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Link invariants via the eta invariant

J. P. LEVINE

Introduction

In their fundamental work on the Index Theorem for bounded manifolds, ATIYAH-PATODI-SINGER introduce a real-valued invariant $\tilde{\eta}(M, \theta)$, associated with a closed oriented odd-dimensional Riemannian manifold M (say connected) and a unitary representation θ of its fundamental group; a basic observation is that $\tilde{\eta}$ gives a *diffeomorphism* invariant of (M, θ) – see [APS II]. It is a consequence of the Index Theorem that if (M, θ) is the boundary of $(V, \bar{\theta})$, then $\tilde{\eta}(M, \theta) =$ signature $(V, \bar{\theta}) - k$ signature V, where k is the dimension of the representation. In the present work we consider pairs (M, α), where M is a (connected) closed oriented odd-dimensional manifold equipped with a G-structure α, i.e. a homomorphism from $\pi_1(M)$ to a group G. We interpret $\tilde{\eta}(M, \theta\alpha)$ as a function $\rho(M, \alpha) : R_k(G) \to \mathbb{R}$, where $R_k(G)$ is the (real) variety (or inverse limit of such, if G is countably generated) of representations of G into the unitary group $U(k)$, $k \geq 1$. In (II.2) we show that $\rho(M, \alpha)$ is piecewise-continuous-more precisely, $R_k(G)$ admits a stratification by subvarieties so that $\rho(M, \alpha)$ is continuous on each open stratum. With an eye to the use of this invariant to study link concordance, we examine the invariance of $\rho(M, \alpha)$ under homology cobordism – in (II.3) we show that $\rho(M, \alpha)$ depends only on the homology cobordism class of (M, α) except on the points of some proper subvariety of a particular type that we call *special*. For example, if $\theta \in R_k(G)$ factors through some group of prime power order then θ cannot lie on any special subvariety. Thus for such θ $\rho(M, \alpha) \cdot \theta$ is a homology cobordism invariant of (M, α) – these are essentially the signature invariants of SMOLINSKY [S]. But the global nature of ρ, and its continuity property, gives this invariant more power than the individual evaluations, as is illustrated by the examples in (III.4,5).

In order to apply ρ to links we first point out that the complements of certain classes of links admit "canonical" G-structures, where G is either a free abelian group $\mathbb{Z}^m$, a free group F, or an "algebraic closure" $\bar{F}$ of F, depending on which

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class of links. The representation varieties of $\mathbb{Z}^m$ and F are well-understood, but we need to study $R_k(\bar{F})$. Our approach, motivated by the use of the dihedral group in [CO], considers certain quotients of F whose algebraic closures are more easily understood. In particular we can construct some rather explicit analytic curves in $R_k(\bar{F})$. This is all done in Chapter I.

To illustrate the scope of these invariants we give two realization theorems (III.3) which, for certain groups G and Hermitian matrices λ with entries in $\mathbb{Z}G$, construct links with G-structures on their complements such that $\rho(M, \alpha) \cdot \theta$ can be computed from the signature of $\theta(\lambda)$. We then make two particular applications. In the first we construct two one-dimensional links which are seen to be non-concordant only by looking at ρ on a proper lower stratum of $R_1(\mathbb{Z}^m)$, the m-torus – the more traditional signature invariants, as well as the ALEXANDER polynomial, fail to detect this. By contrast we prove (in (II.4)) that such examples cannot exist for higher-dimensional links or for one-dimensional links with a mild triviality property – i.e. for such links, ρ contains concordance information only on the open principal stratum of continuity. In the second example we exhibit the phenomenon, first detected by COCHRAN-ORR [CO], of links of any odd-dimension (with vanishing $\bar{\mu}$-invariants in dimension one) which are not concordant to boundary links. For these examples we compute ρ on the analytic curves in $R_k(\bar{F})$ constructed in Chapter I.

Many of the results of this paper were announced in [L4]. In a future work (see also [L4]) we will use signature functions on representation varieties to study the CAPPELL-SHANESON homology surgery Γ-groups of infinite groups. The WALL surgery groups of finite groups are understood largely through the use of this technique but, for infinite groups, the locally constant nature of the signature function makes it less useful – on the other hand this property allows one to globalize and obtain K-theory invariants (see e.g. [Mi]). By contrast, for homology surgery groups the signature function has discontinuities and so is more likely to yield useful information – for the same reason it is unlikely that globalization is possible.

Chapter I: Unitary representation varieties

1. If G is a (discrete) group, then we let $R_k(G)$ denote the set of all k-dimensional unitary representations of G. It is a standard fact that, when G is finitely-generated, $R_k(G)$ is a real algebraic variety. If $x_1, \ldots, x_n$ is a set of generators of G, then $\rho \mapsto (\rho(x_1), \ldots, \rho(x_n))$ imbeds $R_k(G)$ into $U(k) \times \cdots \times U(k)$. Each relation in $\{x_1, \ldots, x_n\}$ defines a real polynomial equation (using $A^{-1} = \bar{A}^T$) and so, if G is finitely-presented, we see $R_k(G)$ displayed as the zeroes of a finite set of real

polynomials. In the infinitely presented case we appeal to the Noetherian property of the real polynomial ring. $R_k(G)$ is the zero set of an infinite set of polynomials, but since the ideal these polynomials generate is finitely generated, we can equally well regard $R_k(G)$ as the zero set of a finite set of polynomials. It is easy to check that the variety obtained is independent (up to isomorphism) of the presentation of G.

Since we will often have to do with infinitely-generated groups, we wish to give $R_k(G)$ the "algebraic" structure induced by the finitely-generated subgroups of G. In other words a function $f : R_k(G) \to \mathbb{R}$ is regular if $f = g \circ i^*$ where $i : H \to G$ is a homomorphism from some finitely-generated group H, $i^* : R_k(G) \to R_k(H)$ the induced function and $g : R_k(H) \to \mathbb{R}$ is regular. Functions into $R_k(G)$ – from some real algebraic variety, or $R_l(H)$, for another group H – are regular if their composition with every regular function $R_k(G) \to \mathbb{R}$ is regular. We adopt similar definitions for (real) analytic functions into or out of $R_k(G)$. It is easy to see that any homomorphism $\phi : G \to H$ induces a regular map $\phi^* : R_k(H) \to R_k(G)$ and the "suspension" $R_k(G) \to R_{k+1}(G)$, defined by the inclusion $U(k) \subseteq U(k+1)$, is regular.

The topology on $R_k(G)$ will always be the "classical" (rather than the Zariski) topology, i.e. that inherited as a subspace of $U(k) \times \cdots \times U(k)$ if G is finitely-generated, or the direct limit topology if G is infinitely-generated.

