Small eigenvalues on Y-pieces and on Riemann surfaces.

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Small eigenvalues on Y-pieces and on Riemann surfaces

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I. Introduction

We treat eigenvalues of the Laplacian on Riemann surfaces whose Gauss curvature is identically -1. We label the eigenvalues in ascending order:

$$0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$$

Each eigenvalue is repeated according to its multiplicity.

We define as *small eigenvalues* those which are less than $\frac{1}{4}$. In particular, 0 is taken to be a small eigenvalue. An introduction to the subject is found, for example, in Chapters 1 and 10 of [6].

The question of how many small eigenvalues can exist on closed Riemann surfaces has been treated in two theorems of [3]:

THEOREM 1. *Given any $\varepsilon > 0$ and integer $g \geq 2$, there exists a closed Riemann surface of genus g with $2g - 2$ eigenvalues smaller than ε.*

THEOREM 2. *A closed Riemann surface of genus $g \geq 2$ has at most $4g - 2$ small eigenvalues.*

In this article we present an improvement of Theorem 2:

THEOREM 3. *A closed Riemann surface of genus $g \geq 2$ has at most $4g - 4$ small eigenvalues.*

These theorems are proved using the principle of monotonicity. Cut the surface M into pieces. Then:

(a) The number of *all* small eigenvalues of all pieces with respect to Neumann boundary conditions is an upper bound for the number of small eigenvalues on M.
(b) The number of *all* small eigenvalues of all pieces with respect to Dirichlet boundary conditions is a lower bound for the number of small eigenvalues on M.

Thus, we must determine the number of small eigenvalues of the pieces.

Considering the fact that a closed Riemann surface of genus g can be cut into $2g-2$ Y-pieces (these are Riemann surfaces of signature $(0,3)$ with closed geodesics as boundary components) or also into $4g-2$ geodesic triangles, the propositions above follow as corollaries of the following more general theorems:

THEOREM 1′. *Given any $\varepsilon > 0$, there exists a Y-piece which has an eigenvalue smaller than ε with respect to Dirichlet boundary conditions.*

THEOREM 2′. *A geodesic triangle has* 0 *as its only small eigenvalue with respect to Neumann boundary conditions.*

THEOREM 3′. *A Y-piece has at most two small eigenvalues with respect to Neumann boundary conditions.*

We proceed as follows with the proof of theorem 3′, our main theorem. In Section II we provide the necessary base which includes information about the small eigenvalues in the right-angled hexagon (hexagons in the hyperbolic plane $\mathbb{H}^2$ with six right angles), the Symmetry-Lemma and the Quadrilateral-Lemma. In Section III we prove the main theorem with two different methods. We also prove that a closed Riemann surface of genus g can be cut into $4g-4$ geodesic triangles. In Section IV we classify the Y-pieces into four types. Finally, in Section V we add some remarks concerning the number of small eigenvalues which can exist on Riemann surfaces.

Notation:

(a) Let S be a Riemann surface. Then $S(N)$ (respectively $S(D)$) denotes the eigenvalue problem on S with respect to Neumann boundary conditions (respectively with respect to Dirichlet boundary conditions). If we have an eignevalue problem on S with respect to mixed boundary conditions (on one portion D of the boundary we have Dirichlet boundary conditions, on the other part we have Neumann boundary conditions), then we write $S(M; D)$.

(b) Let H be a right-angled hexagon. Then there are three pairs of opposite sides which we denote by $a/x, b/y, c/z$, such that among a, b, c there are no neighbors.

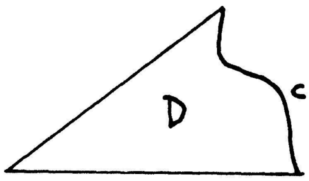
II. Basic Lemmas

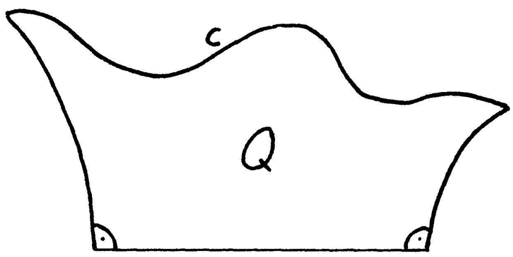
All domains are supposed to be in the hyperbolic plane $\mathbb{H}^2$. We refer the reader to [1] or [5] for results concerning hyperbolic trigonometry.

