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Transcendental submanifolds of $\mathbf{R}^n$

S. Akbulut and H. King

Dedicated to memory of Mario Raimondo

Abstract. In this paper we show how the restriction of the complex algebraic cycles to real part of a complex algebraic set is related to the real algebraic cycles of the real part. As a corollary we give examples of smooth submanifolds of a Euclidean space which can not be isotoped to real parts of complex nonsingular subvarieties in the corresponding projective space.

Algebraic numbers are dense in $\mathbf{R}$. The problem of whether closed smooth submanifolds $M \subset \mathbf{R}^n$ can be approximated by nonsingular algebraic subsets could be viewed as a possible higher dimensional version of this property. By adapting a stronger version of the notion of nonsingularity (complexification is nonsingular) the results of this paper show that this is not the case. By identifying $\mathbf{R}^n \subset \mathbf{RP}^n$ we prove:

THEOREM. *There are closed smooth submanifolds $M \subset \mathbf{R}^n$ which can not be isotoped to the real parts of nonsingular complex algebraic subvarieties of $\mathbf{CP}^n$. Furthermore, we can choose M to be nonsingular real algebraic subsets of $\mathbf{R}^n$.*

Now a brief history: Seifert showed that if $M \subset \mathbf{R}^n$ has trivial normal bundle then it can be isotoped to a nonsingular component of an algebraic subset of $\mathbf{R}^n$ ([S]). Nash proved that in general M can be isotoped to a nonsingular sheet of an algebraic subset of $\mathbf{R}^n$; but the sheets might intersect each other ([N]). In [AK4] it was shown that M can be isotoped to a nonsingular component of an algebraic subset of $\mathbf{R}^n$, i.e. these sheets can be separated. Whether M can be isotoped to (not just to a component of) a nonsingular algebraic subset of $\mathbf{R}^n$ still remains open.

We should emphasize that stably the answers of these problems are known. For example, Nash already proved that M can be isotoped to a nonsingular component of an algebraic set inside of a larger Euclidean space $\mathbf{R}^n \times \mathbf{R}^k$; and later Tognoli showed that in a larger Euclidean space the extra components of the algebraic set can be removed ([T]). In [AK4] and [AK5] it was shown that any $M \subset \mathbf{R}^n$ can be isotoped to a nonsingular algebraic subset of $\mathbf{R}^n \times \mathbf{R}$; more generally M can be isotoped to a nonsingular algebraic subset of $\mathbf{R}^n$ if and only if M is cobordant through immersions to an algebraic subset of $\mathbf{R}^n$.

We obtain the above results as a corollary to our main theorem which says that the restriction of complex algebraic cycles of the complexification of a nonsingular algebraic set consists of the cup products of the real algebraic cycles. Another corollary is that the Ponryagin classes of the tangent and the normal bundle of a nonsingular algebraic set in $\mathbf{R}^n$ are represented by real algebraic subsets, a fact which was known only for the Steifel–Whitney classes.

1. Gysin homomorphism

Let $f: M^m \to N^n$ be a map. The Gysin homomorphism $f_+ : H^*(M) \to H^{*+k}(N)$ is defined by the commutative diagram:

$$\begin{array}{ccc} H^*(M) & \xrightarrow{f_+} & H^{*+k}(N) \\ \downarrow \cong & & \downarrow \cong \\ H_{m-*}(M) & \xrightarrow{f_*} & H_{m-*}(N) \end{array}$$

where $k = n - m$ and the vertical maps are the Poincaré duality isomorphisms. Gysin homomorphism satisfies the following well known properties:

LEMMA 1. *The Gysin homomorphism satisfies the following properties*:
(a) $f_+(f^*(u) \smile v) = u \smile f_+(v)$.
(b) *Given a commuting diagram*:

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \uparrow i & & \uparrow j \\ K & \xrightarrow{g} & L \end{array}$$

where i, j are imbeddings and g is transveral to L with $g^{-1}(L) = K$ then:

$$f^* \circ j_+ = i_+ \circ g^*.$$

LEMMA 2. *Let $f: M \hookrightarrow N$ be an imbedding. Let u_M be the dual of the fundamental class of M in N, and $\chi(v_f)$ be the Euler class of the normal bundle of f then*:
(a) $f_+(1) = u_M$,
(b) $f_+ f^*(x) = x \smile u_M$,
(c) $f^* f_+(1) = f^*(u_M) = \chi(v_f)$,
(d) $f_+(x) \smile f_+(y) = f_+(x \smile y) \smile f_+(1)$.