Examples

(a) $R_1(G)$ is the usual character group of G. If $G = \mathbb{Z}^m$ (free abelian group of rank m), then $R_1(G)$ is the m-torus.

(b) If $G = F^m$, the free group of rank m, then $R_k(G) = U(k) \times \cdots \times U(k)$ the m-fold product.

(c) If G is finite, then $R_k(G)$ is the disjoint union of a finite number of conjugacy classes of sums of irreducible representations.

(d) Suppose $G = D$, the infinite dihedral group with presentation $\{x, t : t^2 = 1, txt^{-1} = x^{-1}\}$. Then $R_2(D)$ has nine components. Eight of them are single conjugacy classes – pull-backs of eight of the ten conjugacy classes of $U(2)$-representations of $\mathbb{Z}/2 \times \mathbb{Z}/2$ via the abelianization $D \to \mathbb{Z}/2 \times \mathbb{Z}/2$. The ninth component is the union of the conjugacy classes of the algebraically imbedded circle $i : S^1 \subseteq R_2(D)$ defined by $i(\omega) \cdot t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $i(\omega) \cdot x = \begin{pmatrix} \omega & 0 \\ 0 & \bar{\omega} \end{pmatrix}$. This component contains the pull-back of the remaining two conjugacy classes of representations of $\mathbb{Z}/2 \times \mathbb{Z}/2$.

(e) For any subring Λ of the real numbers $\mathbb{R}$, we consider an enlargement of the dihedral group D_Λ, defined to be the semi-direct product $\Lambda \times \mathbb{Z}/2$. More specifically D_Λ is the split extension of Λ by $\mathbb{Z}/2$, where conjugation of Λ by the generator t of $\mathbb{Z}/2$ is given by: $t\lambda t^{-1} = -\lambda$ for any $\lambda \in \Lambda$. Then there is an analytic map $\tilde{\imath} : \mathbb{R} \to R_2(D_\Lambda)$, defined by $\tilde{\imath}(s) \cdot t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\tilde{\imath}(s) \cdot \lambda = \begin{pmatrix} e^{2\pi i s\lambda} & 0 \\ 0 & e^{-2\pi i s\lambda} \end{pmatrix}$ for $\lambda \in \Lambda$. If Λ contains $\mathbb{Z}$ *properly*, then $\tilde{\imath}$ is an imbedding. Note that, under the restriction map $R_2(D_\Lambda) \to R_2(D)$, we obtain an infinite cyclic cover $\tilde{\imath}(\mathbb{R}) \to i(S^1)$.

(f) The preceding examples in (d) and (e) can be further generalized. Let Π be a finite group; and Λ a subring of $\mathbb{R}$. Consider the wreath product $\Lambda\S\Pi$ which is, by definition, the semi-direct product $\Lambda\Pi \times \Pi$, where $\Lambda\Pi$ is the group algebra and conjugation of an element λ of $\Lambda\Pi$ by an element $g \in \Pi$ is defined to be $g \cdot \lambda \in \Lambda\Pi$.

For $\Lambda = \mathbb{Z}$, we define an algebraic imbedding $i : T^k \hookrightarrow R_k(\Lambda\S\Pi)$, where $k = |\Pi|$, and T^k is the k-dimensional torus. For the definition we identify the coordinates of $\mathbb{C}^k$ (or $\mathbb{R}^k$) with the elements of Π. This induces an identification $\mathbb{C}^k = \mathbb{C}\Pi$ and $T^k = \mathbb{R}\Pi/\mathbb{Z}\Pi$. Thus there is an induced multiplication $T^k \times \mathbb{Z}\Pi \to T^k$. We also use the obvious identification of T^k with the diagonal unitary matrices (maximal torus of $U(k)$).

We now define i by the formulae:

(i) $(i(\tau) \cdot g) \cdot \gamma = g\gamma$ for $\tau \in T^k$, $g \in \Pi \subseteq \mathbb{Z}\S\Pi$, $\gamma \in \mathbb{C}\Pi$

(ii) $(i(\tau) \cdot \lambda) \cdot \gamma = (\tau\lambda) \cdot \gamma$ for $\tau \in T^k$, $\lambda \in \mathbb{Z}\Pi \subseteq \mathbb{Z}\S\Pi$, $\gamma \in \mathbb{C}\Pi$.

If $\Lambda = \mathbb{R}$ we can define an analytic imbedding $\tilde{\imath} : \mathbb{R}^k = \mathbb{R}\Pi \hookrightarrow R_k(\mathbb{R}\S\Pi)$ by the formulae:

(iii) $(\tilde{\imath}(\alpha) \cdot g) \cdot \gamma = g\gamma$ for $\alpha \in \mathbb{R}\Pi$, $g \in \Pi \subseteq \mathbb{R}\S\Pi$, $\gamma \in \mathbb{C}\Pi$

(iv) $(\tilde{\imath}(\alpha) \cdot \lambda) \cdot \gamma = e(\alpha\lambda) \cdot \gamma$ for $\alpha \in \mathbb{R}\Pi$, $\lambda \in \mathbb{R}\Pi \subseteq \mathbb{R}\S\Pi$, $\gamma \in \mathbb{C}\Pi$

where $e : \mathbb{R}\Pi \to T^k = \mathbb{R}\Pi/\mathbb{Z}\Pi$ is the (exponential) quotient map.

As in *(e)*, the restriction $R_k(\mathbb{R}\S\Pi) \to R_k(\mathbb{Z}\S\Pi)$ induces an infinite cyclic cover $\tilde{\imath}(\mathbb{R}^k) \to i(T^k)$.

(*g*) Finally we mention the flurry of recent activity in the study of $SU(2)$ (and $SU(n)$) representations of knot groups, much of it aimed at the calculation of the Casson invariant and instanton homology of 3-manifolds (see e.g. [B], [F], [K], [KF], [KK]).

2. In this section we recall the notion of *algebraic closure* of a group and after some preparation in sections 3 and 4, give some examples of unitary representations of some of these groups using 1(e), (f). A group G is said to be *algebraically closed* if any contractible system of equations over G has a unique solution in G. A system of equations over $G : x_i = w_i(x_1, \ldots, x_n)$, $1 \le i \le n$, where $w_i = w_i(x_1, \ldots, x_n) \in F * G$, F the free group generated by the indeterminates $\{x_i\}$, is said to be *contractible* if $p(w_i) = 1$, where $p : F * G \to F$ is projection. In other words, w_i is a product of conjugates of elements of G. A *solution* of such a system is a collection of elements $g_1, \ldots, g_n$ in some overgroup of G, such that $g_i = w_i(g_1, \ldots, g_n)$ for all i. The term algebraically closed appears often in group theory literature (see e.g. [Ne]) with rather different meaning than ours. (If we replace the contractible condition with a weaker one – *acyclic* – which means $p(w_i) \in [F, F]$, one obtains a similar theory. It is not known if these two notions of algebraically closed actually differ.)