(*a*) *Right-angled hexagons*

We need two Lemmas from [4] and the Cheeger inequality. Proofs are found in [4] or [9].

LEMMA a. *Let D be a "triangle" of the following kind: two sides of D are geodesic segments, the third one a piecewise smooth curve c. Then $L(c) > Ar(D)$. (L = length, Ar = area)*





LEMMA b. *Let Q be a "quadrilateral" of the following kind: three sides of Q are geodesic segments, which enclose right angles. The fourth side is a piecewise smooth curve c. Then*

$$L(c) > Ar(Q)$$

This Lemma has the following generalization.

LEMMA b′. *The claim of Lemma b holds if one replaces the two right angles of Q by angles α and δ with $\alpha + \delta = \pi$.*

Proof. This change of Q affects neither $L(c)$ nor $Ar(Q)$.

THEOREM (Cheeger inequality). *Let M be a Riemann surface and let λ be the smallest nonzero eigenvalue of M. Then $\lambda \geq \frac{1}{4}h^2$, where h is the isoperimetric constant of Cheeger.*

REMARK. With respect to Neumann boundary conditions, $h(M)$ is defined as follows:

$$h(M) = \inf \frac{L(\Omega)}{\min\{Ar(M_1), Ar(M_2)\}},$$

where the infimum is with respect to all piecewise smooth curves Ω which divide M

into two disjoint subsurfaces M_1 and M_2 with Ω as common boundary. With respect to Dirichlet boundary conditions, $h(M)$ is definded as follows:

$$h(M) = \inf \frac{L(\Omega)}{Ar(M_1)}$$

where Ω is as above with $\partial M_1 \cap \partial M = \phi$. With respect to Neumann boundary conditions, these results of [9] follow:

LEMMA c. *A geodesic triangle has no nonzero small eigenvalue.*

Proof. The Cheeger constant h is greater than 1, by Lemma a.

LEMMA d. *A geodesic quadrilateral has at most two small eigenvalues.*

Proof. Lemma c and principle of monotonicity.

LEMMA e. *A right-angled pentagon has no nonzero small eigenvalue.*

Proof. The Cheeger constant h is greater than 1, by Lemmas a and b.

LEMMA f. *A right-angled hexagon H has at most two small eigenvalues. Moreover, if H has two small eigenvalues, then the nodal line of an eigenfunction of λ_2 connects two opposite sides of H.*

Proof. Lemma e and principle of monotonicity.

(*b*) *Symmetry-Lemma*

SYMMETRY-LEMMA. *Let M be a compact Riemann surface with a (nontrivial) involution Ψ and a symmetrical axis t (composed by geodesic segments) which divides M into two isometric parts A and B and which is composed by fixed points with respect to Ψ. The eigenvalues on $M(N)$ we denote by λ_i. The eigenvalues on $A(N)$ and the eigenvalues on $A(M; t)$ we order in a list and label them μ_i. Then $\lambda_i = \mu_i$, for every $i = 1, 2, 3, \ldots$ Moreover, every eigenfunction on $A(N)$ or on $A(M; t)$ is a restriction of an eigenfunction on $M(N)$.*

Proof. It is easy to show ([9]) that every eigenspace on $M(N)$ has an orthogonal basis of eigenfunctions which are either symmetric or antisymmetric with respect to Ψ. In the following, we suppose that we have on $M(N)$ such an orthogonal basis of eigenfunctions of this kind.