These lemmas are standard facts of algebraic topology; we leave the proofs as an exercise to the reader.

2. Algebraic homology

Let V be a Zariski open real (or complex) algebraic set (defined over $\mathbf{R}$), and $R = \mathbf{Z}_2$ (or $R = \mathbf{Z}$), then we can define algebraic homology groups $H_*^A(V; R)$ to be the subgroup of $H_*(V; R)$ generated by the compact real (or complex) algebraic subsets of V (cf. [AK1]). We define $H_A^*(V; R)$ to be the Poincaré duals of the groups $H_*^A(V; R)$ when defined. The resolution theorem of [H], implies that $H_*^A(V; R)$ is also the subgroup generated by the classes $g_*([S])$ where $g : S \to V$ is an entire rational function, S is a compact nonsingular real (or complex) algebraic set and $[S]$ is the fundamental class of S. Therefore even when V is real, we can define $H_*^A(V; \mathbf{Z})$ to be the subgroup generated by $g_*([S])$ where $g : S \to V$ is an entire rational function from an oriented compact nonsingular real algebraic set and $[S]$ is the fundamental class of S.

We call a real algebraic set V **totally algebraic** if $H_*(V; \mathbf{Z}_2) = H_*^A(V; \mathbf{Z}_2)$. It is known that not all nonsingular algebraic sets are totally algebraic. There are closed smooth manifolds which can not even be diffeomorphic to nonsingular totally algebraic sets ([BD]), even though every closed smooth manifold is homeomorphic to a totally algebraic set ([AK2]). Hence these algebraic homology groups are intimately related to the nonsingularity of the underlying algebraic set.

Recall from [BBK] that, for a compact nonsingular real algebraic set V, $H_{\mathbf{C}-alg}^*(V; \mathbf{Z})$ is defined to be the subgroup of $H^*(V; \mathbf{Z})$ generated by the restriction of the classes of $H_A^*(V_{\mathbf{C}}; \mathbf{Z})$ by the projective nonsingular complexification map $j : V \hookrightarrow V_{\mathbf{C}}$ (this always exists). $H_{\mathbf{C}-alg}^*(V; \mathbf{Z})$ is independent of the complexification $V_{\mathbf{C}}$. Define $H_{\mathbf{C}-alg}^*(V; \mathbf{Z}_2)$ to be the mod 2 reduction of $H_{\mathbf{C}-alg}^*(V; \mathbf{Z})$.

The real algebraic cocycle groups $H_A^*(V, \mathbf{Z}_2)$ play useful role in real algebraic geometry. For example, they carry obstructions to isotoping submanifolds to algebraic subsets (see [AK1]). Likewise the groups $H_{\mathbf{C}-alg}^*(V; \mathbf{Z}_2)$ also appear as obstructions to algebraic approximation problems (see [BK1]). Our main result describes the relation between these groups.

We need the next result in the proof of the main theorem. It is a special case of Fulton's theorem ([F]). In our notation $V_{\mathbf{C}}$ denotes a complex algebraic set defined over $\mathbf{R}$ with real part V. Square bracket such as $[L]$ means the homology class induced by L, and D denotes the Poincaré duality homomorphism.

LEMMA 3. *Let $\pi_{\mathbf{C}} : \tilde{V}_{\mathbf{C}} \to V_{\mathbf{C}}$ be a blowup of a compact nonsingular algebraic set along a nonsingular center $X_{\mathbf{C}} \subset V_{\mathbf{C}}$. Let $L_{\mathbf{C}}$ be an algebraic subset of $V_{\mathbf{C}}$ with $X_{\mathbf{C}} \subset L_{\mathbf{C}}$. Let $\tilde{L}_{\mathbf{C}} \subset \tilde{V}_{\mathbf{C}}$ be the strict transform and $\tilde{X}_{\mathbf{C}}$ be the exceptional locus $\pi_{\mathbf{C}}^{-1}(X_{\mathbf{C}})$. Then there is a proper algebraic subset $Z_{\mathbf{C}} \subset \tilde{X}_{\mathbf{C}}$ such that:*

(a) $D^{-1}[\tilde{L}_{\mathbf{C}}] = \pi^* D^{-1}[L_{\mathbf{C}}] + D^{-1}[Z_{\mathbf{C}}]$.