In [L1] it is shown that every group G admits an essentially unique homomorphism $i : G \to \hat{G}$, where $\hat{G}$ is algebraically closed and i is "initial" among such homomorphisms. $\hat{G}$ is called the algebraic closure of G. If $f : X \to Y$ is a map between finite polyhedra and its cofiber is contractible, i.e. f is a homology equivalence and $f_* \pi_1(X)$ normally generates $\pi_1(Y)$, then f induces an isomorphism $\widehat{\pi_1(X)} \xrightarrow{\approx} \widehat{\pi_1(Y)}$. For any group G, its *nilpotent completion* $\tilde{G}$ is algebraically closed (see [L]). We denote by $\bar{G} \subseteq \tilde{G}$ the subgroup of all elements which are part of a solution to some contractible system of equations *over* G, i.e. $\bar{G}$ is the image of the canonical map $\hat{G} \to \tilde{G}$ extending $G \to \tilde{G}$. We call $\bar{G}$ the *residually nilpotent algebraic closure* of G.

Despite the size of $\hat{G}$ it does seem to have a reasonable collection of unitary representations. In [V1] it is proved, for example, that any unitary representation of a free group F extends to a unitary representation of $\hat{F}$ (in fact, of $\bar{F}$). It will be proved, in a future paper, that for any finitely-generated group G, any unitary representation of G which lies in the "component" of $R_k(G)$ containing the trivial representation, extends over $\hat{G}$. We will be interested in some examples where representations of G extend to many different representations of $\hat{G}$. The topological implications of this phenomenon will arise later in Section (III.5).

Our construction of representations of algebraic closure will rely on the relationship of the algebraic closure of certain groups to the groups D_Λ and $\Lambda \S \Pi$ discussed in example (e) and (f) in Section 1. In fact it is shown in [L] and [CO] that, for the dihedral group D, $\hat{D} = \underline{D_\Lambda}$ where $\Lambda = \mathbb{Z}[\frac{1}{2}]$. We will show that, more generally, for any p-group Π, $\mathbb{Z}\S\Pi \subseteq \widehat{\mathbb{Z}\S\Pi} \subseteq \mathbb{Z}[1/p]\S\Pi$.

3. We will consider a general semi-direct product $G = A \times \Pi$, where Π is any group, A is a left $\mathbb{Z}\Pi$-module and the conjugation action of Π on A in G coincides

with the left multiplication of $\Pi \subseteq \mathbb{Z}\Pi$ on A. Our aim is to give a description of $\bar{G}$ in terms of $\tilde{\Pi}$ and the I-adic completion of A, where $I = I\Pi$ is the augmentation ideal of $\mathbb{Z}\Pi$.

Let $\tilde{A} = \varprojlim A/I^qA$. We show that $\tilde{A}$ is a module over $\mathbb{Z}[\tilde{\pi}]$. First note that $I(\Pi_q) \subseteq (I\Pi)^q$, where $I(\Pi_q)$ is the augmentation ideal of Π_q, the q-th term of the lower central series of Π (defined recursively by $\Pi_1 = \Pi$, $\Pi_q = [\Pi, \Pi_{q-1}]$). To see this, by induction on q, consider a generator $[g, h]$ of Π_q, where $g \in \Pi$, $h \in \Pi_{q-1}$. Then we have

$$[g, h] - 1 = ((g-1)(h-1) - (h-1)(g-1))g^{-1}h^{-1} \in (I\Pi)I(\Pi_{q-1}).$$

But $I(\Pi_{q-1}) \subseteq (I\Pi)^{q-1}$, by induction.

Now A/I^qA is a module over $\mathbb{Z}[\Pi]/(I\Pi)^q$ so it is also a module over $\mathbb{Z}[\Pi]/I(\Pi_q)\cdot\mathbb{Z}[\Pi]$. But this is the same as $\mathbb{Z}[\Pi/\Pi_q]$ – for any group G and normal subgroup N, $\mathbb{Z}[G/N] = \mathbb{Z}G/I(N)\cdot\mathbb{Z}G$. Since $\tilde{\Pi}$ is the inverse unit of $\{\Pi/\Pi_q\}$, we conclude that $\tilde{A}$ is a module over $\mathbb{Z}[\tilde{\Pi}]$.

PROPOSITION 3.1. *$\tilde{A} \times \tilde{\Pi}$ is the nilpotent completion of $A \times \Pi$.*

Proof. First note that $(A \times \Pi)_q = I^{q-1}A \times \Pi_q$. This is a straightforward recursive calculation, using the fact that $I(\Pi_q) \subseteq (I\Pi)^q$. Therefore the lower central series quotients $G/G_q = A \times \Pi/I^{q-1}A \times \Pi_q \approx (A/I^{q-1}A) \times (\Pi/\Pi_q)$ and the result follows by letting $q \to \infty$.

Suppose (λ_{ij}) is an $(n \times n)$-matrix over $\mathbb{Z}\tilde{\Pi}$ with the property $\epsilon(\lambda_{ij}) = \delta_{ij}$ (ϵ is the usual augmentation $\mathbb{Z}\tilde{\Pi} \to \mathbb{Z}$). Then the linear system of equations:

$$\text{(i)} \quad \sum_{j=1}^{n} \lambda_{ij}X_j = \alpha_i, \qquad 1 \le i \le n,$$

has a unique solution in $\tilde{A}$ for any $\alpha_i \in \tilde{A}$. In fact, the recursive formulae:

$$X_{i,q+1} = \alpha_i - \sum_{j=1}^{n} (\lambda_{ij} - \delta_{ij})X_{j,q}, \qquad X_{io} = 0$$

define $\{X_{iq}\} \subseteq A$ satisfying $X_{iq+1} \equiv X_{iq} \bmod I^qA$ and $\Sigma_{j=1}^{n} \lambda_{ij}X_{jq} \equiv \alpha_i \bmod I^qA$. If $R \subseteq \mathbb{Z}\tilde{\Pi}$ is any subring and B an R-submodule of $\tilde{A}$, we denote by $\bar{B}$ the R-submodule of $\tilde{A}$ consisting of all elements which appear as part of the solutions of a system (i) with $\lambda_{ij} \in R$ and $\alpha_i \in B$. This can be alternatively described using the Cohn localization [Co]. If S denotes the set of matrices over R which become non-singular over $\mathbb{Z}$ after augmentation, then R_s is the "localization" of R in which the matrices of S become non-singular. The observation above means that the inclusion $B \subseteq \tilde{A}$ extends to a unique homomorphism $R_s \otimes_R B = B_s \to \tilde{A}$ and $\bar{B}$ is its image.