(i) Let ϕ be a symmetric eigenfunction on $M(N)$. Then $\phi \mid A$ is an eigenfunction on $A(N)$. If ψ is another symmetric eigenfunction on $M(N)$, then

$(\phi \mid A, \psi \mid A) = 0$. Similarly, antisymmetric eigenfunctions ϕ^* and ψ^* on $M(N)$, restricted to A, are eigenfunctions on $A(M; t)$ and $(\phi^* \mid A, \psi^* \mid A) = 0$.

(ii) Now let $\phi_1, \dots, \phi_n$ be an orthogonal basis of the eigenspace of an eigenvalue λ on $A(N), n \geqslant 1$. Let $\phi'_1, \dots, \phi'_n$ be the corresponding symmetric functions on M which are produced by reflection with respect to t of the ϕ_j. The ϕ'_j are pairwise orthogonal and are also orthogonal to all antisymmetric eigenfunctions on $M(N)$. Thus there are symmetric eigenfunctions $\psi'_1, \dots, \psi'_n$ on $M(N)$, for which $(\phi'_j, \psi'_j) \neq 0, j = 1, \dots, n$. We define $\psi_j := \psi'_j \mid A$. Then the ψ_j are eigenfunctions on $A(N)$. Moreover, they are eigenfunctions of the eigenvalue λ, since otherwise $(\phi_j, \psi_j) = (\phi'_j, \psi'_j) = 0, j = 1, \dots, n$. Thus, the ψ_j form an orthogonal basis of the eigenspace of the eigenvalue λ on $A(N)$ and the ϕ_j can be represented in this basis. It follows that the ϕ'_j can be represented in the ψ'_j and are therefore eigenfunctions on $M(N)$.

The proof is analogous for eigenfunctions on $A(M; t)$. □

COROLLARY. *Let H be a right-angled hexagon and let $H(N)$ have two small eigenvalues. Let the nodal line t of an eigenfunction ϕ of λ_2 connect the two opposite sides c and z of H. Reflect H with respect to one of the other four sides of H, producing an octagon A. Then $A(N)$ has three small eigenvalues.*

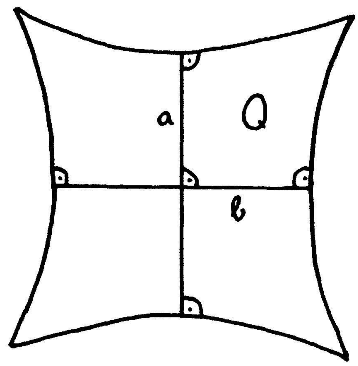
Proof. A is composed of two isometric hexagons H and H'. Define the function ϕ' on H' as the reflection of ϕ. Define the function ψ on A as follows: $\psi \mid H = \phi, \psi \mid H' = \phi'$. Then ψ is an eigenfunction on A with three nodal domains. The corollary then follows by Courant's Nodal Domain Theorem. □

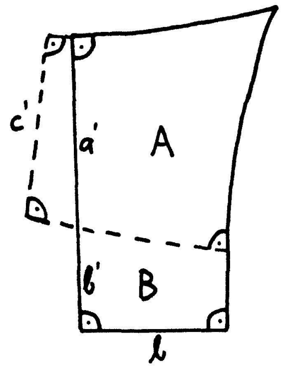
(c) *Quadrilateral-Lemma*

QUADRILATERAL-LEMMA. *Let Q be a geodesic quadrilateral with three right angles. Let a and be be neighbouring sides, each between two right angles. Let $L(a) \geq L(b)$. Then $Q(M; a)$ has no small eigenvalue.*

Proof. Let $Q(M; a)$ have a small eigenvalue λ.

(i) Suppose that $L(a) = L(b)$. We reflect Q with respect to the side a, defining a new quadrilateral Q' which we reflect with respect to the prolongated side b, defining a quadrilateral A. $A(N)$ has two small eigenvalues (because we have also reflected the eigenfunctions). Then, since A has different axes of symmetry, $A(N)$ has three small eigenvalues, contradicting Lemma d in IIa.