(b) $D^{-1}[\tilde{L}] = \pi^* D^{-1}[L] + D^{-1}[Z]$.

Proof. The fact that $D^{-1}[\tilde{L}_{\mathbf{C}}]$ and $\pi^* D^{-1}[L_{\mathbf{C}}]$ differ by a cohomology class supported in $\tilde{X}_{\mathbf{C}}$ is standard (e.g. [AK1], Lemma 2.9.3). More specifically Theorem 6.7 of [F] gives an exact expression for the difference as an algebraic cohomology cycle. □

3. Main results

Now for the rest of the paper let $V \subset \mathbf{RP}^n$ denote a compact nonsingular real algebraic set of dimension v, and $V_{\mathbf{C}} \subset \mathbf{CP}^n$ be a nonsingular projective complexification of V. Let $j : V \hookrightarrow V_{\mathbf{C}}$ denote the inclusion.

Define $\bar{H}^A_{2k}(V_{\mathbf{C}}; \mathbf{Z})$ to be the subgroup of $H^A_{2k}(V_{\mathbf{C}}; \mathbf{Z})$ generated by irreducible complex algebraic subsets defined over $\mathbf{R}$ with k-dimensional real parts. In other words it is generated by the complexification of k-dimensional real algebraic subsets of V in $V_{\mathbf{C}}$. As above, by the resolution theorem, $\bar{H}^A_{2k}(V_{\mathbf{C}}; \mathbf{Z})$ is generated by the classes $g_*([L_{\mathbf{C}}])$, where $L_{\mathbf{C}}$ is a compact irreducible nonsingular complex algebraic set defined over $\mathbf{R}$, i.e., it is the complexification of a k-dimensional real algebraic set L, and $g : L_{\mathbf{C}} \to V_{\mathbf{C}}$ is an entire rational function defined over $\mathbf{R}$, i.e., it is in the form $g = g_{\mathbf{C}}$. Define a subgroup of $H^{2k}_{\mathbf{C}-alg}(V; \mathbf{Z})$ by:

$$\bar{H}^{2k}_{\mathbf{C}-alg}(V; \mathbf{Z}) = j^* \bar{H}^{2k}_A(V_{\mathbf{C}}; \mathbf{Z}).$$

Let $\bar{H}^{2k}_{\mathbf{C}-alg}(V; \mathbf{Z}_2)$ be its mod 2 reduction. The main theorem below implies that this last group is independent of the complexification $V_{\mathbf{C}}$. Finally, define the following natural subgroup:

$$H^k_A(V; \mathbf{Z}_2)^2 = \{\alpha^2 \mid \alpha \in H^k_A(V; \mathbf{Z}_2)\}$$

of $H^{2k}_A(V; \mathbf{Z}_2)$ (since cup product operation preserves algebraic cycles, [AK1])

THEOREM A. *For all k the following hold:*
(a) $H^{2k}_{\mathbf{C}-alg}(V; R) \subset H^{2k}_A(V; R)$, *where* $R = \mathbf{Z}_2$ (*or* $\mathbf{Z}$ *when* V *is orientable*).
(b) $\bar{H}^{2k}_{\mathbf{C}-alg}(V; \mathbf{Z}_2) = H^k_A(V; \mathbf{Z}_2)^2$.

Proof. To prove (a) let $a \in H^{2k}_{\mathbf{C}-alg}(V; R)$ be represented by the restriction of $\alpha \in H^{2k}_A(V_{\mathbf{C}}; R)$. Let $\beta \in H^A_{2v-2k}(V_{\mathbf{C}}; R)$ be the Poincaré dual of α in $V_{\mathbf{C}}$. Recall that the map $j_!$ induced by the restriction and the Poincaré duality maps:

$$\begin{array}{ccc} H^{2k}(V_{\mathbf{C}}; R) & \xrightarrow{j^*} & H^{2k}(V; R) \\ \cong\downarrow & & \downarrow\cong \\ H_{2v-2k}(V_{\mathbf{C}}; R) & \xrightarrow{j_!} & H_{v-2k}(V; R) \end{array}$$

is the homology intersection with the fundamental cycle $[V]$, i.e., $j_!(\beta)$ is obtained by transversally intersecting V and a representative of β.

