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Estimates for the energy of a symplectic map

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1. Introduction

In [9] the author introduced a bi-invariant metric d for the compactly supported symplectic diffeomorphism group and studied its relevance in symplectic geometry. This metric, though at the same time primarily introduced to study generalized symplectic fixed point problems, turned out to be the natural measure of distance in studying to what extent does the boundary of a symplectic manifold reflect properties of its symplectic interior, see [6, 5, 7]. Shortly afterwards, J. Moser (private communication) observed that certain quantities (e.g. mean action) in Aubry–Mather-theory, see [13, 14], depend continuously on this metric. J. Moser and J. C. Sikorav, [15], independently raised the question if it is possible to estimate the d-distance in terms of C^0-data.

The aim of this paper is to provide such an estimate. We also take the opportunity to derive a good estimate for d from below. Related estimates from below can also be obtained by combining results in [3] and [9], see also [2] and [9]. However, the present approach gives some interesting new inequality in symplectic geometry and the proof is quite simple. This inequality can be taken as the single starting point for developing the symplectic C^0-rigidity theory as well as the existence theory for periodic orbits with prescribed energy.

A survey, based on this point of view, describing the recent developments in symplectic geometry and topology, will appear elsewhere, [10]. In order to state the main result we have to introduce some notation. Let $\mathscr{C}$ be the vectorspace of all compactly supported smooth maps $H : [0, 1] \times \mathbb{C}^n \to \mathbb{R}$. For $p \in [1, +\infty]$ we define norms $\| \ \|_p$ on $\mathscr{C}$ by

$$\begin{aligned} \|H\|_p &= \left(\int_0^1 |e_H(t)|^p \, dt\right)^{1/p} && \text{for } p \in [1, +\infty) \\ \|H\|_\infty &= \max_{t \in [0,1]} |e_H(t)| && \text{for } p = +\infty \end{aligned} \tag{1}$$

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with

$$e_H(t) = \max_{x \in \mathbb{C}^n} H(t, x) - \inf_{x \in \mathbb{C}^n} H(t, x). \tag{2}$$

Clearly the following inequality holds for every $H \in \mathscr{C}$

$$\|H\|_1 \leq \|H\|_p \leq \|H\|_\infty.$$

We view $\mathbb{C}^n$ as a real vectorspace with symplectic form $\omega = -\mathrm{Im}\,(\cdot, \cdot)$, where $(\cdot, \cdot)$ is the standard Hermitian inner product. Given $H \in \mathscr{C}$ we define the associated Hamiltonian vectorfield by

$$i_{X_{H_t}}\omega = dH_t,$$

and a symplectic arc $(\Psi_t^H)_{t \in [0,1]}$, by

$$\frac{d}{dt}\Psi_t^H = X_{H_t}(\Psi_t^H), \qquad \Psi_0^H = \mathrm{Id}.$$

Given $H \in \mathscr{C}$ we define the time-1-map Ψ_H by $\Psi_H := \Psi_1^H$. The collection $\mathscr{D}$ of all time-1-maps is a group. We introduce a scale of energies $(E_p)_{p \in [1, +\infty]}$ as follows

$$\begin{aligned} &E_p : \mathscr{D} \to [0, +\infty) \\ &E_p(\Psi) = \inf\{\|H\|_p \mid \Psi_H = \Psi, H \in \mathscr{C}\}. \end{aligned} \tag{3}$$

One easily verifies that

$$\begin{aligned} &E_p(\Psi) = E_p(\Psi^{-1}) = E_p(\Phi\Psi\Phi^{-1}) \\ &E_p(\Psi\Phi) \leq E_p(\Psi) + E_p(\Phi) \end{aligned} \tag{4}$$

for all $\Phi, \Psi \in \mathscr{D}$. The crucial property, which follows from results in [9] is that

$$E_p(\Psi) = 0 \Leftrightarrow \Psi = \mathrm{Id}. \tag{5}$$

This will also be a consequence of theorem 2 below. Using (4) and (5) it follows immediately that

$$d_p(\Psi, \Phi) := E_p(\Psi^{-1}\Phi)$$

defines a bi-invariant metric on $\mathscr{D}$. Observe that the completions of the $(\mathscr{D}, d_p)$ are again groups, since d_p is bi-invariant.

Our first result in this paper is the following estimate from above by C^0-data.

THEOREM 1. *The following estimate holds for* d_∞.

$$d_\infty(\Phi, \Psi) \leq 256 \cdot \operatorname{diam}(\operatorname{supp}(\Phi\Psi^{-1}))\|\Phi - \Psi\|_{C^0} \tag{6}$$

for all $\Phi, \Psi \in \mathscr{D}$. *Here* $\operatorname{diam}(Q)$ *is the diameter of a subset* Q *of* $\mathbb{C}^n$ *and* $\|\Psi - \Phi\|_{C^0} = \sup_{x \in \mathbb{C}^n} |\Psi(x) - \Phi(x)|$.

Since $d_1 \leq d_p \leq d_\infty$ the above estimate holds for every $d_p, p \in [1, +\infty]$. The reader will observe later that the proof of theorem 1 can be adapted for example to estimate the energy in terms of the diameter of the support and certain Sobolev norms. For example in $\mathbb{C}$ the following type of estimate seems to be true for $\Phi, \Psi \in \mathscr{D}$

$$d_\infty(\Phi, \Psi) \leq c \operatorname{diam}(\operatorname{supp}(\Phi^{-1}\Psi))\|\Phi - \Psi\|_{W^{1,2}},$$

with

$$\|\Phi - \Psi\|_{W^{1,2}} = \left(\sum_{|\alpha| \leq 1} \int |D^\alpha\Phi(x) - D^\alpha\Psi(x)|^2 \, dx\right)^{1/2}$$

for a universal constant c. So the $W^{1,2}$-completion of the area preserving maps with support in the unit disk can be considered as a subset of the completion $\tilde{\mathscr{D}}_\infty$ of $(\mathscr{D}, d_\infty)$. Is the abstract inverse of such a map in $\tilde{\mathscr{D}}_\infty$ again of class $W^{1,2}$? It would be interesting to have more estimates of the above type.

In the same way distributions generalize functions, the groups $\tilde{\mathscr{D}}_p$ seem to generalize symplectic maps. It follows from theorem 2 below that the geometry of $\tilde{\mathscr{D}}_p$ is closely tight to phase space geometry.

In order to proceed further we denote by $\mathscr{A}$ the subset of $\mathscr{C}$ consisting of all autonomous Hamiltonians with compact support. For $H \in \mathscr{A}$ all the norms $\|\mathscr{H}\|_p$ coincide. Similarly to a construction given in [12] we define a quantity $c(\mathscr{U})$ for every open subset $\mathscr{U}$ of $\mathbb{C}^n$. Denote by $\mathscr{A}_{ad}$ the subset of $\mathscr{A}$ consisting of all Hamiltonian H, such that every T-periodic solution x of $\dot{x} = X_H(x)$ for $T \in [0, 1]$ is constant. Given an open subset $\mathscr{U}$ of $\mathbb{C}^n$ we define

$$\mathscr{A}_{ad}(\mathscr{U}) = \{H \in \mathscr{A}_{ad} \mid \operatorname{supp}(H) \subset \mathscr{U}\}.$$

We note that $\mathcal{A}_{ad}(\emptyset) = \{0\}$. Then we put

$$c(\mathcal{U}) = \sup\{\|H\|_\infty \mid H \in \mathcal{A}_{ad}(\mathcal{U}) \text{ and } H \leq 0\}. \tag{7}$$

It is an easy exercise, see [12], to show that $c(B^{2n}(1)) \geq \pi$. It is however nontrivial to show that $c(B^{2n}(1)) \leq \pi$. But this will be a corollary of theorem 2.