(ii) Now suppose that $L(a) > L(b)$. We symmetrize Q into a quadrilateral Q' as in the figure: Q' has two sides c and c' with $L(c) = L(c')$. Q is divided by Q' into two parts A and B. Side a is divided by Q' into two parts $a' \subset A$ and $b' \subset B$. Either $A(M; a')$ or $B(M; b')$ must have a small eigenvalue. This is impossible for $B(M; b')$ because of Lemma b of IIa: $B(M; b')$ has Cheeger constant $h > 1$. Thus $A(M; a')$ has a small eigenvalue with eigenfunction ϕ.

Define a function ϕ' on Q' by continuing ϕ on $Q' \backslash Q$ by 0. The Rayleigh-Quotient of ϕ' is less than $\frac{1}{4}$ and thus there is a small eigenvalue on $Q'(M; c')$, contradicting part (i) of this proof. □

REMARK. The Rayleigh-Quotient of f (on a surface M) is defined as

$$\frac{(\operatorname{grad} f, \operatorname{grad} f)}{(f, f)},$$

where (,) denotes the inner product on the Hilbert space $L^2(M)$.

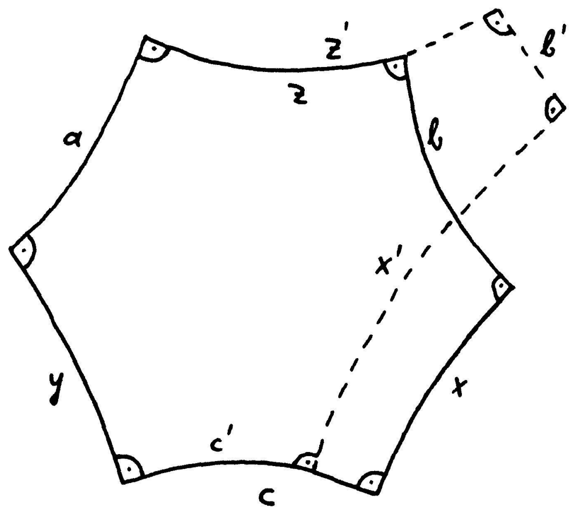
REMARK. The Quadrilateral-Lemma has the following generalization. Its claim holds if the right angle between the sides a and b is replaced by another angle. The proof is similar.

COROLLARY 1. *Let Q be an "infinite" quadrilateral, that is, a quadrilateral with four vertices on $\partial \mathbb{H}^2$. Let a and b be the common orthogonals between opposite sides of Q. Let be $L(a) > L(b)$. Let $Q(N)$ have two small eigenvalues. Then the nodal line t of an eigenfunction of λ_2 lies on b. Moreover* $L(b) < 2 \sinh^{-1}(1)$.

Proof. It follows from hyperbolic trigonometry that a and b are orthogonal and are symmetrical axes of Q; moreover $L(b) < 2 \sinh^{-1}(1)$. The Symmetry-Lemma asserts that t lies either on a or on b. The Quadrilateral-Lemma now proves the claim. □

COROLLARY 2. *Let Q be a quadrilateral with two right angles, with a side c between these two angles and with two vertices on $\partial \mathbb{H}^2$. Let $L(c) \leq 2 \sinh^{-1}(1)$. Then $Q(N)$ has no nonzero small eigenvalue.*

COROLLARY 3. *Let H be a right-angled hexagon. Let $H(M; a, b, c)$ have a small eigenvalue λ. Let H' be another right-angled hexagon with sides a', b', c', x', y', z'. Let $a = a'$, $b > b'$, $c > c'$, $y' = y$. Then $H'(M; a', b', c')$ has a small eigenvalue $\lambda' < \lambda$.*



Proof. Superimpose the two hexagons as shown in the figure. The proof is now the same as the proof of the Quadrilateral-Lemma. □

PENTAGON-LEMMA. *Let P be a right-angled pentagon. Let a be a side of P. Let $P(M; a)$ have a small eigenvalue. Then $L(a) < \sinh^{-1}(1)$.* (Proof [9].)

III. Proof of the main theorem

Every Y-piece M is composed of two isometric right-angled hexagons H_M. The symmetrical axis (composed by three geodesic segments a, b, c which are each a common orthogonal between two boundary components of M) induces an involution Ψ on M.

