\mathbb{H}^2$ , as the quadrilateral in Corollary 2 of IIc. By [7], the small eigenvalues of  $H$  converge into small eigenvalues of  $Q$ , so by the mentioned corollary, the basic side of  $Q$  must be longer than  $2 \sinh^{-1}(1)$ . Thus one of the three sides  $a, b, c$  is longer than  $2 \sinh^{-1}(1)$ .

Analogously, we show that one of the sides  $x, y, z$  of  $H$  must be longer than  $2 \sinh^{-1}(1)$ . The claim follows now by hyperbolic trigonometry which states that in  $H$  the longest side of the triple  $a, b, c$  is opposite the longest side of the triple  $x, y, z$ .  $\square$

*Second proof of the main theorem.* Let  $M$  be a  $Y$ -piece with boundary components  $x', y', z'$ . Let  $P$  be the center of the common orthogonal between  $x'$  and  $y'$ . Let  $s(x', y', \alpha)$  be a non self-intersecting geodesic on  $M$  passing through  $P$  such that one end point lies on  $x'$  and the other lies on  $y'$ , and this geodesic intersects  $x'$  and  $y'$  by an angle  $\alpha \in [0, \pi/2]$ . If  $\alpha = \pi/2$ , then  $s(x', y', \alpha)$  is the common orthogonal between  $x'$  and  $y'$  and is unique. In the other cases, there are two different geodesics both of which we denote by  $s(x', y', \alpha)$  and which are symmetric with respect to the involution  $\Psi$  of  $M$ . If  $\alpha = 0$ , we call  $s(x', y', \alpha)$  the common asymptotic geodesic of  $x'$  and  $y'$ . In this case, of course,  $s(x', y', 0)$  does not intersect  $x'$  or  $y'$ . We now fix  $\alpha \in [0, \pi/2]$  and cut  $M$  along a geodesic  $s(x', y', \alpha)$ . Denote the new surface by  $M'$  and cut  $M'$  along the geodesic  $s(y', z', \alpha)$ , producing an octagon  $A$  with four angles  $\alpha$  and four angles  $\delta = \pi - \alpha$  such that  $\alpha$  and  $\delta$  are always neighbouring angles. Now, of course,  $s(y', z', \alpha)$  is unique on  $M'$ . The geodesic  $y'$  has been cut into two parts  $y_1$  and  $y_2$  which are both sides of  $A$ . The other sides of  $A$  are  $x'$  and  $z'$ , twice  $s(x', y', \alpha)$  and twice  $s(y', z', \alpha)$ .

We now cut  $A$  along the geodesic  $s(x', z', \alpha)$  into two hexagons  $H_1$  and  $H_2$ .

Select  $\alpha$  very small. Then, the two hexagons  $H_1$  and  $H_2$  have three (pairwise non-neighbouring) sides which are very small. It follows that the Cheeger constant



$h$  for  $H_1(N)$  and for  $H_2(N)$  is greater than 1 (compare with Lemma  $b'$  of IIa). Thus these hexagons have no nonzero small eigenvalues. Then by the principle of monotonicity,  $M$  has at most two small eigenvalues.  $\square$

REMARK. Let  $\alpha$  converge to 0. Then the octagon  $A$  in the proof above converges to an “infinite” quadrilateral  $Q$ . By [7] the small eigenvalues of  $A$  tend to small eigenvalues of  $Q$ . Since a quadrilateral has at most two small eigenvalues, by Corollary d of IIa, the claim of the main theorem follows once more.

COROLLARY 2. A closed Riemann surface  $M$  of genus  $g$  can be cut into  $4g - 4$  geodesic triangles.

*Proof.* We cut  $M$  into  $2g - 2$   $Y$ -pieces. Each  $Y$ -piece we cut by asymptotic geodesics  $s(x', y', 0)$ ,  $s(y', z', 0)$  and  $s(x', z', 0)$  into two geodesic triangles (notation as above). Of course, the vertices of these triangles all lie on  $\partial\mathbb{H}^2$ .  $\square$

REMARK. The number  $4g - 4$  in Corollary 2 is minimal; a closed Riemann surface  $M$  of genus  $g$  cannot be cut into less than  $4g - 4$  geodesic triangles since the volume of  $M$  is  $(4g - 4)\pi$  and the volume of a geodesic triangle is at most  $\pi$ .

#### IV. Classification of the $Y$ -pieces

DEFINITION. Let  $M$  be a  $Y$ -piece with hexagon  $H := H_M$  and involution  $\Psi$ . We define the following classification:

TYPE  $S$ .  $M(N)$  has two small eigenvalues. The eigenfunctions of  $\lambda_2(M(N))$  are symmetric with respect to  $\Psi$ .

**TYPE A.**  $M(N)$  has two small eigenvalues. The eigenfunctions of  $\lambda_2(M(N))$  are antisymmetric with respect to  $\Psi$ .

**TYPE D.**  $M(D)$  has a small eigenvalue.

**TYPE K.**  $M(D)$  has no small eigenvalue,  $M(N)$  has no nonzero small eigenvalue.

**PROPOSITION.** *Every Y-piece  $M$  belongs to exactly one of the four types, and there exist Y-pieces of each type.*

*Proof.* Let  $x', y', z'$  be the boundary components of  $M$  and let  $H := H_M$  be the hexagon of  $M$  such that the sides  $x, y, z$  are half of  $x', y', z'$ .