THEOREM 2. *For every* $\Psi \in \mathcal{D}$ *and open subset* $\mathcal{U}$ *of* $\mathbb{C}^n$, *such that* $\Psi(\mathcal{U}) \cap \mathcal{U} = \emptyset$ *we have the estimate*

$$c(\mathcal{U}) \leq E_1(\Psi).$$

In particular

$$\sup\{c(\mathcal{U}) \mid \mathcal{U} \subset \mathbb{C}^n \text{ open}, \Psi(\mathcal{U}) \cap \mathcal{U} = \emptyset\} \leq E_1(\Psi).$$

As a by-product of the proof of theorem 2 we obtain the following result for autonomous Hamiltonians.

THEOREM 3. *Let* $H \in \mathcal{A}_{ad}$, *then for every* $p \in [1, +\infty]$ *we have the equality*

$$E_p(\Psi_H) = \|H\|_\infty.$$

Finally I would like to point out that C. Viterbo [16], motivated by [9], constructed a *homological energy function*. It would be interesting to understand the relationships.

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2. Localization of symplectic maps

Let $S_1, \ldots, S_k$, $k \geq 2$ be subsets of $\mathbb{C}^n$. We say $S_1, \ldots, S_k$ are properly separated provided for every choice of bounded subsets $B_i \subset S_i$, $i = 1, \ldots, k$, there exist parallel hyperplanes $\Sigma_1 := \mathbb{C}^{n-1} \oplus \mathbb{R}, \ldots, \Sigma_l := a_l + \Sigma_1, \ldots, \Sigma_{k-1}$, and a symplectic map $\tau \in \mathcal{D}$, such that the sets $\tau(B_1), \ldots, \tau(B_k)$ are pairwise contained in different components of $\mathbb{C}^n \backslash (\bigcup_{i=1}^{k-1} \Sigma_i)$.

For a given number $p \in [1, +\infty]$ we define the proper displacement energy $e_p(S)$ of a subset S of $\mathbb{C}^n$ as follows:

$$\begin{aligned} e_p(S) = \inf \{a > 0 \mid &\text{For every bounded subset } A \subset S \text{ there} \\ &\text{exists } \Psi \in \mathscr{D} \text{ with } E_p(\Psi) \leq a, \\ &\text{and } A \text{ and } \Psi(A) \text{ are properly} \\ &\text{separated}\}. \end{aligned} \tag{8}$$

Obviously $e_p(S) = e_p(\Psi(S))$ for every $\Psi \in \mathscr{D}$ and $S \subset \mathbb{C}^n$. It is another easy exercise to show that

$$e_p(B^2(1) \times \mathbb{C}^{n-1}) \leq \pi, \qquad \text{for } p \in [1, +\infty] \tag{9}$$

(see [9]). For the proof of theorem 1 the following type of problem turns out to be important. Assume $\Psi \in \mathscr{D}$ and a subset $Q \subset \mathbb{C}^n$ and a number $\lambda \geq 0$ are given. We say $\Psi \mid Q$ is localizable in a subset $\mathscr{U} \subset \mathbb{C}^n$ with E_p-bound λ provided there exists $\Phi \in \mathscr{D}$, such that

$$\begin{aligned} &\Phi \mid Q = \Psi \mid Q \\ &\operatorname{supp}(\Phi) \subset \mathscr{U} \\ &E_p(\Phi) \leq \lambda. \end{aligned} \tag{10}$$

The key point for the proof of theorem 1 is to find on suitable sets Q localizations in convenient sets $\mathscr{U}$ with small energy.

Assume $\Psi \in \mathscr{D}$, $\Psi \neq \mathrm{Id}$, is given. For $x_0 \in \operatorname{supp}(\Psi)$ we denote by $C_{x_0}(\Psi)$ the set of all points $x \in \mathbb{C}^n$ which can be written in the form

$$x = (1-t)x_0 + tz \tag{11}$$

for some $t \in [0, 1]$ and $z \in \operatorname{supp}(\Psi)$. Clearly $C_{x_0}(\Psi) \supset \operatorname{supp}(\Psi)$ and $C_{x_0}(\Psi)$ is starshaped with respect to x_0. We need the following lemma. The proof is straight forward and left to the reader.

LEMMA 4. *Let $\Phi_1, \ldots, \Phi_k$ be elements in $\mathscr{D}$ such that* supp $(\Phi_1), \ldots,$ supp (ϕ_k) *are properly separated. Then*

$$E_\infty\left(\prod_{i=1}^{k} \Phi_k\right) \leq 2 \max \{E_\infty(\Phi_1), \ldots, E_\infty(\Phi_k)\}.$$

In [15] J. C. Sikorav stated the estimated $E_\infty(\Phi_1\Phi_2) \leq \max\{E_\infty(\Phi_1), E_\infty(\Phi_2)\}$, which however is not correct. The next lemma which is due to J. C. Sikorav, [15], relied in its original proof on the above wrong estimate. However replacing it by Lemma 4 all arguments work, only the constant 8 has to be replaced by 16. Sikorav's proof, although quite short, is tricky and the result itself rather counter-intuitive. We give the proof for the convenience of the reader.

LEMMA 5 (Sikorav's Estimate). *Let $H \in \mathscr{C}$ be supported in $[0, 1] \times \mathscr{U}$. Then we have the estimate*

$$E_\infty(\Psi_H) \leq 16 e_\infty(\mathscr{U}).$$

In other words: the energy of a symplectic map Ψ can be estimated through the proper displacement energy of the smallest support one needs to generate Ψ.

Proof. Using the time evolution for the Hamiltonian H we find for given $\tau > 0$ a finite sequence $\Psi_k \in \mathscr{D}$, $k = 0, \ldots, N$ such that

$$\begin{aligned} &\Psi_0 = \mathrm{Id}, \qquad \Psi_N = \Psi_H \\ &\mathrm{supp}(\Psi_k) \subset S \subset \mathscr{U} \qquad \text{for } k = 0, \ldots, N \\ &d_\infty(\Psi_k, \Psi_{k+1}) < \tau \qquad \text{for } k = 0, \ldots, N-1, \end{aligned} \tag{12}$$

where S is a bounded subset of $\mathscr{U}$. We find a sequence $\Phi_0, \ldots, \Phi_{2N}$ in $\mathscr{D}$ with $\Phi_0 = \mathrm{Id}$, $\Phi_1 = \Phi$, such that for $j = 1, \ldots, 2N$ the Φ_j's are pairwise conjugated to each other, and the sets $S_i = \Phi_i(S)$ for $i = 0, \ldots, 2N$ are pairwise properly separated. Here $\Phi \in \mathscr{D}$ is chosen in such a way that

$$\Phi(S) \text{ and } S \text{ are properly separated} \tag{13}$$

and

$$\begin{aligned} E_p(\Phi) &< e_\infty(S) + \tau \\ &\leq e_\infty(\mathscr{U}) + \tau. \end{aligned} \tag{14}$$