By definition β is represented by $g_*([S])$, where S is a compact nonsingular complex algebraic set and $g : S \to V_{\mathbf{C}}$ is an entire rational function. We can ϵ-isotop g to a smooth function $g_0 : S \to V_{\mathbf{C}}$ which is transverse to $V \subset V_{\mathbf{C}}$. By [AK1] Proposition 2.8.8, we can find a nonsingular real algebraic set S' and a rational diffeomorphism $\pi : S' \to S$ and a rational map $F : S' \to V_{\mathbf{C}}$ such that $g_0 \circ \pi$ is ϵ-close to F (here we are viewing S and $V_{\mathbf{C}}$ as real algebraic sets by thinking $\mathbf{C}$ as $\mathbf{R}^2$). Hence F is transverse to V. If $T = F^{-1}(V)$ and $f : T \to V$ is the restriction of F, then $f_*[T]$ represents the Poincaré dual of a.

To see (b) let $a \in \bar{H}^{2k}_{\mathbf{C}-alg}(V; \mathbf{Z}_2)$. Then a is the restriction of a class in $H^{2k}_A(V_{\mathbf{C}}; \mathbf{Z}_2)$ whose Poincaré dual in $H^A_{2v-2k}(V_{\mathbf{C}}; \mathbf{Z}_2)$ can be represented by $g_*[L_{\mathbf{C}}]$, where $g : L_{\mathbf{C}} \to V_{\mathbf{C}}$ is an entire rational function from a nonsingular compact complex algebraic set which is the complexification of a $v-k$ dimensional real algebraic set L.

We first prove (b) under the assumption that g is an inclusion $L_{\mathbf{C}} \subset V_{\mathbf{C}}$ of a nonsingular algebraic subset: We first ϵ-isotop g to a smooth function $g_0 : L_{\mathbf{C}} \to V_{\mathbf{C}}$ which is transverse to $V \subset V_{\mathbf{C}}$. As before, by viewing $L_{\mathbf{C}}$ as a real algebraic set we can find a nonsingular real algebraic set L' and a rational diffeomorphism $\pi : L' \to L_{\mathbf{C}}$ and a rational map $F : L' \to V_{\mathbf{C}}$ such that $g_0 \circ \pi$ is ϵ-close to F. So F is transverse to V. If $T = F^{-1}(V)$ and $f : T \to V$ is the restriction of F, then $f_*[T]$ represents the Poincaré dual of a.

We claim that $f_*[T]$ is also the self intersection of the homology cycle $g_*[L]$ in V. In other words a is the cup product square of the dual of the map $g_*[L]$. To see this observe that the normal bundle of $V \subset V_{\mathbf{C}}$ is isomorphic to the tangent bundle of V (given by the multiplication by $\sqrt{-1}$). Hence the tubular neighborhoods of $V \subset V_{\mathbf{C}}$ and the diagonal $\Delta_V \subset V \times V$ are diffeomorphic. So for the purpose of computing $F^{-1}(V)$ we can assume that $F : L' \to V \times V$. Let $F = (F_1, F_2)$. Then

$$T = F^{-1}(V) = \{x \mid F_1(x) = F_2(x)\}.$$

But since $g \circ \pi$ is close to F and the maps F_1 and F_2 are generic this set is also the transverse self intersection of the homology cycle $g_*[L]$. We see this by looking carefully at the map F. First we look at the map g. We may identify a neighborhood of L in $L_{\mathbf{C}}$ with a neighborhood of the diagonal Δ_L in $L \times L$. Then in a neighborhood of Δ_L, the map g is given by $(x, y) \mapsto (g(x), g(y))$ (since locally g is an inclusion $\mathbf{C}^{v-k} \subset \mathbf{C}^v$). Thus our algebraic approximation F is given by $F(x, y) = (F_1(x, y), F_2(x, y))$ where F_1 approximates $g(x)$ and F_2 approximates $g(y)$. Consequently, $F^{-1}(\Delta_V)$ represents the Poincaré dual of the cup product of the Poincaré duals of $g_*(L)$ with itself.