Proof of the main theorem. Let M be a Y-piece and assume that $M(N)$ have three small eigenvalues.

Let $H := H_M$. Let ϕ and ψ be (mutually orthogonal) eigenfunctions of the two nonzero small eigenvalues of M and suppose that ϕ and ψ are symmetric or antisymmetric with respect to the involution Ψ.

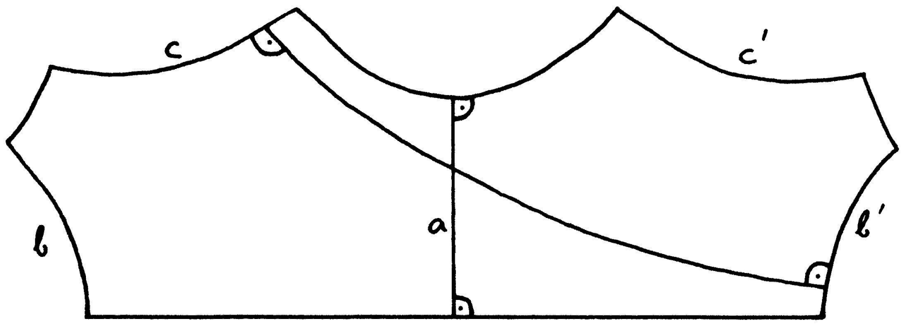
(i) ϕ and ψ cannot both be symmetric with respect to Ψ. Otherwise, by the Symmetry-Lemma, the hexagon H would have three small eigenvalues (with respect to Neumann boundary conditions), contradicting Lemma f of IIa.

(ii) ϕ and ψ cannot both be antisymmetric. Otherwise, ϕ and ψ would have an even number of nodal domains, by antisymmetry, and hence two nodal domains, by Courant's Nodal Domain Theorem. Then the nodal lines of ϕ and ψ would be identically the symmetrical axis of M and ϕ and ψ could not be orthogonal.

It follows that we may assume that ϕ is symmetric and ψ antisymmetric.

(iii) Claim. We can assume without loss of generality that two sides of S are arbitrary small.

Proof. The Symmetry-Lemma says that $H(N)$ has two small eigenvalues and that $H(M; a, b, c)$ has one small eigenvalue. These two conditions we denote by condition N and condition M for H. Let the nodal line of ϕ on M connect the sides c and z of H. We now reflect H with respect to the side a, the result being an octagon A (figure). This we cut along the common orthogonal between the sides c and b' (the reflected b) and the result is two right-angled hexagons, H_1 and H_2. By Corollary 3 of IIc, condition M holds for these two hexagons. By the corollary of IIb, $A(N)$ has three small eigenvalues. Thus, condition N holds for one of the two hexagons by the principle of monotonicity. We now select that hexagon for which the conditions M and N both hold and repeat the process. Thereby, two of the three sides a, b, c are reduced each time. It is easy to show ([9]) that in this way one can make two of the three sides arbitrarily small.



(iv) Thus, supposing the sides a and b of H to be very small, we reflect H with respect to the side c, defining an octagon Q. By the Symmetry-Lemma $Q(N)$ has *three small eigenvalues.* Q has four very small sides a, b, a', b' where a', b' are reflected sides a, b. We cut Q along the common orthogonal between a and b', defining two right-angled hexagons. Both have three very small sides, so that their Cheeger constant h satisfies $h > 1$, by IIa. Thus the hexagons have no nonzero small eigenvalue with respect to Neumann boundary conditions. It follows by the principle of monotonicity that $Q(N)$ has *at most two small eigenvalues*, contradicting the conclusion of the Symmetry-Lemma of above. So the Y-piece M has at most two small eigenvalues. □

COROLLARY 1. *Let H be a right-angled hexagon and let $H(N)$ have two small eigenvalues. Then H has a pair of opposite sides which are both strictly longer than* $2\sinh^{-1}(1)$.