We define for $i = 1, \ldots, N$, maps $\alpha_i \in \mathscr{D}$ by

$$\alpha_i = \Phi_{2i-1}\Psi_i\Phi_{2i-1}^{-1} \tag{15}$$

and for $i = 0, \ldots, N$ maps $\beta_i \in \mathscr{D}$ by

$$\beta_i = \Phi_{2i} \Psi_i^{-1} \Phi_{2i}^{-1}. \tag{16}$$

Clearly we have

$$\begin{aligned} \operatorname{supp}(\alpha_i) &\subset S_{2i-1} \qquad \text{for } i = 1, \ldots, N \\ \operatorname{supp}(\beta_i) &\subset S_{2i} \qquad \text{for } i = 0, \ldots, N. \end{aligned}$$

By construction any two different symplectic maps among the $\alpha_1, \ldots, \alpha_N$, $\beta_1, \ldots, \beta_N$ have properly separated supports and therefore they commute. In view of lemma 4 we obtain

$$E_\infty\left(\prod_{i=1}^{N} \alpha_i \beta_i\right) \leq 2 \max \{E_\infty(\alpha_i \beta_i) \mid i = 1, \ldots, N\} \tag{17}$$

and

$$E_\infty\left(\prod_{i=0}^{N-1} \alpha_{i+1} \beta_i\right) \leq 2 \max \{E_\infty(\alpha_{i+1} \beta_i) \mid i = 0, \ldots, N-1\}. \tag{18}$$

We have for $i = 1, \ldots, N$

$$\begin{aligned} \alpha_i \beta_i &= \Phi_{2i-1} \Psi_i \Phi_{2i-1}^{-1} \Phi_{2i} \Psi_i^{-1} \Phi_{2i}^{-1} \\ &=: (\theta_i (\Phi_{2i-1}^{-1} \Phi_{2i}) \theta_i^{-1})(\Phi_{2i-1} \Phi_{2i}^{-1}) \end{aligned} \tag{19}$$

for a suitable $\theta_i \in \mathscr{D}$, and for $i = 0, \ldots, N-1$

$$\begin{aligned} \alpha_{i+1} \beta_i &= \Phi_{2i+1} \Psi_{i+1} \Phi_{2i+1}^{-1} \Phi_{2i} \Psi_i^{-1} \Phi_{2i}^{-1} \\ &= (\theta_{i+1} (\Phi_{2i+1}^{-1} \Phi_{2i}) \theta_{i+1}^{-1})(\Phi_{2i+1} \Psi_{i+1} \Psi_i^{-1} \Phi_{2i}^{-1}) \\ &= (\theta_{i+1} (\Phi_{2i+1}^{-1} \Phi_{2i}) \theta_{i+1}^{-1})(\Phi_{2i+1} (\Psi_{i+1} \Psi_i^{-1}) \Phi_{2i+1}^{-1})(\Phi_{2i+1} \Phi_{2i}^{-1}). \end{aligned} \tag{20}$$

Using conjugacy invariance and the triangle inequalities for E_∞ we obtain from equations (14), (19) and (20) since the Φ_i were all conjugated for $i \geq 1$:

$$\begin{aligned} E_\infty(\alpha_i \beta_i) &\leq E_\infty(\Phi_{2i-1}^{-1} \Phi_{2i}) + E_\infty(\Phi_{2i-1} \Phi_{2i}^{-1}) \\ &\leq 4 E_\infty(\Phi) \\ &\leq 4(e_\infty(\mathscr{U}) + \tau) \end{aligned} \tag{21}$$

and

$$\begin{aligned} E_\infty(\alpha_{i+1}\beta_i) &\le E_\infty(\Phi_{2i+1}^{-1}\Phi_{2i}) + E_\infty(\Psi_{i+1}\Psi_i^{-1}) + E_\infty(\Phi_{2i+1}\Phi_{2i}^{-1}) \\ &\le 4E_\infty(\Phi) + \tau \\ &\le 4(e_\infty(\mathscr{U})) + 5\tau. \end{aligned} \tag{22}$$

Combining this with (17) and (18) gives

$$\begin{aligned} E_\infty\left(\prod_{i=1}^{N} \alpha_i\beta_i\right) &\le 2(4e_\infty(\mathscr{U}) + 4\tau) \\ E_\infty\left(\prod_{i=0}^{N-1} \alpha_{i+1}\beta_i\right) &\le 2(4e_\infty(\mathscr{U}) + 5\tau). \end{aligned} \tag{23}$$

Next we observe that $\Psi_N^{-1} = \Psi_H^{-1}$ and β_N are conjugated so that

$$E_\infty(\Psi_H) = E_\infty(\Psi_H^{-1}) = E_\infty(\beta_N). \tag{24}$$

We write

$$\begin{aligned} \beta_N &= \beta_N\left(\prod_{i=0}^{N-1} \alpha_{i+1}\beta_i\right)\left(\prod_{i=0}^{N-1} \alpha_{i+1}\beta_i\right)^{-1} \\ &= \left(\prod_{i=1}^{N} \alpha_i\beta_i\right)\beta_0\left(\prod_{i=1}^{N-1} \alpha_{i+1}\beta_i\right)^{-1} \\ &= \left(\prod_{i=1}^{N} \alpha_i\beta_i\right)\left(\prod_{i=0}^{N-1} \alpha_{i+1}\beta_i\right)^{-1}. \end{aligned}$$

Combining this with (23) we deduce

$$\begin{aligned} E_\infty(\Psi_H) &\le E_\infty\left(\prod_{i=1}^{N} \alpha_i\beta_i\right) + E_\infty\left(\prod_{i=0}^{N-1} \alpha_{i+1}\beta_i\right) \\ &\le 16e_\infty(\mathscr{U}) + 18\tau. \end{aligned}$$

Since $\tau > 0$ was arbitrarily chosen

$$E_\infty(\Psi_H) \le 16e_\infty(\mathscr{U}). \qquad \square$$

Now we are ready to state the localization result.

PROPOSITION 6. *Let* $\Psi \neq \mathrm{Id}$, $\Psi \in \mathscr{D}$, *and* $\delta > \|\Psi - \mathrm{Id}\|_{C^0}$. *Suppose* Q *is an open nonempty subset of* $\mathbb{C}^n$ *intersecting* $\operatorname{supp}(\Psi)$, *and* $x_0 \in Q \cap \operatorname{int}(\operatorname{supp}(\Psi))$. *Then* $\Psi \mid Q$ *is localizable in* $B_\delta(Q) \cap C_{x_0}(\Psi)$ *with* E_∞*-bound* $\lambda_\infty = 16 e_\infty(B_\delta(Q) \cap C_{x_0}(\Psi))$. *Here* $B_\delta(Q)$ *is the open* δ*-neighbourhood of the set* Q.