PROPOSITION 3.2. $\overline{A \times \Pi} = \overline{(\mathbb{Z}\bar{\Pi})A} \times \bar{\Pi}$.

Note. In this formula, $\overline{A \times \Pi}$ and $\bar{\Pi}$ mean the residually nilpotent algebraic closures, while $\overline{(\mathbb{Z}\bar{\Pi})A}$ is the module localization defined just above.

Proof. Suppose $(\alpha, g) \in \overline{A \times \Pi} \subseteq A \times \Pi = \tilde{A} \times \tilde{\Pi}$. To understand α and g we examine a system of equation over $A \times \Pi$, denoting the indeterminates (X_i, x_i). The system breaks up into two systems – corresponding to the variables $\{X_i\}$, $\{x_i\}$. The system over Π, obtained by projecting the original system, is contractible if the original system is and, in this case, will have unique solutions $x_i = g_i \in \bar{\Pi}$. Making this substitution in the original system results in a system of linear equations (i), where $\alpha_i \in (\mathbb{Z}\bar{\Pi})A$, $\lambda_{ij} \in \mathbb{Z}\bar{\Pi}$. The contractibility of the original system implies $\epsilon(\lambda_{ij}) = \delta_{ij}$ and so the solutions lie in $\overline{(\mathbb{Z}\bar{\Pi})A}$.

To complete the proof we will show that, for any $\alpha \in \overline{(\mathbb{Z}\bar{\Pi})A}$, the element $(\alpha, 1) \in \overline{(\mathbb{Z}\bar{\Pi})A} \times \bar{\Pi}$ is part of a solution of some contractible system of equations over $A \times \Pi$. Suppose we have a linear system (i) with $\lambda_{ij} \in \mathbb{Z}\bar{\Pi}$, $\alpha_i \in (\mathbb{Z}\bar{\Pi})A$ and $\epsilon(\lambda_{ij}) = \delta_{ij}$ whose solution set contains α. Since $\overline{(\mathbb{Z}\bar{\Pi})A}$ is a $\mathbb{Z}\bar{\Pi}$-module it suffices to consider the case where every $\alpha_i \in A$. Write out $\lambda_{ij} = \delta_{ij} + \Sigma_r c_{ijr}(g_{ijr} - 1)$, where $c_{ijr} \in \mathbb{Z}$ and $g_{ijr} \in \bar{\Pi}$. Then consider the following system of equations:

$$\text{(ii)} \quad (X_i, 1) \prod_{j,r} ((0, g_{ijr})(X_j, 1)(0, g_{ijr}^{-1})(-X_j, 1))^{c_{ijr}} = (\alpha_i, 1)$$

where the ordering of the terms in the product can be chosen at will. This is a contractible system over $(\mathbb{Z}\bar{\Pi})A \times \bar{\Pi}$ with indeterminates $\{(X_i, 1)\}$ which corresponds precisely to the linear system (i). Now there is some contractible system of equations over Π:

$$\text{(iii)} \quad x_i = w_i(x_1, \ldots, x_n), \qquad 1 \leq i \leq n$$

such that each g_{ijr} is a member of the solution set of (iii). (The single system (iii) is obtained by putting together the individual systems which give rise to each g_{ijr}.) We may now substitute for each g_{ijr} appearing in (ii) the variable x_l from (iii) such that $x_l = g_{ijr}$ is part of the solution. Now (ii) contains two sets of indeterminates: $\{(X_i, 1), (0, x_l)\}$. If we identify the variable $(0, x_l)$ in (ii) with x_l in (iii), then the combined system (ii), (iii) is a contractible system over $A \times \Pi$ whose solution set contains $(\alpha, 1)$, as desired.

4. We now specialize to the case of Π a p-group. As a consequence of Propositions 3.1 and 3.2 we prove:

THEOREM 4.1. *If Π is a p-group and A a left $\mathbb{Z}\Pi$-module, then there is a natural inclusion $\widetilde{A \times \Pi} \subseteq A_p \times \Pi$, extending the identity on $A \times \Pi$, under which $A \times \Pi \subseteq A_{(p)} \times \Pi$.*

Notation. $A_p = \mathbb{Z}_p \otimes_{\mathbb{Z}} A = \mathbb{Z}_p\Pi \otimes_{\mathbb{Z}\Pi} A$ and $A_{(p)} = \mathbb{Z}_{(p)} \otimes A = \mathbb{Z}_{(p)}\Pi \otimes A$, where $\mathbb{Z}_p$ is the ring of p-adic integers and $\mathbb{Z}_{(p)}$ is the ring of rational p-adic integers i.e. $\mathbb{Z}_p \cap \mathbb{Q}$.

LEMMA 4.2. *If Π is a p-group, then $(I\Pi)^n \subseteq pI\Pi$, for some positive integer n, and $p^k(I\Pi) \subseteq (I\Pi)^2$, for some positive integer k.*

Proof. Let $R = \mathbb{Z}/p$; we must prove that the augmentation ideal $I\Pi \subseteq R\Pi$ is nilpotent, i.e. $(I\Pi)^n = 0$ for some n. Suppose Π is cyclic of order p with generator t. Then $I\Pi = (t-1)$ and so $(I\Pi)^p = ((t-1)^p) = (t^p - 1) = 0$. We now proceed by induction on the order of Π. Let N be a cyclic central subgroup of order p and set $\Pi' = \Pi/N$. By induction $(I\Pi)^n \subseteq \operatorname{Ker}\{R\Pi \to R\Pi'\} = \mathbb{Z}\Pi \cdot IN$, for some n. So $(I\Pi)^{np} \subseteq (\mathbb{Z}\Pi \cdot IN)^p = \mathbb{Z}\Pi \cdot (IN)^p = 0$, since N is central and of order p.

To prove the second inclusion we first note the simple formula: for any $g \in \Pi$, $g^r - 1 \equiv r(g-1) \bmod (I\Pi)^2$. This follows by induction on r: $g^r - 1 = g(g^{r-1} - 1) + g - 1 \equiv g^{r-1} - 1 + g - 1 \pmod{(I\Pi)^2} \equiv (r-1)(g-1) + g - 1 \pmod{(I\Pi)^2}$. Now suppose $g^{p^k} = 1$ for every $g \in \Pi$. Then $p^k(g-1) \in (I\Pi)^2$ for any $g \in \Pi$.