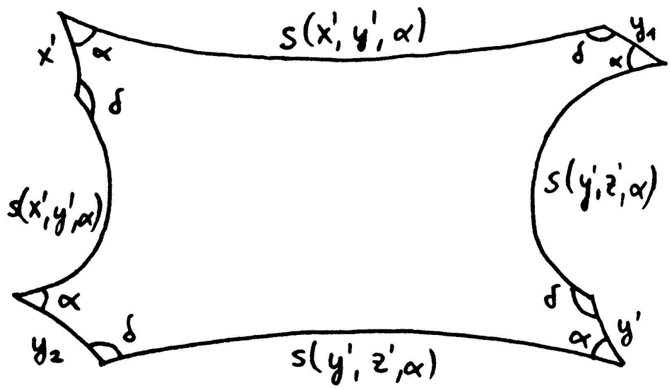
Proof. We iterate the process of the above proof. During this, the three sides a, b, c of H do not get longer. Repeating the process arbitrarily often, the hexagon H converges into a quadrilateral Q with two vertices on $\partial\mathbb{H}^2$, as the quadrilateral in Corollary 2 of IIc. By [7], the small eigenvalues of H converge into small eigenvalues of Q, so by the mentioned corollary, the basic side of Q must be longer than $2\sinh^{-1}(1)$. Thus one of the three sides a, b, c is longer than $2\sinh^{-1}(1)$.

Analogously, we show that one of the sides x, y, z of H must be longer than $2\sinh^{-1}(1)$. The claim follows now by hyperbolic trigonometry which states that in H the longest side of the triple a, b, c is opposite the longest side of the triple x, y, z. □

Second proof of the main theorem. Let M be a Y-piece with boundary components x', y', z'. Let P be the center of the common orthogonal between x' and y'. Let $s(x', y', \alpha)$ be a non self-intersecting geodesic on M passing through P such that one end point lies on x' and the other lies on y', and this geodesic intersects x' and y' by an angle $\alpha \in [0, \pi/2]$. If $\alpha = \pi/2$, then $s(x', y', \alpha)$ is the common orthogonal between x' and y' and is unique. In the other cases, there are two different geodesics both of which we denote by $s(x', y', \alpha)$ and which are symmetric with respect to the involution Ψ of M. If $\alpha = 0$, we call $s(x', y', \alpha)$ the common asymptotic geodesic of x' and y'. In this case, of course, $s(x', y', 0)$ does not intersect x' or y'. We now fix $\alpha \in [0, \pi/2]$ and cut M along a geodesic $s(x', y', \alpha)$. Denote the new surface by M' and cut M' along the geodesic $s(y', z', \alpha)$, producing an octagon A with four angles α and four angles $\delta = \pi - \alpha$ such that α and δ are always neighbouring angles. Now, of course, $s(y', z', \alpha)$ is unique on M'. The geodesic y' has been cut into two parts y_1 and y_2 which are both sides of A. The other sides of A are x' and z', twice $s(x', y', \alpha)$ and twice $s(y', z', \alpha)$.

We now cut A along the geodesic $s(x', z', \alpha)$ into two hexagons H_1 and H_2.

Select α very small. Then, the two hexagons H_1 and H_2 have three (pairwise non-neighbouring) sides which are very small. It follows that the Cheeger constant



h for $H_1(N)$ and for $H_2(N)$ is greater than 1 (compare with Lemma b' of IIa). Thus these hexagons have no nonzero small eigenvalues. Then by the principle of monotonicity, M has at most two small eigenvalues. □

REMARK. Let α converge to 0. Then the octagon A in the proof above converges to an "infinite" quadrilateral Q. By [7] the small eigenvalues of A tend to small eigenvalues of Q. Since a quadrilateral has at most two small eigenvalues, by Corollary d of IIa, the claim of the main theorem follows once more.