Proof. Let $t \to \Psi_t \in \mathscr{D}$, $t \in [0, 1]$, be a smooth arc connecting Id with the given Ψ. For $s \in [0, 1)$ and $t \in [0, 1]$ we define $\Psi_t^s \in \mathscr{D}$ by

$$\Psi_t^s(x) = s x_0 + (1-s)\Psi_t(x_0 + (1-s)^{-1}(x - x_0)). \tag{25}$$

Pick $R > 0$ such that

$$\operatorname{supp}(\Psi_t) \subset B_R(x_0) \qquad \text{for all } t \in [0, 1].$$

We observe that

$$\operatorname{supp}(\Psi_t^s) \subset B_{(1-s)R}(x_0) \tag{26}$$

for all $(t, s) \in [0, 1] \times [0, 1)$. (Note that definition (25) is in some sense the "inverse Alexander trick".) Moreover

$$\operatorname{supp}(\Psi_1^s) \subset C_{x_0}(\Psi) \tag{27}$$

for all $s \in [0, 1)$. Fix a $s_0 \in (0, 1)$ such that

$$\begin{aligned} &B_{(1-s_0)R}(x_0) \subset C_{x_0}(\Psi) \cap Q \\ &(1 - s_0)R \leq \tfrac{1}{3}\|\mathrm{Id} - \Psi\|_{C^0}. \end{aligned} \tag{28}$$

We define a subset $\Gamma \subset \mathbb{R}^2$ by

$$\Gamma = ([0, 1] \times [s_0, 1)) \cup (\{1\} \times [0, 1)).$$

We take a smooth map $\beta := (a, b) : [0, 1] \to \mathbb{R}^2$ satisfying $\beta([0, 1]) \subset \Gamma$ and moreover

$$\begin{aligned} &\beta(0) = (0, s_0) \\ &\beta([0, \tfrac{1}{2}]) \subset [0, 1] \times [s_0, 1) \\ &\beta([\tfrac{1}{2}, 1]) \subset \{1\} \times [0, 1) \\ &\beta(1) = (1, 0). \end{aligned} \tag{29}$$

Using β and $t \to \Psi_t$ we define a smooth arc $t \to \Phi_t$, $t \in [0, 1]$, by

$$\Phi_t := \Psi_{a(t)}^{b(t)}. \tag{30}$$

By the preceding discussion we have

$$\begin{aligned}
&\bullet \operatorname{supp}(\Phi_t) \subset C_{x_0}(\Psi) \qquad \text{for all } t \in [0, 1].\\
&\bullet \operatorname{supp}(\Phi_t) \subset B_{\epsilon/3}(x_0) \subset C_{x_0}(\Psi) \cap Q \qquad \text{for all } t \in [0, \tfrac{1}{2}]\\
&\quad \text{with } \epsilon = \min\{3R(1 - s_0), \|\mathrm{Id} - \Psi\|_{C_0}\}.\\
&\bullet |\Phi_t(x) - x| \leq (1 - b(t))\|\mathrm{Id} - \Psi\|_{C^0} \qquad \text{for all } x \in \mathbb{C}^n\\
&\quad \text{and } t \in [\tfrac{1}{2}, 1].
\end{aligned} \tag{31}$$

Let $\hat{H}$ be the Hamiltonian in $\mathscr{C}$ generating $t \to \Phi_t$. Since $C_{x_0}(\Psi)$ is starshaped, we have $\hat{H}(t, x) = 0$ for all $t \in [0, 1]$ and $x \in \mathbb{C}^n \backslash C_{x_0}(\Psi)$, i.e. $\operatorname{supp}(\hat{H}) \subset [0, 1] \times C_{x_0}(\Psi)$. Let $\gamma : \mathbb{C}^n \to [0, 1]$ be a smooth function satisfying

$$\begin{aligned}
&\gamma \mid \bar{B}_{\|\mathrm{Id} - \Psi\|_{C^0}}(Q) \equiv 1\\
&\gamma \mid (\mathbb{C}^n \backslash \bar{B}_{\hat{\delta}}(Q)) \equiv 0
\end{aligned} \tag{32}$$

for some $\hat{\delta} \in (\|\mathrm{Id} - \Psi\|_{C^0}, \delta)$. In view of equation (31) we note that

$$\Phi_t(Q) \subset B_{\|\mathrm{Id} - \Psi\|_{C^0}}(Q) \tag{33}$$

for all $t \in [0, 1]$. We define a new Hamiltonian $H \in \mathscr{C}$ by

$$H(t, x) = \gamma(x)\hat{H}(t, x). \tag{34}$$

In view of (33) we must have

$$\Psi_H \mid Q \equiv \Phi_1 \mid Q \equiv \Psi \mid Q.$$

Using (31) and (32) we see that

$$\operatorname{supp}(H) \subset [0, 1] \times (C_{x_0}(\Psi) \cap B_\delta(Q)). \tag{35}$$

Hence we have shown the existence of a Hamiltonian H satisfying (35) and

$$\Psi_H \mid Q \equiv \Psi \mid Q, \qquad \operatorname{supp}(\Psi_H) \subset (C_{x_0}(\Psi) \cap B_\delta(Q)).$$

From lemma 5 we obtain

$$E_\infty(\Psi_H) \le 16 e_\infty(C_{x_0}(\Psi) \cap B_\delta(Q)). \tag{36}$$

□

Assume $\mathscr{U} = (a_1, a_2) \oplus i(b_1, b_2) \oplus \mathbb{C}^{n-1}$ with $-\infty < a_1 < a_2 < +\infty$, $-\infty < b_1 < b_2 < +\infty$. Given any $\Psi \in \mathscr{D}$ with $\operatorname{supp}(\Psi) \subset \mathscr{U}$ we can use the first part of the proof of proposition 6 to construct a Hamiltonian $H \in \mathscr{C}$ with $\operatorname{supp}(H) \subset [0, 1] \times \mathscr{U}$ and $\Psi_H = \Psi$. This is of course possible since $\mathscr{U}$ is starshaped. It is an easy exercise similar to (9) that

$$e_\infty((a_1, a_2) \oplus i(b_1, b_2) \oplus \mathbb{C}^{n-1}) \le (a_2 - a_1)(b_2 - b_1). \tag{37}$$

Hence we obtain in view of lemma 5 the following corollary:

COROLLARY 7. *Assume* $\Psi \in \mathscr{D}$ *with*

$$\operatorname{supp}(\Psi) \subset (a_1, a_2) \oplus i(b_1, b_2) \oplus \mathbb{C}^{n-1}.$$

Then $E_\infty(\Psi) \le 16(b_2 - b_1)(a_2 - a_1)$.

3. The C^0-Estimate

Using the results from section 2, theorem 1 can be quite easily deduced.

Proof of theorem 1. Let $\Psi \in \mathscr{D}$, $\Psi \ne \mathrm{Id}$ and put $\epsilon = \|\Psi - \mathrm{Id}\|_{C^0}$. Pick a $\delta > \epsilon$ and choose a sequence $(a_k) \subset \mathbb{R}$ satisfying

$$a_0 = 0, \quad a_{k+1} - a_k = 2\delta. \tag{38}$$

We define $Q_k \subset \mathbb{C}^n$ by

$$Q_k = (a_k - \tau, a_k + \tau) \oplus \mathbb{C}^{n-1} \tag{39}$$

for some small $\tau > 0$ satisfying

$$\tau + \epsilon < \delta. \tag{40}$$

Define $\Sigma = \{k \in \mathbb{Z} \mid Q_k \cap \operatorname{supp}(\Psi) \neq \varnothing\}$ and pick $x_k \in Q_k \cap \operatorname{int}(\operatorname{supp}(\Psi))$ for $k \in \Sigma$ such that

$$B_R(x_k) \supset \operatorname{supp}(\Psi) \tag{41}$$

with $R = \operatorname{diam}(\operatorname{supp}(\Psi))$. Clearly for $k \in \Sigma$ we have

$$C_{x_k}(\Psi) \subset B_R(x_k). \tag{42}$$

Pick a $\tilde{\delta}$ satisfying

$$\epsilon < \tilde{\delta} \quad \text{and} \quad \tilde{\delta} + \tau < \delta. \tag{43}$$

By the localization proposition we find Φ_k for $k \in \Sigma$ satisfying

$$\begin{aligned} &\Phi_k \mid Q_k \equiv \Psi \mid Q \\ &\operatorname{supp}(\Phi_k) < B_{\tilde{\delta}}(Q_k) \cap C_{x_k}(\Psi) \\ &E_\infty(\Phi_k) \leq 16 e_\infty(B_{\tilde{\delta}}(Q_k) \cap C_{x_k}(\Psi)). \end{aligned} \tag{44}$$