Conversely if $a \in H_A^k(V; \mathbf{Z}_2)^2$, then $a = \alpha^2$ with $\alpha \in H_A^k(V; \mathbf{Z}_2)$. The dual of α is represented by a $v-k$ dimensional real algebraic set $L \subset V$. Let $L_\mathbf{C} \subset V_\mathbf{C}$ be the complexification of L. Then by applying the above argument to the inclusion map $g : L_\mathbf{C} \to V_\mathbf{C}$ we see that the restriction of the dual of $g_*[L_\mathbf{C}]$ to V is α^2, i.e., $a \in \bar{H}^{2k}_{\mathbf{C}-alg}(V; \mathbf{Z}_2)$.

Now in the general case, a is represented by restricting the dual of the fundamental class $[L_\mathbf{C}]$ of a possibly singular algebraic subset $L_\mathbf{C} \subset V_\mathbf{C}$. Let $\pi_\mathbf{C} : \tilde{V}_\mathbf{C} \to V_\mathbf{C}$ be a resolution of $V_\mathbf{C}$ turning $L_\mathbf{C}$ into a nonsingular subset $\tilde{L}_\mathbf{C} \subset \tilde{V}_\mathbf{C}$. In particular the restriction map $\pi : \tilde{V} \to V$ resolves L to $\tilde{L}$. Since $\pi_\mathbf{C}$ and π are degree one maps in $\mathbf{Z}$ and $\mathbf{Z}_2$ coefficients respectively, the following commutes:

$$\begin{array}{ccccccccc}
H_{2v-2k}(\tilde{V}_\mathbf{C}) & \xleftarrow{D} & H^{2k}(\tilde{V}_\mathbf{C}) & \xrightarrow{j^*} & H^{2k}(\tilde{V}) & \xleftarrow{Sq^k} & H^k(\tilde{V}) & \xrightarrow{D} & H_{v-k}(\tilde{V}) \\
\downarrow \pi_* & & \uparrow \pi^* & & \uparrow \pi^* & & \uparrow \pi^* & & \downarrow \pi_* \\
H_{2v-2k}(V_\mathbf{C}) & \xleftarrow{D} & H^{2k}(V_\mathbf{C}) & \xrightarrow{j^*} & H^{2k}(V) & \xleftarrow{Sq^k} & H^k(V) & \xrightarrow{D} & H_{v-k}(V)
\end{array}$$

In this abbreviated diagram, the two left vertical maps are induced by $\pi_\mathbf{C}$, and the homologies of the complex algebraic sets are taken with $\mathbf{Z}$ coefficients and the real algebraic sets with $\mathbf{Z}_2$ coefficients. D denotes the Poincaré duality isomorphisms, and Sq^k is the Steenrod square, i.e. in our case $Sq^k(\theta) = \theta^2$. Also j^* denotes the composition map: $\mathbf{Z}_2$ reduction followed by the restriction. By the previous nonsingular case we have:

$$j^*D^{-1}[\tilde{L}_\mathbf{C}] = Sq^k D^{-1}[\tilde{L}] \qquad (*)$$

We need to show that $j^*D^{-1}[L_\mathbf{C}] = Sq^k D^{-1}[L]$; but since $\pi_\mathbf{C}$ is a composition of blowups along nonsingular centers, it suffices to prove the equality in the case where $\pi_\mathbf{C}$ is a single blowup along a nonsingular center $X_\mathbf{C} \subset V_\mathbf{C}$. We will prove this by induction on the dimension of V.

By substituting Lemma 3 to the identity $(*)$ we see for some algebraic subset $Z_\mathbf{C}$ of the exceptional locus $\tilde{X}_\mathbf{C} = \pi_\mathbf{C}^{-1}(X_\mathbf{C})$ (so $Z \subset \tilde{X} = \pi^{-1}(X)$)

$$j^*\pi^*D^{-1}[L_\mathbf{C}] + j^*D^{-1}[Z_\mathbf{C}] = Sq^k\pi^*D^{-1}[L] + Sq^kD^{-1}[Z], \qquad \text{hence}$$

$$\pi^*(j^*D^{-1}[L_\mathbf{C}] - Sq^kD^{-1}[L]) + j^*D^{-1}[Z_\mathbf{C}] - Sq^kD^{-1}[Z] = 0$$