As a consequence of this lemma the p-adic topology and the $(I\Pi)$-adic topology on $\mathbb{Z}\Pi$ coincide on $I\Pi$. Thus $\widetilde{I\Pi} = \lim_q I\Pi/(I\Pi)^q$ coincides with $(I\Pi)_p = \lim_k I\Pi/p^k(I\Pi)$. Since $\widetilde{\mathbb{Z}\Pi} / \widetilde{I\Pi} = \mathbb{Z}$ and $(\mathbb{Z}\Pi)_p/(I\Pi)_p = \mathbb{Z}_p$, we have $\widetilde{\mathbb{Z}\Pi} \subseteq (\mathbb{Z}\Pi)_p$. Now, by Proposition 3.1, $\widetilde{A \times \Pi} = \tilde{A} \times \tilde{\Pi} = \tilde{A} \times \Pi$, since Π is a p-group. Recall $\Pi = \bar{\Pi} = \tilde{\Pi}$ for any nilpotent group. So $\tilde{A} = \widetilde{\mathbb{Z}\Pi} \otimes_{\mathbb{Z}\Pi} A \subseteq (\mathbb{Z}\Pi)_p \otimes_{\mathbb{Z}\Pi} A = A_p$ and the first assertion of Theorem 4.1 follows.

To prove the second assertion we need:

LEMMA 4.3. *Let Π be a p-group and A a left $\mathbb{Z}\Pi$-module such that* $\mathbb{Z} \otimes_{\mathbb{Z}\Pi} A = 0$. Then $\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} A = 0$. *In particular, if $(\lambda_{ij}) = \lambda$ is a square matrix over $\mathbb{Z}\Pi$ such that $\epsilon(\lambda)$ is non-singular over $\mathbb{Z}$, then λ is non-singular over $\mathbb{Z}_{(p)}\Pi$.*

This lemma implies immediately that, under the inclusion $\mathbb{Z}\Pi \to \mathbb{Z}_p\Pi$ established above, we have $\overline{\mathbb{Z}\Pi} \subseteq \mathbb{Z}_{(p)}\Pi$. Thus the second assertion of Theorem 4.1 follows from Proposition 3.2.

Proof of Lemma 4.3. We first note that the second assertion follows from the first by considering A to be the $\mathbb{Z}\Pi$-module with presentation matrix λ.

Let $R = \mathbb{Z}/p$ again and let $A' = R \otimes_{\mathbb{Z}} A = R\Pi \otimes_{\mathbb{Z}\Pi} A$. So $R \otimes_{R\Pi} A' = 0$. We will prove that, for any $R\Pi$-module B, that $R \otimes_{R\Pi} B = 0$ implies $B = 0$. If $A' = 0$, then A consists entirely of elements of finite order prime to p. But then $\mathbb{Z}_{(p)} \otimes A = 0$.

Suppose Π is cyclic of order p with generator t, then $R \otimes_{R\Pi} B = 0$ implies that $t - 1$ is an epimorphism of B. But $(t-1)^p = t^p - 1 = 0$ and so $B = 0$. We proceed by induction on $|\Pi|$. Again let N be a central cyclic subgroup of order p and $\Pi' = \Pi/N$. Let $B' = R\Pi' \otimes_{R\Pi} B$; then $R \otimes_{R\Pi'} B' = R \otimes_{R\Pi} B = 0$ and so, by induction, we have $B' = 0$. If we now consider B as an RN-module and note that $R \otimes_{RN} B = R\Pi' \otimes_{R\Pi} B = B' = 0$. But we know the lemma is true for N and so we conclude $B = 0$.

We can now combine $\overline{\text{Theorem}}$ 4.1 with the representations of 1(f), to define the analytic map $\check{\imath} : \mathbb{R}^k \to R_k(\overline{\mathbb{Z}\S\Pi})$, where $k = |\Pi|$, using $\check{\imath}$ defined in 1(f) followed by the restriction of representations $\mathbb{R}\S\Pi$ to $\overline{\mathbb{Z}\S\Pi}$ via ϕ from Theorem 4.1.

Chapter II: The signature invariant

1. In [APS I, II] an invariant, which we denote $\tilde{\eta}_\alpha(M)$, is defined for a closed smooth oriented connected odd-dimensional manifold M and a unitary representation $\alpha : \pi_1(M) \to U(k)$. We give a brief outline. If M is *Riemannian*, an invariant $\eta_\alpha(M) \in \mathbb{R}$ is defined from the spectrum of a certain self-adjoint elliptic linear differential operator. The following theorem is of paramount importance:

INDEX THEOREM [APS II]. *If $M = \partial N$ where N is a connected compact Riemannian oriented manifold, and α extends over $\Pi_1(N)$ then*:

$$\operatorname{sign}_\alpha(N) = k \int_N L(p) - \eta_\alpha(M)$$

where $\operatorname{sign}_\alpha(N)$ *is the* (*twisted*) *α-signature of N (α is also used to denote its extension over $\Pi_1(N)$) and $L(p)$ is the Hirzebruch L-polynomial in the Pontriagin forms of N.*

We will recall the definition of $\text{sign}_\alpha(N)$ below. As an immediate consequence of the Index Theorem, one concludes that $\tilde{\eta}_\alpha(M) = \eta_\alpha(M) - k\eta_o(M)$, where o denotes the trivial representation, is a diffeomorphism invariant of (M, α). Then the Index Theorem implies the formula:

(1) $\tilde{\eta}_\alpha(M) = k \text{ sign }(N) - \text{sign}_\alpha(N)$.

Thus $\tilde{\eta}_\alpha(M)$ is an integer if M bounds but, by contrast, [APS II] supplies the following example:

(2) If $M = S^1$ and $\alpha : \Pi_1(S^1) \to U(1)$ is defined by $\alpha(t) = (e^{2\pi i a})$, for a suitable generator t of $\Pi_1(S^1)$ and a real number a, then:

$$\tilde{\eta}_\alpha(M)\begin{cases} 1-2a & 0<a<1 \\ 0 & a=1 \end{cases}.$$

Let's recall the definition of $\text{sign}_\alpha(N)$ – see [APS II] or [N] also.

We adopt the following convention. If $\alpha : \Pi \to U(k)$ is a representation and A is a left $\mathbb{Z}\Pi$ or $\mathbb{C}\Pi$-module, then $\mathbb{C}^k \otimes_\alpha A$ denotes $C^k_\alpha \otimes_{\mathbb{C}\Pi} A$ where $\mathbb{C}^k_\alpha$ is $\mathbb{C}^k$ with the right $\mathbb{C}\Pi$-module structure defined by the formula $v \cdot g = v\alpha(g)$; v is interpreted as a row-vector.

