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**Autor:** Félix, Y.

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# Rational category of the space of sections of a nilpotent bundle

Y. FÉLIX*

*Abstract.* Denote by $\zeta : F \to E \xrightarrow{p} B$ a nilpotent fibration where $F$ is a 1-connected space of finite category and $B$ a finite c.w. complex with non trivial rational cohomology. In this note we compute the rational category of the space $\Gamma_*$ of continuous pointed sections of $\zeta$.

## §1. Introduction

In 1956, R. Thom studied the homotopy type of the space $F_f^X$ of continuous maps of $X$ into $F$ homotopic to a given map $f$. He computed explicitly the cohomology of $F_f^X$ when $F$ is a product of Eilenberg–Mac Lane spaces [12].

Later on, following ideas of Sullivan, A. Haefliger gave the rational minimal model of the space of sections of a nilpotent bundle [7]. This model has been extensively studied by K. Shibata and M. Vigué-Poirrier [14]. In particular, M. Vigué-Poirrier noted that, if the dimension of $X$ is less than the connectivity of $Y$, then the rational homotopy Lie algebra of $Y^X$ is isomorphic as a Lie algebra to $H^*(X; \mathbb{Q}) \otimes (\pi_*(\Omega Y) \otimes \mathbb{Q})$.

The aim of this paper is to show that the category of the space of continuous maps from $X$ into $Y$, and more generally of pointed sections of a fibration, is often infinite.

To be more precise, we prove in fact the following two theorems.

THEOREM 1. *Let $\Gamma_*$ be the space of continuous pointed sections of a nilpotent fibration $F \to E \to B$. Suppose that*

(1) $\Gamma_* \neq \phi$

(2) *$F$ is a nilpotent space of finite category*

(3) $H^+(B; \mathbb{Q}) \neq 0$ *and* $\dim H^*(B; \mathbb{Q}) < \infty$

(4) $\dim (\pi_*(F) \otimes \mathbb{Q})$ *is infinite.*

*Then,* $\text{cat}\,(\Gamma_*) = \infty$.

* Chercheur qualifié FNRS.

In the case where the fibration is trivial, the result is more precise:

THEOREM 2. *Let $X$ be a finite nilpotent c.w. complex and $Y$ be a nilpotent space. We suppose that the rational cohomologies of $X$ and $Y$ are not trivial and one of the two following conditions is satisfied*:

(1) $\dim \pi_*(Y) \otimes \mathbb{Q} = \infty$

(2) $\dim \pi_*(Y) \otimes \mathbb{Q} < \infty$ *and there are odd integers $p$ and $q$ such that* $H^p(X; \mathbb{Q}) \neq 0$, $\pi_q(Y) \otimes \mathbb{Q} \neq 0$ *and* $q - p \geq 2$.

*Then the functional space $Y^X$ has infinite category.*

This result clearly yields the following corollary previously proved by E. Fadell and S. Husseini.

COROLLARY [3]. *If $Y$ is a* 1-*connected space of finite category, such that* $\tilde{H}^*(Y; \mathbb{Q}) \neq 0$; *then the free loop space $Y^{S^1}$ has infinite category.*

The organization of the paper is as follows. We first recall the construction of the Sullivan–Haefliger model for the space of continuous (resp. pointed) sections $\Gamma$ (resp. $\Gamma_*$). We then show how to deduce the two theorems from the model. We also deduce a way to compute explicitly the rational homotopy groups of $\Gamma$.

In fact, if $X$ is a 1-connected space and $X_0$ its rationalization, we have the inequality $\operatorname{cat} X_0 \leq \operatorname{cat} X$ [13]. The integer $\operatorname{cat}(X_0)$ is called the rational category of $X$ and is denoted $\operatorname{cat}_0 X$. Its relevance comes from the fact that $\operatorname{cat}_0(X)$ can be obtained from the minimal model of the space [5].

## §2. The Sullivan–Haefliger model

Let $\zeta : F \to E \xrightarrow{p} B$ be a fibration. We suppose that $B$ is a finite nilpotent c.w. complex and $F$ is a nilpotent space with finite Betti numbers. We suppose that $\Gamma_* \neq \phi$.