We note that the

$$B_{\tilde{\delta}}(Q_k) = (a_k - \tau - \tilde{\delta}, a_k + \tau + \tilde{\delta}) \oplus i\mathbb{R} \oplus \mathbb{C}^{n-1} \qquad .$$

are mutually properly separated. The same is then true for the sets $B_{\tilde{\delta}}(Q_k) \cap C_{x_0}(\Psi)$, $k \in \Sigma$. We note that for $k \in \Sigma$ and a suitable choice of $b_k \in \mathbb{R}$ we have with $R = \operatorname{diam}(\operatorname{supp}(\Psi))$

$$B_{\tilde{\delta}}(Q_k) \cap C_{x_k}(\Psi) \subset (a_k - \tau - \tilde{\delta}, a_k + \tau + \tilde{\delta}) \oplus i(b_k - R, b_k + R) \oplus \mathbb{C}^{n-1}. \tag{45}$$

For different $k, j \in \Sigma$ the supports of Φ_j and Φ_k are properly separated. Hence from equations (44), (45), lemma 4 and corollary 7

$$\begin{aligned} E_\infty\left(\prod_{k \in \Sigma} \Phi_k\right) &\leq 2 \max\{E_\infty(\Phi_k) \mid k \in \Sigma\} \\ &\leq 32(2(\tau + \tilde{\delta}) \cdot 2R) \\ &= 128 \cdot R(\tau + \tilde{\delta}) \\ &\leq 128 \cdot R\delta. \end{aligned} \tag{46}$$

Next we write $\Psi = \Psi(\Pi_{k\in\Sigma}\Phi_k)^{-1}(\Pi_{k\in\Sigma}\Phi_k)$ and estimate in view of equation (46)

$$\begin{aligned} E_\infty(\Psi) &\leq E_\infty\left(\Psi\left(\prod_{k\in\Sigma}\Phi_k\right)^{-1}\right) + E_\infty\left(\prod_{k\in\Sigma}\Phi_k\right) \\ &\leq E_\infty\left(\Psi\left(\prod_{k\in\Sigma}\Phi_k\right)^{-1}\right) + 128\cdot R\delta. \end{aligned} \tag{47}$$

On Q_k for any k in $\mathbb{Z}$ we have with $\theta = \Pi_{k\in\Sigma}\Phi_k$

$$\Psi\theta^{-1}(x) = x.$$

Moreover, for every $k\in\Sigma$ the support of Φ_k was contained in $B_{\tilde\delta}(Q_k)\cap C_{x_k}(\Psi)$. Hence $\Psi\theta^{-1}$ can be written as a finite product of maps in $\mathscr{D}$, say $\gamma_1,\dots,\gamma_l$, with mutually properly separated supports contained in sets of the form

$$\mathscr{U}_j := [\hat a_j - (\delta-\tau), \hat a_j + (\delta-\tau)] \oplus i(\hat b_j - R, \hat b_j + R) \oplus \mathbb{C}^{n-1} \tag{48}$$

for suitable $\hat a_j, \hat b_j \in \mathbb{R}$. Hence arguing via corollary 7 along the previous lines

$$\begin{aligned} E_\infty(\Psi\theta^{-1}) &\leq 2\max\{E_\infty(\gamma_j)\mid j=1,\dots,l\} \\ &\leq 2\max\{16e_\infty(16e_\infty(\mathscr{U}_j)\mid j=1,\dots,l\} \\ &\leq 32\cdot(2(\delta-\tau)\cdot 2R) \\ &= 128\cdot R(\delta-\tau) \\ &\leq 128\cdot R\delta. \end{aligned} \tag{49}$$

Combining now (47) and (49) we obtain

$$E_\infty(\Psi) \leq 256\cdot R\delta. \tag{50}$$

Since $R = \text{diam}(\text{supp}(\Psi))$ and δ was an arbitrarily chosen number greater than $\|\text{Id}-\Psi\|_{C^0}$ we have arrived at

$$E_\infty(\Psi) \leq 256\cdot \text{diam}(\text{supp}(\Psi))\|\text{Id}-\Psi\|_{C^0}. \tag{51}$$

From this we deduce

$$\begin{aligned} d_\infty(\Phi, \Psi) &= d_\infty(\Phi^{-1}, \Psi^{-1}) \\ &= E_\infty(\Phi\Psi^{-1}) \\ &\leq 256 \cdot \operatorname{diam}(\operatorname{supp}(\Phi\Psi^{-1}))\, \|\mathrm{Id} - \Phi\Psi^{-1}\|_{C^0} \\ &= 256 \cdot \operatorname{diam}(\operatorname{supp}(\Phi\Psi^{-1}))\, \|\Psi - \Phi\|_{C^0}. \qquad \square \end{aligned}$$

4. Functional analysis of the action integral

The method we are employing is close to [7, 3, 4, 11, 12] and utilizes the variational approach to strongly indefinite functionals going back to Benci and Rabinowitz, [1] and the author [8]. We denote by $\mathscr{B}$ the Hilbertspace consisting of all functions $u \in L^2((0, 2); \mathbb{C}^n)$ with Fourier series

$$u = \sum_{k \in \mathbb{Z}} x_k\, e^{\pi i k t}, \qquad x_k \in \mathbb{C}^n$$

satisfying the summability condition

$$\sum |x_k|^2 |k| < \infty.$$

As norm we take

$$\|u\|^2 = 2\pi \sum |k|\,|x_k|^2 + 2|x_0|^2.$$

Clearly, $\|\ \|$ is induced by some inner product $(\cdot, \cdot)$. $\mathscr{B}$ has an orthogonal decomposition $\mathscr{B} = \mathscr{B}^- \oplus \mathscr{B}^0 \oplus \mathscr{B}^+$ given by

$$u = u^- + u^0 + u^+ = \sum_{k<0} x_k\, e^{\pi i k t} + x_0 + \sum_{k>0} x_k\, e^{\pi i k t}.$$

We denote the corresponding orthogonal projections by P^-, P^0 and P^+. The action integral is the quadratic form $a : \mathscr{B} \to \mathbb{R}$ defined by

$$a(u) = -\frac{1}{2}\|P^- u\|^2 + \frac{1}{2}\|P^+ u\|^2. \tag{52}$$

If $u : \mathbb{R}/2\mathbb{Z} \to \mathbb{C}^n$ is a smooth 2-periodic loop we have

$$a(u) = \frac{1}{2}\int_0^2 \langle -i\dot{u}, u\rangle \, dt, \tag{53}$$

where $\langle \cdot, \cdot \rangle = \mathrm{Re}\,(\cdot, \cdot)$ is the standard real inner product on $\mathbb{C}^n$. Note that the right hand side of equation (53) is the classical action integral.