Now we can define $H_*(N; \alpha)$ to be the homology of the chain complex $\mathbb{C}^k \otimes_\alpha C_*(\tilde{N})$, where $\tilde{N}$ is the universal covering space of N, if $\Pi = \Pi_1(N)$. $H_*(N; \alpha)$ supports an intersection pairing via the following pairing on the chain level:

(3) $\langle v_1 \otimes c_1, v_2 \otimes c_2 \rangle = v_1 \alpha(\langle c_1, c_2 \rangle)\bar{v}_2^\tau$

where $\langle c_1, c_2 \rangle$ is the equivariant intersection pairing on $\tilde{N}$ with values in $\mathbb{Z}\Pi$, v^τ is the transpose of v and $^-$ denotes complex conjugation. If $\dim N = 2q$, then (3) induces a $(-1)^q$-Hermitian pairing on the complex vector space $H_q(N; \alpha)$. More generally, one obtains a *non-singular* pairing of $H_i(N; \alpha)$ with $H_{2q-i}(N, \partial N; \alpha)$ – using Poincaré duality.

If G is a group, then a *G-manifold* will be a pair (M, α), where M is a compact oriented manifold and α a collection of homomorphisms $\alpha_i : \pi_1(M_i) \to G$, where $\{M_i\}$ are the components of M, each α_i defined up to an inner automorphism of G.

Now suppose (M, α) is a G-manifold, where M is also closed and odd-dimensional. For any $\theta \in R_k(G)$, the composition $\theta\alpha$ gives a unitary representation of $\pi_1(M)$ (or $\pi_1(M_i)$, for each component of M) and so $\tilde{\eta}_{\theta\alpha}(M) \in \mathbb{R}$ is defined. We can thus define:

$$\rho(M, \alpha) : R_k(G) \to \mathbb{R}$$

by $\rho(M, \alpha) \cdot \theta = \tilde{\eta}_{\theta\alpha}(M)$.

2. Our first result is that $\rho(M, \alpha)$ is "piecewise continuous." The discontinuities will be *subvarieties* of $R_k(G)$. When G is finitely-generated, and so $R_k(G)$ is an ordinary (real) algebraic variety, then a subvariety is the zero set of a regular function (or, equivalently, a finite set of functions). To cover the case of G infinitely-generated we define a subvariety to be, in general, the zero set of a regular function.

THEOREM 2.1. *If (M, α) is a G-manifold, M closed odd-dimensional, then there exists a stratification: $R_k(G) = \Sigma_0 \supseteq \Sigma_1 \supseteq \cdots \Sigma_i \supseteq \Sigma_{i+1} \supseteq \cdots$ of finite length (i.e. Σ_i is empty for some i), where each Σ_i is a subvariety of $R_k(G)$, such that $\rho(M, \alpha) \mid \Sigma_i - \Sigma_{i+1}$ is continuous for every $i \geq 0$. The discontinuities of $\rho(M, \alpha)$ are all given by integer jumps, i.e. when reduced* mod $\mathbb{Z}$, *$\rho(M, \alpha)$ is continuous.*

We will call $\{\Sigma_i\}$ a *continuity stratification* for (M, α). Σ_1 will be called a *singular locus* and $\Sigma_0 - \Sigma_1$ a domain of continuity, if Σ_1 is a proper subvariety. Of course there are many possible continuity stratifications, although it is possible to define a minimal one when G is finitely generated.

Proof. The particular continuity stratification we propose is obtained as follows. Consider, for any $\theta \in R_k(G)$, the number:

$$(1) \quad r(\theta) = \sum_{i=0}^{\infty} \dim_{\mathbb{C}} H_i(M; \theta\alpha).$$

We will prove that, for any r, the subset of $R_k(G)$ defined by

$$(2) \quad \Sigma_r = \{\theta \in R_k(G) : r(\theta) \geq r\}$$

is a subvariety of $R_k(G)$. Note that $\Sigma_0 = R_k(G)$ and $\Sigma_r = \phi$ if $r > kN$, where N is the total number of simplices in a triangulation of M.

To see that Σ_r is a subvariety, consider the free $\mathbb{Z}G$-chain complex $\{C_i(\tilde{M}), \partial_i\}$, where $\tilde{M}$ is the regular G-covering of M defined by α. Each $\partial_s : C_s(\tilde{M}) \to C_{s-1}(\tilde{M})$ is represented by a matrix (λ_{ij}^s) over $\mathbb{Z}G$, and so $H_*(M; \theta\alpha)$ is the homology of the chain complex $\mathbb{C}^k \otimes_\theta C_*(\tilde{M})$ whose boundary operators are represented by the complex matrices $(\theta(\lambda_{ij}^s))$. Each $\theta(\lambda_{ij}^s)$ is, itself, a matrix and these form blocks in the larger matrix. Since

$$\dim H_s(M; \theta\alpha) = \dim(\mathbb{C}^k \otimes_\theta C_s(\tilde{M})) - \operatorname{rank}(\theta(\lambda_{ij}^s)) - \operatorname{rank}(\theta(\lambda_{ij}^{s+1}))$$

we have $r(\theta) = kN - 2\,\Sigma_s \operatorname{rank}(\theta(\lambda_{ij}^s))$. If $T(\theta)$ is the block sum of the matrices $(\theta(\lambda_{ij}^s))$, over all s, then

(3) $\Sigma_r = \{\theta : \operatorname{rank} T(\theta) \le \frac{1}{2}(kN - r)\}$.

Since the entries of $T(\theta)$, and therefore its minors, are regular functions of θ, and Σ_r is defined, according to (3), by the vanishing of minors, we conclude that Σ_r is a subvariety.

We must now prove that $\rho(M, \alpha)$ is continuous on the sets $V_r = \{\theta : \dim H_*(M; \theta\alpha) = r\}$. We may as well assume that M is connected (since $\rho(M, \alpha)$ is additive under disjoint union) and $G = \pi_1(M)$, $\alpha =$ identity, since, if we use the notation $\rho(M)$ for $\rho(M, \alpha)$ in this special case, we have $\rho(M, \alpha) = \rho(M) \circ \alpha^*$, where $\alpha^* : R_k(G) \to R_k(\pi_1(M))$ – the function induced by α – is regular.