Let $(A, d_A) \to (A \otimes \Lambda V, D) \to (\Lambda V, d)$ be a minimal K.S. model of $\zeta$ [9]. As $\Gamma_* \neq \phi$, we can also suppose that the differential $D$ satisfies:

$$D(V) \subset A \otimes \Lambda^+ V.$$

$B$ is a finite nilpotent c.w. complex. Therefore we can average $A$ is a finite dimensional graded $\mathbb{Q}$-vector space. Denote by $S$ a graded supplementary subspace of the graded vector space formed by the cocycles in $A$. This gives a direct sum decomposition of $A : A = S \oplus d(S) \oplus T$. We then choose a homogeneous basis

$(a_i)_{i \in I}$ of $A$ by taking the union of homogeneous bases of $S$, $d(S)$ and $T$. The graded dual vector space of $A$ will be denoted by $A^\vee$:

$$(A^\vee)_n = \operatorname{Hom}(A^n, \mathbb{Q}).$$

$A^\vee$ is naturally equipped with the dual basis $a_i^*$:

$$\langle a_i^*; a_j \rangle = \delta_{ij}.$$

We now look at the map of algebras defined by:

$$\varepsilon : A \otimes \Lambda V \to A \otimes \Lambda(A^\vee \otimes V) : \varepsilon(v) = \sum_{i \in I} a_i \otimes (a_i^* \otimes v); \quad \varepsilon(a) = a, \quad a \in A.$$

In [7], A. Haefliger shows how to put a uniquely defined differential $d_A \otimes \delta$ on $A \otimes \Lambda(A^\vee \otimes V)$ in such a way that $\varepsilon$ becomes a morphism of commutative differential graded algebras. Let $W$ be the quotient of $A^\vee \otimes V$ by the subspace of elements of degree $<0$, and by the subspace formed by the $\delta$-cocycles in degree 0.

A short computation shows that $\delta(1 \otimes v) = 1 \otimes dv$, so that the injection $V \cong \mathbb{Q} \otimes V \hookrightarrow A^\vee \otimes V$ induces a K.S. extension:

$$\theta : (\Lambda V, d) \to (\Lambda W, \delta) \to (\Lambda(W/V), \bar{\delta}).$$

THEOREM A (Haefliger, [7]). *$\theta$ is a model for the canonical fibration $\Gamma_* \to \Gamma \xrightarrow{p} F$ where $p$ denotes the evaluation on the basis point of $B$.*

With this model, we can for instance give a rational analogue of the Cohen–Taylor theorem [2].

PROPOSITION. *Let $X$ be a finite wedge of spheres of dimension less than $m$ ($X = \vee_{i=1}^r S^{n_i}$) and $Y$ be a $(m+2)$-connected space, then we have a rational homotopy equivalence*

$$(Y^X)_* \cong \prod_{i=1}^r (Y^{S^{n_i}})_*.$$

*Proof.* The Haefliger model for $(Y^X)_*$ is $(\Lambda(H_*^+(X; \mathbb{Q}) \otimes (\pi_*(Y))^\vee), 0)$. □

## §3. The rational homotopy Lie algebra of a space

If $S$ is a nilpotent space with finite Betti numbers, the minimal model of $S$ is a free commutative differential graded algebra $(\Lambda Z, d)$. The graded vector spaces $Z^*$ and $\mathrm{Hom}(\pi_*(S), \mathbb{Q})$ are then isomorphic [11].

The differential $d$ always decomposes in the form $d = d_2 + d_3 + \cdots$, where $d_i(Z) \subset \Lambda^i Z$. This gives on $s^{-1}\,\mathrm{Hom}(Z, \mathbb{Q})$ a structure of Lie algebra by putting:

$$\langle d_2 z; f, g\rangle = (-1)^{\deg(f)+1}\langle sz; [s^{-1}f, s^{-1}g]\rangle$$

$z \in Z; f, g \in \mathrm{Hom}(Z, \mathbb{Q})$.

It is a result of Andrews and Arkowitz [1] that this Lie algebra is isomorphic to the Lie algebra $L_S = \pi_*(\Omega S) \otimes \mathbb{Q}$ obtained on the rational homotopy groups by means of the Whitehead product. An extensive study of $L_S$ has been made these last years with for instance the following result:

THEOREM B ([6], [4]). *If $S$ is a space of finite category, then every nilpotent ideal $I$ of $L_S$ is finite dimensional.*

We now want to compute $L_\Gamma$ for a given fibration. With the notations of §2, we decompose the differentials $D$ and $\delta$ in the form

$$D = D_1 + D_2 + \cdots \qquad D_i(V) \subset A \otimes \Lambda^i(V)$$

$$\delta = \delta_1 + \delta_2 + \cdots \qquad \delta_i(A^\vee \otimes V) \subset \Lambda^i(A^\vee \otimes V)$$

$\delta_1$ is completely defined by $d_A$ and $D_1$. In fact, put

$$D_1 v_r = \sum_s \alpha_{rs} v_s, \quad \alpha_{rs} \in A^+.$$

We then have:

$$(*)\ \sum_i (-1)^{\deg(a_i)} a_i \otimes \delta_1(a_i^* \otimes v_r) = -\sum_i d_A(a_i) \otimes (a_i^* \otimes v_r) + \sum_i \sum_s \alpha_{rs} \cdot a_i \otimes (a_i^* \otimes v_s)$$