For pairs $(H, K) \in \mathscr{A} \times \mathscr{C}$, where $\mathscr{A} \times \mathscr{C}$ carries the norm $\|(H, K)\|_1 = \|H\|_1 + \|K\|_1$, we define an associated smooth functional $b_{(H,K)}$ by

$$b_{(H,K)} : \mathscr{B} \to \mathbb{R}$$

$$b_{(H,K)}(u) = \int_0^1 H(u(t))\, dt + \int_1^2 K_{t-1}(u(t))\, dt. \tag{54}$$

Moreover, the gradient of $b_{H,K}$ denoted by $b'_{(H,K)} : \mathscr{B} \to \mathscr{B}$ has a relatively compact image. This follows since H and K are compactly supported and smooth and $\mathscr{B}$ is compactly embedded into $L^p((0, 2); \mathbb{C}^n)$ for every $p \in [1, +\infty)$. Also $b_{(H,K)}(\mathscr{B})$ is bounded in $\mathbb{R}$. Using a variant of a construction in [3] we introduce a special subgroup $\mathscr{G}$ of the homeomorphism group homeo $(\mathscr{B})$ of $\mathscr{B}$. We say a homeomorphism $h : \mathscr{B} \to \mathscr{B}$ belongs to $\mathscr{G}$ provided h, h^{-1} map bounded sets into bounded sets and there exist continuous maps $\gamma^{\pm} : \mathscr{B} \to \mathbb{R}$, $K : \mathscr{B} \to \mathscr{B}$ having the following properties. $\gamma^{\pm}$ and K map bounded sets into relatively compact sets. Moreover there exists a constant $R = R(h) > 0$, such that $K(u) = 0$ and $\gamma^{\pm}(u) = 0$ for all $u \in \mathscr{B}^+$ satisfying $\|u\| \geq R$. Moreover h has the representation

$$h(u) = e^{\gamma^-(u)}u^- + u^0 + e^{\gamma^+(u)}u^+ + K(u). \tag{55}$$

It follows immediately from its definition and elementary properties of nonlinear compact operators that $\mathscr{G}$ is a group. In the following we shall need

LEMMA 8. *For every $h \in \mathscr{G}$ we have*

$$h(\mathscr{B}^+) \cap (\mathscr{B}^- \oplus \mathscr{B}^0) \neq \varnothing.$$

Proof. We have to find $u \in \mathscr{B}^+$ such that $P^+h(u) = 0$. This is equivalent to

$$0 = u + e^{-\gamma^+(u)}P^+K(u)$$
$$=: u + Tu$$

for some $u \in \mathscr{B}^+$. $T : \mathscr{B}^+ \to \mathscr{B}^+$ is a nonlinear compact operator with $\overline{T(\mathscr{B}^+)}$ being compact. Hence via Schauder's fixed theorem we find a $u \in \mathscr{B}^+$ with $u = -T(u)$. □

We define a map $\alpha : \mathscr{A} \times \mathscr{C} \to \mathbb{R}$ as follows. For $(H, K) \in \mathscr{A} \times \mathscr{C}$ we put

$$a_{(H,K)} := a - b_{(H,K)} \tag{56}$$

and define

$$\alpha(H, K) = \sup_{h \in G} \inf_{u \in \mathscr{B}^+} a_{(H,K)}(h(u)). \tag{57}$$

Let us define for $(H, K) \in \mathscr{A} \times \mathscr{C}$ two real numbers $q^-(H, K) \leq 0$ and $q^+(H, K) \geq 0$ by

$$\begin{aligned} q^-(H, K) &= \inf_{x \in \mathbb{C}^n} H(x) + \int_0^1 \inf_{x \in \mathbb{C}^n} K(t, x)\, dt \\ q^+(H, K) &= \sup_{x \in \mathbb{C}^n} H(x) + \int_0^1 \sup_{x \in \mathbb{C}^n} K(t, x)\, dt. \end{aligned} \tag{58}$$

Observe that

$$\|H\|_1 + \|K\|_1 = q^+(H, K) - q^-(H, K). \tag{59}$$

LEMMA 9. *For $(H, K) \in \mathscr{A} \times \mathscr{C}$ we have*

$$\begin{aligned} &-q^+(H, K) \leq \alpha(H, K) \leq -q^-(H, K) \\ &-q^+(O, K) + \alpha(H, O) \leq \alpha(H, K) \leq \alpha(H, O) - q^-(O, K). \end{aligned} \tag{60}$$

Moreover

$$|\alpha(H_2, K_2) - \alpha(H_1, K_1)| \leq \|H_2 - H_1\| + \|K_2 - K_1\|_1. \tag{61}$$

Proof. We have for $u \in \mathscr{B}^+$

$$\begin{aligned} a_{(H,K)}(u) &= \frac{1}{2}\|u\|^2 - b_{(H,K)}(u) \\ &\geq \frac{1}{2}\|u\|^2 - q^+(H, K). \end{aligned}$$

Hence

$$\begin{aligned}\alpha(H, K) &\geq \inf_{u \in \mathscr{B}^+} \left(\frac{1}{2} \|u\|^2 - q^+(H, K) \right) \\ &\geq -q^+(H, K).\end{aligned} \tag{62}$$

For $v \in \mathscr{B}^- \oplus \mathscr{B}^0$ we estimate

$$\begin{aligned}a_{(H,K)}(v) &= -\frac{1}{2} \|v^-\|^2 - b_{(H,K)}(v^- + v^0) \\ &\leq -q^-(H, K).\end{aligned} \tag{63}$$

In view of lemma 8 we have for every $h \in \mathscr{G}$ $h(\mathscr{B}^+) \cap (\mathscr{B}^- \oplus \mathscr{B}^0) \neq \varnothing$. Hence in view of (63)

$$\begin{aligned}\inf_{u \in \mathscr{B}^+} a_{(H,K)}(h(u)) &\leq \sup_{v \in \mathscr{B}^- \oplus \mathscr{B}^0} a_{(H,K)}(v) \\ &\leq -q^-(H, K).\end{aligned} \tag{64}$$

So we obtain from equation (62) and (64) and the definition of α the first part of the assertion (60) of lemma 9. The second part can be proved similarly. The trivial estimate

$$\begin{aligned}&|a_{(H_2,K_2)}(u) - a_{(H_1,K_1)}(u)| \\ &= \left| \int_0^1 (H_2 - H_1)(u(t))\, dt + \int_1^2 ((K_2)_{t-1}(u(t)) - (K_1)_{t-1}(u(t)))\, dt \right| \\ &\leq \|H_2 - K_1\|_1 + \|K_2 - K_1\|_1\end{aligned}$$

gives immediately

$$|\alpha(H_2, K_2) - \alpha(H_1, K_1)| \leq \|H_2 - H_1\|_1 + \|K_2 - K_1\|_1.$$

□

LEMMA 10. *Let* $(H, K) \in \mathscr{A} \times \mathscr{C}$ *and assume* $(u_k) \subset \mathscr{B}$, $d \in \mathbb{R}$, *such that*

$$a_{(H,K)}(u_k) \to 0 \quad \text{in } \mathscr{B}, \qquad a_{(H,K)}(u_k) \to d. \tag{65}$$

Then there exists $u \in \mathscr{B}$ *satisfying*

$$a'_{(H,K)}(u) = 0, \qquad a_{(H,K)}(u) = d.$$

Note that this is a version of the so-called Palais–Smale condition. However for the maps $(H, K) \in \mathscr{A} \times \mathscr{C}$ the condition (65) will in general not imply that (u_k) has a converging subsequence.

