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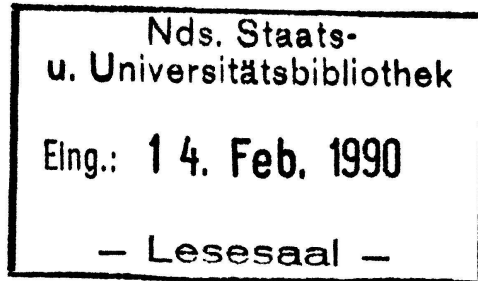
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# On the periodic spectrum of the 1-dimensional Schrödinger operator

TH. KAPPELER



## 1. Introduction

Let us consider Hill's equation

$$-y''(x) + q(x)y(x) = \lambda y(x) \quad (x \text{ in } \mathbf{R}) \tag{1}$$

$$y(x + 2) = y(x) \quad (x \text{ in } \mathbf{R}) \tag{2}$$

where  $q$  is a potential in  $L^2[0, 1]$ , periodically extended to all of  $\mathbf{R}$ . It is well known that those  $\lambda$ 's for which (1)–(2) admits a non zero solution, form a non decreasing sequence of real numbers  $\lambda_k = \lambda_k(q)$  ( $k \geq 0$ ), written with multiplicities.  $(\lambda_k)_{k \geq 0}$  is called the periodic spectrum of  $q$ . Observe that, for convenience, the period in (2) has been chosen equal to 2 rather than 1 in order to include the so called antiperiodic eigenvalues as well. For  $q$  in  $L^2[0, 1]$ , the isospectral set  $\text{Iso}(q)$  is defined to be the set of all potentials  $p$  in  $L^2[0, 1]$  such that  $\lambda_k(p) = \lambda_k(q)$  ( $k \geq 0$ ) and  $G(q)$  denotes the set of all potentials  $p$  in  $L^2[0, 1]$  with the same gaps as  $q$ , i.e.  $\lambda_0(p) = \lambda_0(q)$  and  $\lambda_{2k}(p) - \lambda_{2k-1}(p) = \lambda_{2k}(q) - \lambda_{2k-1}(q)$  ( $k \geq 1$ ). Then  $\text{Iso}(q) \subseteq G(q)$ .

This paper presents an elementary proof of a result due to J. Garnett and E. Trubowitz [GT1] which says that the converse inclusion  $G(q) \subseteq \text{Iso}(q)$  also holds:

**THEOREM** (Garnett, Trubowitz). *For all  $q$  in  $L^2[0, 1]$ ,  $\text{Iso}(q) = G(q)$ .*

In [GT1], this theorem is proved by applying harmonic measure arguments to the identification, due to Marcenko and Ostrovskii [MO], of band configurations with certain slit quarter planes. In this paper it is shown that the theorem is a direct consequence of the spectral theory for even potentials  $q$  in  $L^2[0, 1]$  (i.e.  $q(x) = q(1 - x)$ ), as it is presented in the beautiful paper [GT2], using analysis in Hilbert space.

## 2. Proof of theorem

First observe that due to the fact that  $\lambda_k(q+c) = \lambda_k(q) + c$  ( $k \geq 0$ ;  $c$  real) it suffices to prove the theorem for potentials  $q$  in  $V$  where  $V$  is given by  $V := \{q \in L^2[0, 1] : \lambda_0(q) = 0\}$ .

Let  $q$  be a fixed element in  $V$ . Clearly, for  $p$  in  $G(q)$ ,  $\text{Iso}(p) \subset G(q)$  and thus  $G(q) = \bigcup \text{Iso}(p)$  where the union extends over all  $p$  in  $G(q)$ .

Denote by  $\mu_n = \mu_n(p)$  ( $n \geq 1$ ) and  $\nu_n = \nu_n(p)$  ( $n \geq 0$ ) the Dirichlet and Neuman spectrum of  $p$  in  $L^2[0, 1]$ , that is the spectrum of (1) for the boundary conditions  $y(0)=0, y(1)=0$  and  $y'(0)=0, y'(1)=0$  respectively. It is well known (cf. e.g. [MW]) that  $\nu_0 \leq \lambda_0$  and  $\lambda_{2n-1} \leq \mu_n, \nu_n \leq \lambda_{2n}$  ( $n \geq 1$ ). By the lemma below one can find for a given  $p$  in  $V$  an element  $p_{\max}$  in  $\text{Iso}(p) \cap E$  with the properties that  $\mu_n(p_{\max}) = \lambda_{2n}(p)$  ( $n \geq 1$ ),  $\nu_0(p_{\max}) = \lambda_0(p)$  and  $\nu_n(p_{\max}) = \lambda_{2n-1}(p)$  ( $n \geq 1$ ) where  $E$  denotes the subspace of  $L^2[0, 1]$  of all even potentials  $p$  (i.e.  $p(x) = p(1-x)$ ). Thus for all  $n \geq 1$ ,  $\mu_n(p_{\max}) - \nu_n(p_{\max}) = \lambda_{2n}(q) - \lambda_{2n-1}(q)$ . For  $p$  in  $L^2[0, 1]$ , define  $\sigma(p)$  to be the sequence  $(\mu_n(p) - \nu_n(p))_{n \geq 1}$ . From the asymptotics (cf. e.g. [PT])  $\mu_n(p), \nu_n(p) = n^2\pi^2 + \int_0^1 p(x) dx + a_n, b_n$ , where  $\sum (a_n^2 + b_n^2) < \infty$ , one concludes that  $\sigma(p)$  is an element in  $l^2$ . In [GT2] it is proved that the restriction of  $\sigma$  to  $V \cap E$  is 1-1. Therefore  $p_{\max} = q_{\max}$  for all  $p$  in  $G(q)$  and thus  $\text{Iso}(p) = \text{Iso}(q)$ . This implies that  $G(q) = \text{Iso}(q)$ .