The homology of $(A^\vee \otimes V, \delta_1)$ and $(A_+^\vee \otimes V, \delta_1)$ are respectively isomorphic to the vector spaces of indecomposable elements of the minimal models of $\Gamma$ and $\Gamma_*$:

We thus have isomorphisms:

(1) $H^*(A^\vee \otimes V, \delta_1) \cong (\pi_*(\Gamma) \otimes \mathbb{Q})^\vee$

and

(2) $H^*(A_+^\vee \otimes V, \delta_1) \cong (\pi_*(\Gamma_*) \otimes \mathbb{Q})^\vee$.

Moreover, the short exact sequence of complexes

$$0 \to (V, 0) \to (A^\vee \otimes V, \delta_1) \to (A_+^\vee \otimes V, \delta_1) \to 0$$

induces in homology a long exact sequence isomorphic to the dual of the homotopy long exact sequence of the fibration $\Gamma_* \to \Gamma \to F$ [9]:

$$\begin{array}{ccccccccc} \cdots \to H^q(A_+^\vee \otimes V, \delta_1) & \xrightarrow{\Delta} & V^{q+1} & \longrightarrow & H^{q+1}(A^\vee \otimes V, \delta_1) & \longrightarrow & H^{q+1}(A_+^\vee \otimes V, \delta_1) \to \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdots \to (\pi_q(\Gamma_*))^\vee & \longrightarrow & (\pi_{q+1}(F))^\vee & \longrightarrow & (\pi_{q+1}(\Gamma))^\vee & \longrightarrow & (\pi_{q+1}(\Gamma_*))^\vee \to \cdots \end{array}$$

REMARK. "$D_1$ is differential" can be expressed by the fact that the matrix $\mathbf{a}$ consisting of the $\alpha_{rs}$ satisfies $\mathbf{a}^2 + d\mathbf{a} = 0$.

## §4. Proof of Theorem 1

We use the notations of §§2 and 3, $(\Lambda V, d) \to (\Lambda W, \delta) \to (\Lambda(W/V, \bar{\delta})$ is a model of the fibration $\Gamma_* \to \Gamma \xrightarrow{p} F$. We consider the linear map

$$D_1 : V \to A \otimes V.$$

There are two cases: Either there exists an infinite sequence of homogeneous linearly independent elements $v_1, v_2, \ldots$ belonging to $V$ such that $D_1(v_i)$ doesn't belong to $D_1(A^+ \otimes V)$, or we can suppose that there exists an integer $N$ such that for every $v$ in $V$ of degree larger than $N$, $D(v)$ belongs to $A \otimes \Lambda^{\geq 2} V$.

We take a K.S. basis $(v_i)_{i \geq 1}$ of $V$: $D(v_i) \in A \otimes (v_j)_{j<i}$. Write

$$D_1(v_n) = \sum_{r=1}^{s} \alpha_r \cdot v_r.$$

We obtain $d_A(\alpha_s) = 0$. If $[\alpha_s] = 0$, then $\alpha_s = d_A(b)$ and $D_1(v_n - b \cdot v_s) \in A \otimes (v_j)_{j<s}$.

We then replace $v_n$ by $v'_n = v_n - b \cdot v_s$. If $D_1(v)$ does not belong to $D_1(A^+ \otimes V)$, we can suppose $[a_s] \neq 0$. In this case, formula (*) gives the equality

$$\delta_1(\alpha_s^* \otimes v_s) - (-1)^{\deg(a_s)} 1 \otimes v_n.$$

This means that the element $1 \otimes v_n$ belongs to the image of $\Delta$ in the dual homotopy long exact sequence. Recall that the elements in the image of $\Delta$ are called the Gottlieb elements of the fibration and let us come back to our dichotomy:

In the first case, the $v_i$ are Gottlieb elements of $\Gamma_*$ [5]. By [5], we know that the category of a space is greater or equal to the number of its linearly independent Gottlieb elements, so that cat $(\Gamma_*) = \infty$.

In the second case, put $n = \max\{p$ such that $A^p \neq 0\}$. For $q > n + N$, formula (*) yields the isomorphisms

$$H^q((A^\vee \otimes V), \delta_1) = (H(A, d_A)^\vee \otimes V)^q.$$

The injections $\delta(A_p^\vee \otimes V) \subset \Lambda(A^\vee_{<p} \otimes V)$, valid for $p > 0$, imply that the Lie algebra $L = (H(A, d_A)^\vee_+ \otimes V)^{>n+N}$ is a nilpotent Lie algebra of infinite dimension, which is impossible by Theorem B. □

## §5. Proof of Theorem 2

In this case, the fibration $(Y^X)_* \xrightarrow{i} (Y^X) \to Y$ admits a section and so $\pi_*(i) \otimes \mathbb{Q}$ is injective. It then results from [5] that $\text{cat}_0((Y^X)_*) \leq \text{cat}_0(Y^X)$. If (1) is satisfied, we deduce from Theorem 1 that $\text{cat}_0(Y^X)$ is infinite.