Proof. Since $b'_{(H,K)}(\mathscr{B})$ is precompact and $a'(u) = -u^- + u^+$, it follows immediately after taking a subsequence that

$$u_k^\pm \to u^\pm \in \mathscr{B}^\pm.$$

Since $\mathscr{B}^0$ is finite dimensional we are done if (u_k^0) is bounded. So without loss of generality if suffices to study the case

$$u_k^\pm \to u^\pm \in \mathscr{B}^\pm, \qquad |u_k^0| \to +\infty. \tag{66}$$

From equation (66) we deduce immediately that

$$b'_{(H,K)}(u_k) \to 0 \quad \text{in } \mathscr{B}$$

$$b_{(H,K)}(u_k) \to 0 \quad \text{in } \mathbb{R},$$

since H and K have compact support. Hence $u^\pm = 0$ and we conclude

$$a_{(H,K)}(u_k) \to 0.$$

That is $d = 0$. Let $c \in \mathbb{C}^n$ with $[0, 1] \times \{c\} \cap [([0, 1] \times \operatorname{supp}(H)) \cup \operatorname{supp}(K)] = \emptyset$. Then the constant loop $\mathbb{R}/2\mathbb{Z} \to \mathbb{C}^n : t \to c$ is a critical point of $a_{(H,K)}$ and $a_{(H,K)}(c) = 0$.

The key technical result for proving theorem 2 is the following construction of a *selection function*:

PROPOSITION 11. *There exists a not necessarily continuous map $\beta : \mathscr{A} \times \mathscr{C} \to \mathscr{B}$ satisfying*

$$\alpha'_{(H,K)}(\beta(H, K)) = 0$$

$$a_{(H,K)}(\beta(H, K)) = \alpha(H, K).$$

Proof. It is enough to find for every $(H, K) \in \mathscr{A} \times \mathscr{C}$ a critical point $u_{(H,K)}$ of $a_{(H,K)}$ with

$$a_{(H,K)}(u_{(H,K)}) = \alpha(H, K).$$

Arguing indirectly let us assume $\alpha(H, K) \in \mathbb{R}$ is not a critical level. In view of lemma 10 we find $\epsilon > 0$ such that

$$\|a'_{(H,K)}(u)\| \geq \epsilon \qquad \text{if } a_{(H,K)}(u) \in [\alpha(H, K) - \epsilon, \alpha(H, K) + \epsilon]. \tag{67}$$

Take a smooth map $\sigma : \mathscr{B} \to [0, 1]$ satisfying

$$\begin{aligned} \sigma(u) &= 1 \qquad \text{for } a_{(H,K)}(u) \in [a(H, K) - \epsilon, \alpha(H, K) + \epsilon] \\ \sigma(u) &= 0 \qquad \text{for } a_{(H,K)}(u) \notin [\alpha(H, K) - 2\epsilon, \alpha(H, K) + 2\epsilon]. \end{aligned} \tag{68}$$

Consider the ordinary differential equation in $\mathscr{B}$ given by

$$\begin{aligned} \dot{u} &= \sigma(u)a'_{(H,K)}(u) \\ &=: G(u). \end{aligned} \tag{69}$$

If $u \in \mathscr{B}^+$ and $\|u\|$ is large we have $a_{(H,K)}(u) \geq \alpha(H, K) + 2\epsilon$ and consequently by (68) $G(u) = 0$. (69) generates a global flow denoted by

$$\mathbb{R} \times \mathscr{B} \to \mathscr{B} : (s, u) \to u * s. \tag{70}$$

In view of (67) we find a real number $T > 0$ such that the map $h : \mathscr{B} \to \mathscr{B}$ defined by $h(u) = u * T$ satisfies:

$$\text{If } a_{(H,K)}(u) \geq \alpha - \epsilon \quad \text{then } \alpha_{(H,K)}(h(u)) \geq \alpha + \epsilon, \tag{71}$$

with $\alpha = \alpha(H, K)$. We note that $G(u) = -\sigma(u)u^- + \sigma(u)u^+ - \sigma(u)b'_{(H,K)}(u)$. For given $u \in \mathscr{B}$ define $A_u(t) = -\sigma(u * t)P^- + \sigma(u * t)P^+$ and $f_u(t) = -\sigma(u * t)b'_{(H,K)}(u * t)$. If $t \to u(t)$ solves $\dot{u} = G(u)$ with $u(0) = u_0$, it solves the linear inhomogeneous system

$$\begin{aligned} \dot{u}(t) &= A_{u_0}(t)u(t) + f_{u_0}(t) \\ u(0) &= u_0. \end{aligned} \tag{72}$$

Now using the variation of constant formula we see that $h \in \mathscr{G}$. By the definition of $\alpha(H, K)$ we find $h_0 \in \mathscr{G}$ such that

$$\inf_{u \in \mathscr{B}^+} a_{(H,K)}(h_0(u)) \geq \alpha - \epsilon. \tag{73}$$

Since $k = h \circ h_0 \in \mathscr{G}$ it follows from equation (71), (73) and the definition of $\alpha(H, K)$

$$\begin{aligned} \alpha(H, K) &\geq \inf_{u \in \mathscr{B}^+} a_{(H,K)}(k(u)) \\ &\geq \alpha(H, K) + \epsilon. \end{aligned} \tag{74}$$

This contradiction proves the proposition. □

Using proposition 11 and Sard's theorem we will be able to investigate the behaviour of α on certain subsets of $\mathscr{A} \times \mathscr{C}$.

LEMMA 12. *For all* $H \in \mathscr{A}_{ad}$ *we have*

$$\alpha(H, 0) = -\min_{x \in \mathbb{C}^n} H(x) = -q^-(H, 0).$$

In particular, if in addition $H \leq 0$ *we have* $\alpha(H, 0) = \|H\|_1$.

Proof. A critical point u of $a_{(H,K)}$ satisfies

$$\begin{aligned} &u : [0, 2] \to \mathbb{C}^n \text{ is continuous} \\ &u(0) = u(2) \\ &\dot{u} = X_H(u) \quad \text{on } (0, 1) \\ &\dot{u} = X_{K_{t-1}(u)} \quad \text{on } (1, 2). \end{aligned} \tag{75}$$

If $K = 0$ we have $u \mid (1, 2) \equiv \text{const}$, so that $u(0) = u(1)$. This means $u \mid [0, 1]$ can be extended to a smooth 1-periodic solution of $\dot{x} = X_H(x)$. Since $H \in \mathscr{A}_{ad}$ all those solutions are constant. Consequently, denoting by Cr (H, K) the set of critical levels for $a_{(H,K)}$ we must have

$$\text{Cr}\,(H, 0) \subset \{-H(m) \mid dH(m) = 0\}. \tag{76}$$

The right hand side is obviously a compact subset of $\mathbb{R}$ and by Sard's theorem nowhere dense. Let $c \in \mathbb{C}^n \backslash \{0\}$ and denote by $H^\theta \in \mathscr{A}_{ad}$ the Hamiltonian

$$H^\theta(x) = H(x - \theta c)$$

for $\theta \in [0, +\infty)$. Clearly Cr $(H^\theta, 0) =$ Cr $(H, 0)$. Since $\theta \to \alpha(H^\theta, 0)$ is continuous and (76) holds this map must be constant. Hence we may assume without loss of generality that

$$H(0) = \inf_{x \in \mathbb{C}^n} H(x). \tag{77}$$

Now consider $\alpha(\tau H, 0)$, where we assume (77). Clearly for $\tau \in [0, 1]$

$$\tau H(0) = \tau \inf_{x \in \mathbb{C}^n} H(x).$$

The map $\tau \to \alpha(\tau H, 0)$ is continuous and since for $\tau \in [0, 1]$ $\tau H \in \mathscr{A}_{ad}$ we infer from the preceding discussion