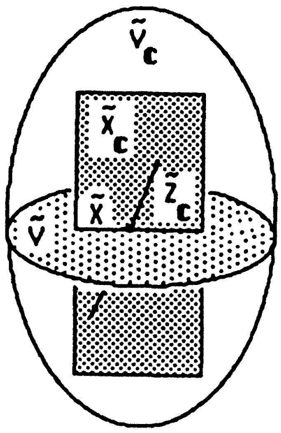
Since π is degree one π^* is an injection, hence it suffices to prove $j^*D^{-1}[Z_\mathbf{C}] = Sq^kD^{-1}[Z]$. To see this, consider the inclusions:

$$\begin{array}{ccc}
\tilde{V}_\mathbf{C} & \xleftarrow{j} & \tilde{V} \\
\uparrow i & & \uparrow \tilde{i} \\
\tilde{X}_\mathbf{C} & \xleftarrow{\tilde{j}} & \tilde{X}
\end{array}$$

By making i transversal to $\tilde{V}$ and calling $i^{-1}(\tilde{V}) = Q$ we obtain the inclusions:

$$\begin{array}{ccc} \tilde{V}_{\mathbf{C}} & \xleftarrow{\ j\ } & \tilde{V} \\ \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle I} \\ \tilde{X}_{\mathbf{C}} & \xleftarrow{\ J\ } & Q \end{array}$$

By the above discussion on the nonsingular case in fact $Q = \chi(v_{\tilde{i}}) = \tilde{X} \cap \tilde{X}$, i.e. Q is the transverse self intersection of $\tilde{X}$ in $\tilde{V}$. Also $I = \tilde{i} \circ i_0$ and $J = \tilde{j} \circ i_0$ where $i_0 : Q \hookrightarrow \tilde{X}$ is the inclusion.



Define $\Phi : H^{2k-2}(\tilde{X}; \mathbf{Z}_2) \to H^{2k}(\tilde{V}; \mathbf{Z}_2)$ be the map $\Phi(x) = u_{\tilde{X}} \smile \tilde{i}_+(x)$. We claim that the following diagram commutes:

$$\begin{array}{ccccccccc} H_{2v-2k}(\tilde{V}_{\mathbf{C}}) & \xleftarrow{D} & H^{2k}(\tilde{V}_{\mathbf{C}}) & \xrightarrow{j^*} & H^{2k}(\tilde{V}) & \xleftarrow{Sq^k} & H^k(\tilde{V}) & \xrightarrow{D} & H_{v-k}(\tilde{V}) \\ \uparrow{\scriptstyle i_*} & & \uparrow{\scriptstyle i_+} & & \uparrow{\scriptstyle \Phi} & & \uparrow{\scriptstyle \tilde{i}_+} & & \uparrow{\scriptstyle \tilde{i}_*} \\ H_{2v-2k}(\tilde{X}_{\mathbf{C}}) & \xleftarrow{D} & H^{2k-2}(\tilde{X}_{\mathbf{C}}) & \xrightarrow{\tilde{j}^*} & H^{2k-2}(\tilde{X}) & \xleftarrow{Sq^{k-1}} & H^{k-1}(\tilde{X}) & \xrightarrow{D} & H_{v-k}(\tilde{X}) \end{array}$$

As above the homologies and cohomologies of the complex algebraic sets are taken with $\mathbf{Z}$ coefficients and the real algebraic sets with $\mathbf{Z}_2$ coefficients, and j^* denotes the composition map: $\mathbf{Z}_2$ reduction followed by the map induced by j.

Now given this, we can finish the proof as follows: Since $Z_{\mathbf{C}}$ lies in $X_{\mathbf{C}}$ we can write $[Z_{\mathbf{C}}] = i_*[Z_{\mathbf{C}}]$ and $[Z] = \tilde{i}_*[Z]$. Since $\dim(\tilde{X}) = v - 1$ by induction $\tilde{j}^*D^{-1}[Z_{\mathbf{C}}] = Sq^{k-1}D^{-1}[Z]$. This with the commutativity of the diagram implies $j^*D^{-1}i_*[Z_{\mathbf{C}}] = Sq^kD^{-1}i_*[Z]$.