**LEMMA.** *Let  $p$  be in  $L^2[0, 1]$ . Then there exists a potential  $p_{\max}$  in  $\text{Iso}(p)$ , such that*

- (1)  $\mu_n(p_{\max}) = \lambda_{2n}(p)$  ( $n \geq 1$ )
- (2)  $\nu_0(p_{\max}) = \lambda_0(p)$  and  $\nu_n(p_{\max}) = \lambda_{2n-1}(p)$  ( $n \geq 1$ )
- (3)  $p_{\max}$  is even, i.e. an element in  $E$ .

*Proof.* For  $p$  in  $L^2[0, 1]$  with only a finite number of simple periodic eigenvalues, the existence of  $p_{\max}$  with property (1) together with  $\|p_{\max}\|_{L^2} = \|p\|_{L^2}$  is a direct consequence of results presented in [M, M]. By standard arguments one proves (2) and (3). To be more precise, denote by  $y_1(x, \lambda)$  and  $y_2(x, \lambda)$  the fundamental solutions of (1), i.e. the solutions  $y(x, \lambda)$  of (1) with the initial conditions  $y(0, \lambda) = 1, y'(0, \lambda) = 0$  and  $y(0, \lambda) = 0, y'(0, \lambda) = 1$  respectively. For  $\lambda_{2n}(p_{\max}) = \mu_n(p_{\max})$ ,  $y_2(x, \mu_n)$  is the corresponding eigenfunction and thus  $y_2'(2, \mu_n) = 1$ . By investigating the Floquet matrix

$$F(\lambda) = \begin{pmatrix} y_1(1, \lambda) & y_2(1, \lambda) \\ y_1'(1, \lambda) & y_2'(1, \lambda) \end{pmatrix}$$

one concludes that  $|y_2'(1, \mu_n)| = 1$ . Combining Corollary 2.2 and Lemma 3.4 in [PT], it follows that  $p_{\max}$  is even. Using this fact together with reflection one now

verifies that  $\nu_n(p_{\max})$  is a periodic eigenvalue of  $p_{\max}$  ( $n \geq 0$ ). From  $\nu_0 \leq \lambda_0$  and  $\lambda_{2n-1} \leq \mu_n$ ,  $\nu_n \leq \lambda_{2n}$  ( $n \geq 1$ ) it then follows that  $\nu_0(p_{\max}) = \lambda_0(p_{\max})$  and  $\nu_n(p_{\max}) = \lambda_{2n-1}(p_{\max})$  ( $n \geq 1$ ).

Towards the general case, choose a sequence  $(p_n)_{n \geq 1}$  of potentials in  $L^2[0, 1]$  such that  $p = \lim_{n \rightarrow \infty} p_n$  in the norm topology of  $L^2[0, 1]$  and such that, for  $n \geq 1$ ,  $p_n$  has only a finite number of periodic eigenvalues. (Cf. [CK] for an elementary proof concerning the existence of such a sequence). This implies that  $\lim_{n \rightarrow \infty} \lambda_k(p_n) = \lambda_k(p)$  ( $k \geq 0$ ) as the eigenvalues depend continuously on the potential. Denote by  $q_n$  the potential  $(p_n)_{\max}$  in  $\text{Iso}(p_n)$ . Then  $\lambda_{2k}(p_n) = \mu_k(q_n)$  ( $k \geq 1$ ),  $\lambda_0(p_n) = \nu_0(q_n)$  and  $\lambda_{2k-1}(p_n) = \nu_{2k-1}(q_n)$  ( $k \geq 1$ ). Moreover  $\|p_n\|_{L^2[0,1]} = \|q_n\|_{L^2[0,1]}$ . Thus  $(q_n)_{n \geq 1}$  is a sequence, bounded in  $L^2[0, 1]$ ; without loss of any generality we may assume that  $(q_n)_{n \geq 1}$  converges weakly to a potential  $q$  in  $L^2[0, 1]$ . As  $E$  is a closed subspace of  $L^2$ ,  $q$  must be an element of  $E$ . Now use that all eigenvalues  $\lambda_k$  ( $k \geq 0$ ),  $\nu_k$  ( $k \geq 0$ ) and  $\mu_k$  ( $k \geq 1$ ) depend continuously on the potential with respect to the weak topology of  $L^2[0, 1]$  to conclude that  $\lambda_k(q) = \lim_{n \rightarrow \infty} \lambda_k(q_n) = \lim_{n \rightarrow \infty} \lambda_k(p_n) = \lambda_k(p)$  ( $k \geq 0$ ) as well as  $\mu_k(q) = \lambda_{2k}(p)$  ( $k \geq 1$ ),  $\nu_0(q) = \lambda_0(p)$  and  $\nu_k(q) = \lambda_{2k-1}(p)$  ( $k \geq 1$ ). Thus  $p_{\max} := q$  has the desired properties.

REMARK. The proof of the lemma shows that for a given  $q$  in  $L^2[0, 1]$  one has  $\|p\|_{L^2} = \|q\|_{L^2}$  for all potentials  $p$  in  $\text{Iso}(q)$ . For  $q$  in  $C^\infty(\mathbf{R}/\mathbf{Z})$  this is a consequence of results in [MT].

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