$$\alpha(\tau H, 0) \in \{ -\tau H(m) \mid dH(m) = 0\}. \tag{78}$$

Hence for $\tau \in [0, 1]$ the map $\tau \to \alpha(\tau H, 0)$ satisfies

$$\alpha(\tau H, 0) = -\tau H(m) \tag{79}$$

for a suitable $m \in \mathbb{C}^n$ with $dH(m) = 0$. If we can show that $m = 0$ is a good choice we obtain

$$\alpha(H, 0) = - \inf_{x \in \mathbb{C}^n} H(x)$$

and the lemma is proved. For a suitable $\tau_0 > 0$ small the following holds

$$\tau H(0) \leq \tau H(x) \leq \tau H(0) + \pi |x|^2 \qquad \text{for all } x \in \mathbb{C}^n \tag{80}$$

provided $\tau \in [0, \tau_0]$. Hence for $\tau \in [0, \tau_0]$

$$\begin{aligned} a(\tau H, 0) &\geq \inf_{u \in \mathscr{B}^+} \left[a(u) - \int_0^1 \pi |u|^2 \, dt - \tau H(0) \right] \\ &\geq -\tau H(0) + \inf_{u \in \mathscr{B}^+} \left[a(u) - \int_0^2 \pi |u|^2 \, dt \right] \\ &= -\tau H(0). \end{aligned} \tag{81}$$

On the other hand lemma 9 gives

$$\alpha(\tau H, 0) \leq -\tau H(0) \tag{82}$$

for all $\tau \in [0, 1]$. From equations (81) and (82) we obtain

$$\alpha(\tau H, 0) = -\tau \inf_{x \in \mathbb{C}^n} H(x) \qquad \text{for } \tau \in [0, \tau_0].$$

Hence $\alpha(\tau H, 0] = -\tau H(0)$ for all $\tau \in [0, 1]$ by the preceding discussions. □

Let us denote by $\mathscr{A}(\mathscr{U})$ the collection of all $H \in \mathscr{A}$ with $\operatorname{supp}(H) \subset \mathscr{U}$.

LEMMA 13. *Let $\mathscr{U} \subset \mathbb{C}^n$ be a subset and $K \in \mathscr{C}$ such that $\Psi_K(\mathscr{U}) \cap \mathscr{U} = \emptyset$. Then*

$$\alpha(H, K) = \alpha(0, K)$$

for all $H \in \mathscr{A}(\mathscr{U})$.

Proof. For $\tau \in [0, 1]$ we have $\tau H \in \mathscr{A}(\mathscr{U})$. Moreover the map $\tau \to \alpha(\tau H, K)$ is continuous. We show that its image lies in a nowhere dense subset and consequently has to be constant. Assume u_τ is a critical point of $a_{(\tau H, K)}$. Then $u_\tau : [0, 2] \to \mathbb{C}^n$ is continuous, $u_\tau(0) = u_\tau(2)$, and

$$\dot{u}_\tau = \tau X_H(u_\tau) \quad \text{on } (0, 1)$$

$$\dot{u}_\tau = X_{K_{t-1}}(u_\tau) \quad \text{on } (1, 2).$$

Since $\Psi_k(\mathscr{U}) \cap \mathscr{U} = \emptyset$ and $\operatorname{supp}(\tau H) \subset \mathscr{U}$ we must have $u_\tau \mid [0, 1] \equiv \text{const} \notin \mathscr{U}$. Hence

$$a_{(\tau H, K)}(u_\tau) = a_{(0, K)}(u_\tau) \in \operatorname{Cr}(0, K).$$

$\operatorname{Cr}(0, K)$ is the set of critical levels for a smooth functional on a Hilbert space having a Fredholm type gradient. Moreover it is a compact set. This can be used easily to write $\operatorname{Cr}(0, K)$ as the countable union of critical levels for smooth finite dimensional functionals (using the implicit function theorem near critical points). Hence $\operatorname{Cr}(0, K)$ is compact and has an empty interior. Consequently the map $\tau \to \alpha(\tau H, K)$ has to be constant, i.e.

$$\alpha(0, K) = \alpha(H, K). \qquad \square$$

5. Estimates from below

The proofs of theorems 2 and 3 follow now quite easily by combining the results in the previous section

Proof of theorem 2. Let $\tau > 0$ and pick $K \in \mathscr{C}$ with

$$\Psi_K = \Psi, \qquad \|K\|_1 \leq E_1(\Psi) + \tau. \tag{83}$$

Let $\mathscr{U}$ be any open set with $\Psi(\mathscr{U}) \cap \mathscr{U} = \emptyset$ and pick any $H \in \mathscr{A}_{ad}(\mathscr{U})$ with $H \leq 0$.
In view of lemmata 9, 12 and 13 we estimate

$$\begin{aligned} -q^-(0, K) &\geq \alpha(0, K) \\ &= \alpha(H, K) \\ &\geq \alpha(H, 0) - q^+(0, K) \\ &= \|H\|_1 - q^+(0, K). \end{aligned}$$

Hence

$$\|K\|_1 \geq \|H\|_1.$$

Consequently $\|K\|_1 \geq c(\mathscr{U})$ and since $\tau > 0$ was arbitrarily given

$$E_1(\Psi) \geq c(\mathscr{U}). \qquad \square$$

Now we prove theorem 3.

Proof of theorem 3. We can phrase the statement of theorem 3 alternatively as follows: If $H\colon \mathbb{C}^n \to \mathbb{R}$ is a compactly supported Hamiltonian in $\mathscr{A}_{ad}$ and $K \in \mathscr{C}$ such that $\Psi_H = \Psi_K$, then $\|H\|_1 \leq \|K\|_1$. Of course this implies that

$$E_p(\Psi_H) \geq E_1(\Psi_H) = \|H\|_1.$$

Since however $\|H\|_p = \|H\|_\infty$ the conclusion of theorem 3 follows.
We observe that $\Psi_{-H} = \Psi_H^{-1}$. Hence

$$\Psi_K \Psi_{-H}(x) = x$$

for all $x \in \mathbb{C}^n$. Hence the set of critical points for $a_{(-H,K)}$ is path connected. Consequently, since the gradient is Fredholm type, $\mathrm{Cr}\,(-H, K) \subset \mathbb{R}$ consists of a single point. Since H and K have compact support it follows that $\mathrm{Cr}\,(-H, K) = \{0\}$ for all pairs $(H, K) \in \mathscr{A}_{ad} \times \mathscr{C}$, which satisfy $\Psi_K = \Psi_H$. We estimate

$$\begin{aligned} 0 &= \alpha(-H, K) \\ &\geq \alpha(-H, 0) - q^+(0, K) \\ &= q^+(H, 0) - q^+(0, K). \end{aligned}$$

Hence

$$q^+(0, K) \geq q^+(H, 0). \tag{84}$$

Define $\bar{K}(t, x) = -K(t, \Psi_t^K(x))$. Then $\Psi_{\bar{K}} = \Psi_K^{-1}$. Since $\Psi_{\bar{K}} = \Psi_{-H}$ we obtain similarly to (84)

$$q^+(0, \bar{K}) \geq q^+(-H, 0). \tag{85}$$

Observe that

$$q^+(0, \bar{K}) = -q^-(0, K)$$

$$q^+(-H, 0) = -q^-(H, 0).$$

Hence (85) implies

$$q^-(H, 0) \geq q^-(0, K). \tag{86}$$

Combining (84) and (86) gives

$$\|K\|_1 \geq \|H\|_1$$

and the proof is complete. Note that we actually have proved the stronger statement that the q^+- and q^--parts satisfy the inequalities (84) and (86).

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