It remains to check the commutativity of the diagram. By Lemma 2(a), (d)

$$Sq^k\tilde{i}_+(x) = \tilde{i}_+(x) \smile \tilde{i}_+(x) = \tilde{i}_+(x^2) \smile \tilde{i}_+(1) = \Phi(x^2) = \Phi Sq^{k-1}(x).$$

By Lemma 1(b) and Lemma 2(b)

$$j^*i_+(x) = I_+J^*(x) = (\tilde{i} \circ i_0)_+(\tilde{j} \circ i_0)^*(x) = \tilde{i}_+(i_0)_+i_0^*(\tilde{j}^*(x)) = \tilde{i}_+(\tilde{j}^*(x) \smile u_Q).$$

Being over $\mathbf{Z}_2$ coefficients, in the last term we can commute the cup products, also by using Lemma 2(c), (a) and Lemma 1(a):

$$\tilde{i}_+(u_Q \smile \tilde{j}^*(x)) = \tilde{i}_+(\tilde{i}^*\tilde{i}_+(1) \smile \tilde{j}^*(x)) = \tilde{i}_+(\tilde{i}^*(u_{\tilde{X}}) \smile \tilde{j}^*(x)) = u_{\tilde{X}} \smile \tilde{i}_+(\tilde{j}^*(x)).$$

Hence we have shown $j^*i_+(x) = \Phi\tilde{j}^*(x)$.

Finally to start the induction observe that for algebraic sets V of dimension $v - k + 1$ any homology class $[L]$ of dimension $v - k$ has a nonsingular representative, so the proof in this case follows from the first part of the theorem. To see this observation, pick a codimension one closed smooth submanifold $S \subset V$ homologous to L. Then since the homology class $[S] = [L] \in H^A_{v-k}(V; \mathbf{Z}_2)$ is algebraic, the submanifold S can be isotoped to a nonsingular algebraic subset (e.g. [AK1] Theorem 2.8.2). □

Remark. By defining $H^{\mathbf{C}-alg}_{2k}(V; \mathbf{Z}) = j_! H^A_{2k}(V_{\mathbf{C}}; Z)$, even when V is nonorientable we can restate (a) in a slightly stronger form:

$$H^{\mathbf{C}-alg}_{2k}(V; \mathbf{Z}) \subset H^A_{2k}(V; \mathbf{Z}).$$

A useful corollary to the theorem is that we can estimate the number of complex algebraic cycles of a nonsingular complex algebraic set in terms of the real algebraic cycles of the real part:

COROLLARY 1. rank $H^{2k}_A(V_{\mathbf{C}}; \mathbf{Z}) \geq$ rank $H^k_A(V; \mathbf{Z}_2)^2$.

It is well known that the duals of Steifel–Whitney classes of any compact nonsingular real algebraic set are represented by algebraic subsets. This is because the Grassmanian $G(n, k)$ of unoriented k planes in $\mathbf{R}^n$ is a nonsingular algebraic set in such a way that all the Steifel–Whitney classes are represented by algebraic subsets and the (tangent and normal) Gauss map $\alpha : V \to G(n, k)$ is entire rational (cf., [AK3], [AK1]). It is also well known that the Chern classes of a complex algebraic set are algebraic (cf., [F]). Since $p_k(V) = (-1)^k j^* c_{2k}(V_{\mathbf{C}})$ then Pontryagin classes are in $H^*_{\mathbf{C}-alg}(V; Z)$.

COROLLARY 2. *The duals of Pontryagin classes of V are represented by real algebraic subsets of V (in the unoriented case dual means the dual of* mod 2 *reduction with $\mathbf{Z}_2$ coefficient).*

Recall that under the additional assumption: either $2k \leq 2v - n$ or $V_{\mathbf{C}}$ is a complete intersection, for all $2k < v$ the group $H^{2k}_{\mathbf{C}-alg}(V_{\mathbf{C}}; \mathbf{Z})$ is equal to the image of the restriction homomorphism (see [BBK]):

$$H^{2k}(\mathbf{RP}^n; \mathbf{Z}) \to H^{2k}(V; \mathbf{Z}).$$

COROLLARY 3. *If $V \subset \mathbf{R}^n$ (here we are identifying $\mathbf{R}^n \subset \mathbf{RP}^n$) and either $2k \leq 2v - n$ or $V_{\mathbf{C}}$ is a complete intersection, then no element $\alpha \in H^k(V; \mathbf{Z}_2)$ with $2k < v$ and $\alpha^2 \neq 0$ can be algebraic.*

This corollary has the following amusing consequence:

THEOREM B. *There exist closed smooth submanifolds $M \subset \mathbf{R}^n$ which can not be isotoped to the real parts of any nonsingular complex algebraic subvarieties of $\mathbf{CP}^n$.*

Proof. Pick $M^m \subset \mathbf{R}^n$ with $n = 2m - s$, and $c_k \in H^k(M; \mathbf{Z}_2)$ such that:
(i) $k \leq s/2$.
(ii) $c_k^2 \neq 0$.
(iii) c_k is either a Steifel–Whitney class or a $\mathbf{Z}_2$ reduction of a Ponryagin class of the tangent or normal bundle.