$R_k(\pi_1(M))$ is well-known to be closely related to the class of k-dimensional flat bundles over M. For each principal $U(k)$-bundle ξ over M, let $A(\xi)$ denote the space of flat connexions on ξ and $B(\xi) = A(\xi)$ modulo the action of the gauge group of bundle automorphisms of ξ. Then the disjoint union of $\{B(\xi)\}$ is homeomorphic to the quotient $R_k(\pi_1(M))$/conjugation. Suppose we choose a Riemannian metric for M. Then the η-invariant $\eta_c(M)$ can be defined, for any $c \in A(\xi)$, by considering the linear elliptic self-adjoint differential operator $E_c = \pm(*D_c - D_c*)$ on $\Omega_{\text{even}}(\xi)$, where D_c is the covariant derivative defined by c, $*$ the duality involution defined by the metric.

Let $\theta_0 \in V_r$ and $c_0 \in A(\xi)$ a corresponding connexion. To show continuity of $\rho(M) \mid V_r$ at θ_0, we can instead consider $\eta_c(M)$ as a function of $c \in V_r'$ near c_0, where V_r' is the set of c such that E_c has nullity r. Note that $\eta_c(M) - \rho(M) \cdot \theta$ is constant, for corresponding c, θ, and the nullspace of E_c corresponds to $H_*(M, \theta)$ by Hodge Theory (see [APSII]). Now choose $\epsilon > 0$ so that E_{c_0} has no non-zero eigenvalues λ with $|\lambda| \le \epsilon$ and let W be a neighborhood of c_0 so that $\pm\epsilon$ is not an eigenvalue of E_c for any $c \in W$. We can follow [APS III, p. 74ff] and write $\eta_c(M) = \eta_c' + \eta_c''$ for $c \in W$, corresponding to eigen-values λ with $|\lambda| < \epsilon$ and $|\lambda| > \epsilon$, respectively. Now η_c'' is, up to a constant, the η-function of an invertible operator if $c \in W$ and, as shown in [APS III], is therefore a differentiable function of c. On the other hand, η_c' is just a finite sum of the signs of those eigenvalues λ of E_c with $|\lambda| < \epsilon$. If $d(c)$ denotes the total dimension of the eigenspaces of E_c for eigen-values λ with $|\lambda| < \epsilon$, then d is locally constant on W. Since $d(c_0) = r$, we have $d(c) = r$ in some neighborhood $W' \subseteq W$ of c_0. But then, if $c \in W' \cap V'_r$, E_c has no eigenvalues λ with $0 < |\lambda| < \epsilon$ and so $\eta'_c = 0$. So we conclude that $\eta_c(M)$ is continuous in $W' \cap V'_r$.

The continuous function $\tilde{\rho}(M, \alpha) : R_k(G) \to \mathbb{R}/\mathbb{Z}$, defined by reducing $\rho(M, \alpha)$ mod $\mathbb{Z}$, is well-understood. It is locally constant when $\dim M \equiv 3 \bmod 4$ and differs from a locally constant function by an explicit formula depending only on the determinant $R_k(G) \to R_1(G)$. (See, e.g. a forthcoming paper of M. Farber and the author.) Furthermore $\tilde{\rho}(M, \alpha)$ depends only on the G-bordism class of (M, α), by the Index theorem.

3. We propose to investigate the extent to which $\rho(M, \alpha)$ is an invariant of *homology G-bordism*. We say (M, α) and (M', α') are homology G-bordant if there is G-manifold (N, β) such that $\partial N = M' - M$ and $\beta \mid \pi_1(M) = \alpha$, $\beta \mid \pi_1(M') = \alpha'$, up to inner automorphism, and $H_*(N, M) = H_*(N, M') = 0$. It will turn out that, in this case, $\rho(M, \alpha) = \rho(M', \alpha')$ except on a subvariety of $R_k(G)$ of a certain type.

Let A be a finitely-presented $\mathbb{C}G$-module. We define a subvariety Σ_A of $R_k(G)$ by

$$\Sigma_A = \{\theta : \mathbb{C}^k \otimes_\theta A \neq 0\}.$$

To see that this is a subvariety, consider a presentation matrix (λ_{ij}) for A. Then, if (λ_{ij}) is an $(m \times n)$-matrix – i.e. A has n generators and m relations – $\mathbb{C}^k \otimes_\theta A$ is the quotient of $\mathbb{C}^{nk}$ by the row-space of the complex $(mk) \times (nk)$-matrix $(\theta(\lambda_{ij}))$. Thus Σ_A is the zero set of all the $nk \times nk$ minors of $(\theta(\lambda_{ij}))$, and each minor is clearly a regular function on $R_k(G)$.

We define a *special subvariety* of $R_k(G)$ to be a subvariety of the form Σ_A, where $A = \mathbb{C} \otimes_{\mathbb{Z}} A'$ for some finitely-presented $\mathbb{Z}G$-module A' satisfying:

(1) $\mathbb{Z} \otimes_{\mathbb{Z}G} A' = 0.$

In particular, if (λ_{ij}) is any square matrix over $\mathbb{Z}G$ such that $(\epsilon(\lambda_{ij}))$ is unimodular, where $\epsilon : \mathbb{Z}G \to \mathbb{Z}$ is the usual augmentation, then $\{\theta : \det(\theta(\lambda_{ij})) = 0\}$ is a special subvariety. If $f(\theta) = \det(\theta(\lambda_{ij}))$, we refer to f as a *special function.* Since $\Sigma_{A \oplus B} = \Sigma_A \cup \Sigma_B$, the union of two special subvarieties is special. Note that a special subvariety of $R_k(G)$ is invariant under conjugation by any element of $U(k)$. If $k = 1$, then for any special subvariety Σ_A there is an element $\lambda \in \mathbb{Z}G$ such that $\epsilon(\lambda) = 1$ and $\theta(\lambda) = 0$ for any $\theta \in \Sigma_A$. If (λ_{ij}) is an $m \times n$ presentation matrix of A then some integral linear combination of the $(n \times n)$-minors of $(\epsilon(\lambda_{ij}))$ equals 1. This is just a polynomial in the entries of $(\epsilon(\lambda_{ij}))$ and we choose λ to be the same polynomial, replacing each occurrence of $\epsilon(\lambda_{ij})$ of λ_{ij}. Because $k = 1$, $\theta(\lambda)$ is a linear combination of the $(n \times n)$-minors of $(\theta(\lambda_{ij}))$, for any $\theta \in R_1(G)$.

We point out the following important property:

PROPOSITION 3.1. *A special subvariety contains no point of $R_k(G)$ which factors through a representation of a group of prime power order.*

Proof. This is an immediate consequence of lemma (I.4.3). Suppose we have a homomorphism $G \to P$, where P is a p-group, and θ is induced by $\theta' \in R_k(P)$. If A' is a $\mathbb{Z}G$-module satisfying (1), let $B' = \mathbb{Z}P \otimes_{\mathbb{Z}G} A'$. Then it follows from lemma (I.4.3) that $B = \mathbb{C} \otimes_{\mathbb{Z}} B' = 0$. Therefore $\mathbb{C}^k \otimes_\theta A = C^k \otimes_{\theta'} B = 0$.