If (2) is satisfied, choose a non homologically trivial cycle $\alpha$ in $A^p$ and a nonzero element $v$ in $V^q$. We now have $D_1 = 0$. The formula (*) shows that $a^* \otimes v$ is a $\delta_1$-cycle which is not a $\delta_1$-boundary. $\alpha^* \otimes v$ defines thus a nonzero indecomposable element in the minimal model of $Y^X$. The definition of $\delta$, as given in §2 implies the following formulas (**) and (***):

$$(**) \quad \sum_i (-)^{\deg(a_i)} a_i \otimes \delta(a_i^* \otimes t) = -\sum_i d_A(a_i) \otimes (a_i^* \otimes t) + \varepsilon(D(t)).$$

$$(***) \quad \varepsilon(v_1 \cdot v_2 \cdots v_r) = \sum_{i_1} \sum_{i_2} \cdots \sum_{i_r} a_{i_1} \cdot a_{i_2} \cdots a_{i_r} \otimes (a_{i_1}^* \otimes v_1)(a_{i_2}^* \otimes v_2) \cdots (a_{i_r}^* \otimes v_r).$$

As $\alpha^2 = 0$ formula (***) shows that $(\alpha^* \otimes v)^n$ can never appear in the decomposition of $\varepsilon(v_q \cdot v_2 \cdots v_r)$, and so by (**), in the differential of an element $\beta^* \otimes t$. It then results from ([8] Proposition 1) that $\text{cat}_0(Y^X) = \infty$. □

If the dimension of $X$ is less than the connectivity of $Y$, the result we obtain is better.

THEOREM 3. *Let $X$ be a nilpotent space such that there exists an integer $k \geq 1$ with $H^p(X; \mathbb{Q}) = 0$, $p > k$ and $H^k(X; \mathbb{Q}) \neq 0$ and let $Y$ be a $(m-1)$-connected space, non contractible over $\mathbb{Q}$, with $m \geq k+2$, then the functional space $Y^X$ has finite rational category iff the three following conditions are satisfied*:

(1) $\pi_*(Y) \otimes \mathbb{Q} = \pi_{\mathrm{odd}}(Y) \otimes \mathbb{Q}$.

(2) $H^*(X; \mathbb{Q}) = H^{\mathrm{even}}(X; \mathbb{Q})$.

(3) $\dim \pi_*(Y) \otimes \mathbb{Q} < \infty$.

*Proof.* By Theorem 2, Condition 3 is necessary. In this case, we have $\dim \pi_{\mathrm{even}}(Y) \otimes \mathbb{Q} \leq \dim \pi_{\mathrm{odd}}(Y) \otimes \mathbb{Q}$ ([8], Proposition 1), so that $\pi_{\mathrm{odd}}(Y) \otimes \mathbb{Q} \neq 0$. By Theorem 2, the second condition is thus also necessary.

Suppose thus that $H^*(X; \mathbb{Q}) = H^{\mathrm{even}}(X; \mathbb{Q})$. We can suppose that $A^{>k} = 0$. If $\pi_{\mathrm{even}}(Y) \otimes \mathbb{Q} \neq 0$, let's choose a cycle $\alpha$ of $A^k$ defining a nonzero element of $H^k(X; \mathbb{Q})$ and let's choose a nonzero element $v$ in $V^{\mathrm{even}}$. Then, formula (***) shows that no power of $a^* \otimes v$ can appear in the decomposition of $\varepsilon(v_1 \cdot v_2 \cdots v_r)$ for any choice of $v_1, v_2, \ldots, v_r$. Now by (**) no power of $a^* \otimes v$ appear in the expression of a boundary, so that, by ([8]) the category of $Y^X$ has to be infinite.

On the other hand, if the three conditions are satisfied,

$$\pi_*(Y^X) \otimes \mathbb{Q} \cong \pi_{\mathrm{odd}}(Y^S) \otimes \mathbb{Q}$$

is finite dimensional and concentrated in odd degrees. The minimal model of $Y^X$ is thus finite dimensional. This implies that $Y^X$ has the rational homotopy type of a finite c.w. complex. □

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*Institut Mathématique*
*2, chemin du Cyclotron*
*1348 Louvain-la-Neuve*
*Belgique*

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