We claim that M can not be isotopic to the real part V of a nonsingular complex algebraic set in $\mathbf{CP}^n$. Otherwise, by Corollary 3 the class c_k could not be algebraic; on the other hand by Corollary 2 and the preceding discussion c_k would have to be algebraic. Contradiction.

It remains to find examples of M satisfying the above properties. Real or Complex projective spaces could be imbedded in this way ([J]). For example $\mathbf{RP}^{10} \subset \mathbf{R}^{18}$ ([Ha]), in which case we take $k = 1$ and c_1 the first Steifel–Whitney class $w_1(M)$. More generally, for any s there exists m such that there are imbeddings $\mathbf{RP}^m \subset \mathbf{R}^{2m-s}$ ([MM]). We claim that we can choose some of these M to be a nonsingular algebraic subset of $\mathbf{R}^n$. To see that first choose q so that $\mathbf{RP}^q \subset \mathbf{R}^{2q-5}$. If q is even we choose $M = \mathbf{RP}^q$ with $k = 1$ and $c_1 = w_1(M)$, otherwise we choose $M = \mathbf{RP}^{q-1} \subset \mathbf{RP}^q \subset \mathbf{R}^{2(q-1)-3}$. In any case, in our example we can assume that $M^m \subset \mathbf{R}^{2m-3}$. Hence by [AK4] we can isotop M to a nonsingular algebraic subset of $\mathbf{R}^{2m-2}$. □

On the positive side we can prove the following:

COROLLARY 4. *If $M \subset \mathbf{RP}^n$ is a topological complete intersection, that is if it is an intersection $\bigcap L_i$ of smooth codimension one submanifolds of $\mathbf{RP}^n$ in general position, then M is isotopic to the real part V of a nonsingular complex complete* *intersection* $V_{\mathbf{C}}$ *in* $\mathbf{CP}^n$. *Furthermore, when* $M \subset \mathbf{R}^n$ *then for any* $\alpha \in H_A^k(V; \mathbf{Z}_2)$ *with* $2k < v$ *has the property* $\alpha^2 = 0$.

Proof. We first isotop each L_i to a nonsingular hypersurface V_i in $\mathbf{RP}^n$ (this is possible since the group $H_{n-1}(\mathbf{RP}^n; \mathbf{Z}_2)$ is algebraic, see [AK1]), then change the coefficients of the defining equations of each V_i a little so that the complex solutions become nonsingular and transverse to each other without affecting the isotopy type of $\bigcap V_i \approx M$. The last requirement follows from the above discussion. □

As an application to this theorem we see that we can isotop $\mathbf{RP}^3$ a nonsingular real algebraic subset V of $\mathbf{R}^5$ such that $H_A^1(V; \mathbf{Z}_2) = 0$, a fact previously proven in [BBK]. According to [BK2] any closed smooth manifold M is diffeomorphic to a nonsingular algebraic set V with $H^2_{\mathbf{C}-alg}(V; \mathbf{Z}) = H^2(M; \mathbf{Z})$. According to [BD] if M is a smooth manifold approximating to a large finite skeleton of $K(\mathbf{Z}_2, 2)$, then for any nonsingular algebraic set V diffeomorphic to M we must have $H_A^2(V; \mathbf{Z}_2) = 0$. These two results together appear to contradict (a) of Theorem A. The reason they are consistent is that $H^2(K(\mathbf{Z}_2, 2); \mathbf{Z})$ and hence its $\mathbf{Z}_2$ reduction is zero.

Remark. Some results in this paper were announced in [A]. The reader should be warned that in [A] the distinction between the groups $\bar{H}^{2k}_{\mathbf{C}-alg}(V; \mathbf{Z})$ and $H^{2k}_{\mathbf{C}-alg}(V; \mathbf{Z})$ is intentionally suppressed. Also G. Mikhalkin independently observed a special case of (b) of Theorem A.

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