(i)  $H(a, b, c)$  and  $H(x, y, z)$  cannot both have a small eigenvalue. If the Cheeger constant of  $H(a, b, c)$  is  $< 1$ , then  $a + b + c < \pi$ .

But also  $a + b + c + x + y + z > 2\pi$ , and the claim follows.

(ii) The following relations hold:

$M$  is of type  $S \Leftrightarrow H(N)$  has two small eigenvalues.

$M$  is of type  $A \Leftrightarrow H(a, b, c)$  has a small eigenvalue.

$M$  is of type  $D \Leftrightarrow H(x, y, z)$  has a small eigenvalue.

By the main theorem and by part (i), it follows that  $M$  belongs to exactly one of the four types.

(iii) Let  $\varepsilon > 0$ . As  $a, b, c$  (respectively  $x, y, z$ ) can be made arbitrarily small, there are right-angled hexagons  $H$  such that the lowest eigenvalue of  $H(a, b, c)$  (respectively of  $H(x, y, z)$ ) is less than  $\varepsilon$ .

Furthermore, one of the common orthogonals between two opposite sides of a right-angled hexagon can be made arbitrary small, and thus there are hexagons  $H$  such that the smallest nonzero eigenvalue of  $H(N)$  is lesser than  $\varepsilon$ .

As an example of type  $K$ , we may take a right-angled hexagon  $H$  such that all six sides of  $H$  have the same length.  $\square$

The following is a criterion for distinguishing between type  $S$  and type  $A$ .

**LEMMA.** *Let  $M$  be a Y-piece with hexagon  $H := H_M$  and let  $M(N)$  have two small eigenvalues. If  $H$  has a pair of opposite sides which are both strictly longer than  $2 \sinh^{-1}(1)$ , then  $M$  is of type  $S$ .*

*Otherwise,  $M$  is of type  $A$ .*

*Proof.* Compare with Corollary 1 of III.

## V. Outlook

The question of the existence of closed Riemann surfaces of genus  $g$  with more than  $2g - 2$  small eigenvalues is still an open question. To this, we will add a few remarks.

(a) In the proof of Theorem 1 of the introduction, if the required surface is constructed of  $Y$ -pieces of type  $D$ , it follows by the proposition of IV that this surface has no more than  $2g - 2$  small eigenvalues.

(b) Naturally, one tries to cut a closed Riemann surface into  $Y$ -pieces of type  $D$  or of type  $K$ . To do so, one needs criteria which indicate when a  $Y$ -piece is of one of these types. Hence the following is crucial. Let  $Q$  be an "infinite" quadrilateral with symmetrical axes  $a$  and  $b$  and  $L(b) < L(a)$ . Let  $Q(N)$  have two small eigenvalues. What is the upper bound for the length of  $b$ ?

Corollary 1 of IIa says that  $L(b) < 2 \sinh^{-1}(1) = 1,76 \dots$ , but this reflects only the fact  $L(b) < L(a)$ . On the other hand, our numerical experiments indicate that  $L(b) < 0,9$ . It should be possible to improve theoretically the upper bound for the length of  $b$ .

## REFERENCES

- [1] A. BEARDON, *The Geometry of Discrete Groups*; Springer, 1983.
- [2] P. BÉRARD, *Spectral Geometry: Direct and Inverse Problems*, Springer, 1986.
- [3] P. BUSER, *Riemannsche Flächen mit Eigenwerten in  $(0, \frac{1}{4})$* . *Comment. Math. Helv.* 52, 25–34 (1977).
- [4] P. BUSER, *On Cheeger's Inequality  $\lambda_1 \geq 1/4h^2$* . *Amer. Math. Soc. Proc. Symp. Pure Math.* 36, 29–77 (1980).
- [5] P. BUSER, *Riemannsche Flächen und Längenspektrum vom trigonometrischen Standpunkt, Habilitationsschrift*, Bonn, 1980.
- [6] I. CHAVEL, *Eigenvalues in Riemannian Geometry*, Academic Press, 1984.
- [7] B. COLBOIS, G. COURTOIS, *Les valeurs propres inférieures à  $\frac{1}{4}$  des surfaces de Riemann de petit rayon d'injectivité*. *Comment. Math. Helv.* 64, 349–362 (1989).
- [8] J. DODZIUK, T. PIGNATARO, B. RANDOL, D. SULLIVAN, *Estimating Small Eigenvalues of Riemann Surfaces. The legacy of Sonya Kovalevskaya* (Cambridge, Mass., and Amherst, Mass., 1985) *Contemp. Math.* 64, 93–121 (1987).
- [9] P. SCHMUTZ, *Zur Anzahl kleiner Eigenwerte auf Riemannschen Flächen* (Dissertation) Zürich, 1989.
- [10] R. SCHOEN, S. WOLPERT, S. T. YAU, *Geometric bounds on the low eigenvalues of a compact Riemann surface*. *Amer. Math. Soc. Proc. Symp. Pure Math.* 36, 279–285 (1980).

Paul Schmutz  
DMA-EPFL  
CH-1015 Lausanne, Switzerland

Received November 23, 1989