COROLLARY 2. *A closed Riemann surface M of genus g can be cut into* $4g-4$ *geodesic triangles.*

Proof. We cut M into $2g-2$ Y-pieces. Each Y-piece we cut by asymptotic geodesics $s(x', y', 0)$, $s(y', z', 0)$ and $s(x', z', 0)$ into two geodesic triangles (notation as above). Of course, the vertices of these triangles all lie on $\partial\mathbb{H}^2$. □

REMARK. The number $4g-4$ in Corollary 2 is minimal; a closed Riemann surface M of genus g cannot be cut into less than $4g-4$ geodesic triangles since the volume of M is $(4g-4)\pi$ and the volume of a geodesic triangle is at most π.

IV. Classification of the Y-pieces

DEFINITION. Let M be a Y-piece with hexagon $H := H_M$ and involution Ψ. We define the following classification:

TYPE S. $M(N)$ has two small eigenvalues. The eigenfunctions of $\lambda_2(M(N))$ are symmetric with respect to Ψ.

TYPE A. $M(N)$ has two small eigenvalues. The eigenfunctions of $\lambda_2(M(N))$ are antisymmetric with respec to Ψ.

TYPE D. $M(D)$ has a small eigenvalue.

TYPE K. $M(D)$ has no small eigenvalue, $M(N)$ has no nonzero small eigenvalue.

PROPOSITION. *Every Y-piece M belongs to exactly one of the four types, and there exist Y-pieces of each type.*

Proof. Let x', y', z' be the boundary components of M and let $H := H_M$ be the hexagon of M such that the sides x, y, z are half of x', y', z'.

(i) $H(a, b, c)$ and $H(x, y, z)$ cannot both have a small eigenvalue. If the Cheeger constant of $H(a, b, c)$ is < 1, then $a + b + c < \pi$.

But also $a + b + c + x + y + z > 2\pi$, and the claim follows.

(ii) The following relations hold:

M is of type $S \Leftrightarrow H(N)$ has two small eigenvalues.

M is of type $A \Leftrightarrow H(a, b, c)$ has a small eigenvalue.

M is of type $D \Leftrightarrow H(x, y, z)$ has a small eigenvalue.

By the main theorem and by part (i), it follows that M belongs to exactly one of the four types.

(iii) Let $\varepsilon > 0$. As a, b, c (respectively x, y, z) can be made arbitrarily small, there are right-angled hexagons H such that the lowest eigenvalue of $H(a, b, c)$ (respectively of $H(x, y, z)$) is less than ε.

Furthermore, one of the common orthogonals between two opposite sides of a right-angled hexagon can be made arbitrary small, and thus there are hexagons H such that the smallest nonzero eigenvalue of $H(N)$ is lesser than ε.

As an example of type K, we may take a right-angled hexagon H such that all six sides of H have the same length. □

The following is a criterion for distinguishing between type S and type A.

LEMMA. *Let M be a Y-piece with hexagon* $H := H_M$ *and let* $M(N)$ *have two small eigenvalues. If H has a pair of opposite sides which are both strictly longer than* $2\sinh^{-1}(1)$, *then M is of type S.*

Otherwise, M is of type A.

Proof. Compare with Corollary 1 of III.

V. Outlook

The question of the existence of closed Riemann surfaces of genus g with more than $2g-2$ small eigenvalues is still an open question. To this, we will add a few remarks.

(a) In the proof of Theorem 1 of the introduction, if the required surface is constructed of Y-pieces of type D, it follows by the proposition of IV that this surface has no more than $2g-2$ small eigenvalues.

(b) Naturally, one tries to cut a closed Riemann surface into Y-pieces of type D or of type K. To do so, one needs criteria which indicate when a Y-piece is of one of these types. Hence the following is crucial. Let Q be an "infinite" quadrilateral with symmetrical axes a and b and $L(b) < L(a)$. Let $Q(N)$ have two small eigenvalues. What is the upper bound for the length of b?

Corollary 1 of IIa says that $L(b) < 2\sinh^{-1}(1) = 1,76\ldots$, but this reflects only the fact $L(b) < L(a)$. On the other hand, our numerical experiments indicate that $L(b) < 0,9$. It should be possible to improve theoretically the upper bound for the length of b.

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