Denote by $P_k(G) \subseteq R_k(G)$ the set of all θ which factor through some group of prime power order. $P_k(G)$ is often a dense subset of $R_k(G)$ – e.g. If $G = \mathbb{Z}^m$, then $P_1(G) \subseteq R_1(G) = T^m$, the m-torus, is the set of all m-tuples $(z_1, \ldots, z_m)$ where each z_i is a p-th power root of unity (some prime p). In $U(k)$ the elements of prime-power order are dense – they are the elements whose eigen-values are all powers of some single prime. Thus $P_k(F)$ is dense in $R_k(F)$ for a free group F. Also $P_2(D)$ is dense in $R_2(D)$ but, by contrast, $P_2(D_\Lambda)$ consists only of those representations induced from $R_2(\mathbb{Z}/2)$ by the canonical homomorphism $D_\Lambda \to \mathbb{Z}/2$, if $\frac{1}{2} \in \Lambda$ – see (I.1(e)). For finite groups G which are not a product of groups of prime power order, it is easy to see that $P_k(G)$ is smaller than (and, therefore, not dense in) $R_k(G)$.

Our interest in special subvarieties stems from:

PROPOSITION 3.2. *Suppose C is a free chain complex over $\mathbb{Z}G$, finitely generated in each dimension, and suppose $\bar{C} = \mathbb{Z} \otimes_{\mathbb{Z}G} C$ satisfies:*

(2) $H_q(\bar{C}) = 0,\quad$ *for $m \geq q > n$, and $H_n(\bar{C})$ is torsion-free.*

Then there is a special subvariety $\Sigma \subseteq R_k(G)$ such that $H_q(C; \theta) = 0$, for $m \geq q > n$, if $\theta \notin \Sigma$.

Proof. We begin with the standard construction of a chain contraction: $\bar{s}_q : \bar{C}_q \to \bar{C}_{q+1}$, for $m \geq q \geq n$, satisfying

(3) $\partial \bar{s}_q + \bar{s}_{q+1}\partial = 1\quad$ for $m \geq q > n$ and $\partial \bar{s}_n \mid \partial C_{n+1} = 1$.

Define $\bar{s}_n : \partial\bar{C}_{n+1} \to \bar{C}_{n+1}$ so that $\partial \bar{s}_n = 1$; since $H_n(\bar{C})$ is torsion-free, $\partial\bar{C}_{n+1}$ is a direct summand of $\bar{C}_n$ and so we can extend $\bar{s}_n$ over $\bar{C}_n$. Now assume $\bar{s}_q$ is defined for $n \leq q < l \leq m$, so that (3) holds for $l > q > n$. As a consequence $\partial \circ (s_{l-1}\partial - 1) = 0$. Since $H_l(\bar{C}) = 0$, $\operatorname{Im}(\bar{s}_{l-1}\partial - 1) \subseteq \partial\bar{C}_{l+1}$ and so, since $\bar{C}_l$ is free, we can construct $\bar{s}_l$ as desired.

Now choose homomorphisms $s_q : C_q \to C_{q+1}$, for $m \geq q \geq n$, so that $1 \otimes s_q = \bar{s}_q$ (C_q is free). For any $\theta \in R_k(G)$ define $s_q^\theta : \mathbb{C}^k \otimes_\theta C_q \to \mathbb{C}^k \otimes_\theta C_q$ to be $1 \otimes s_q$. The endomorphism $\partial s_q^\theta + s_{q+1}^\theta \partial$ is an isomorphism – in fact, the identity – for $m \geq q > n$ when θ is the trivial representation, since it is then just k copies of $\partial \bar{s}_q + \bar{s}_{q+1}\partial$. If we define $f_q(\theta) = \det(\partial s_q^\theta + s_{q+1}^\theta \partial)$, then f_q is a special function for $m \geq q > n$. If we define $\Sigma = \Sigma_{n+1} \cup \cdots \cup \Sigma_m$, where Σ_q is the zero set of f_q, then Σ is a special subvariety and $\partial s_q^\theta + s_{q+1}^\theta \partial$ is an isomorphism for $m \geq q > n$ if $\theta \notin \Sigma$. We see that this implies $H_q(C; \theta) = 0$ (for $m \geq q > n$). Let $\phi_q^\theta = \partial s_q^\theta + s_{q+1}^\theta \partial$, an isomorphism if $\theta \notin \Sigma$. It is clear that $\phi_q^\theta(\operatorname{Ker} \partial) \subseteq \operatorname{Im} \partial$ and so $\dim_{\mathbb{C}} \operatorname{Ker} \partial \leq \dim_{\mathbb{C}} \operatorname{Im} \partial$. On the other hand $\operatorname{Im} \partial \subseteq \operatorname{Ker} \partial$, and so $\operatorname{Im} \partial$ must equal $\operatorname{Ker} \partial$.

COROLLARY 3.3. *Suppose (M, α) and (N, β) are homology cobordant G-manifolds. Then, for some large subset (i.e. complement of a special subvariety) L of $R_k(G)$, $\rho(M, \alpha) \mid L = \rho(N, \beta) \mid L$.*

Combining this Corollary with Proposition 3.1, we have the following. Suppose $i : V \to R_k(G)$ is an analytic map from a connected analytic manifold V such that $i(V)$ contains at least one point of $P_k(G)$ – for example, the maps i and $\check{\imath}$ of (I.1.(d), (e), (f)) and (I.4) have this property. Then $\rho(M, \alpha) \circ i = \rho(N, \beta) \circ i$ off some *proper* analytic subvariety of V – in particular $\rho(M, \alpha) \circ i = \rho(N, \beta) \circ i$ on an open, dense subset of V.

Proof of Corollary. Let (W, γ) be a homology cobordism between (M, α) and (N, β). Then $H_*(W, M) = H_*(W, N) = 0$ and so, by Prop. 3.2, $H_*(W, M\colon \theta\gamma) = 0$ in a large subset of $R_k(G)$. In particular $\operatorname{sign}_{\theta\gamma}(W) = 0$, for all such θ, since this is the signature of a Hermitian form actually defined on Image $\{H_*(W; \theta\gamma) \to H_*(W, M; \theta\gamma)\}$ – similarly $\operatorname{sign}(W) = 0$. It then follows immediately from the Index Theorem (II.1(1)) that $\rho(M, \alpha) \cdot \theta = \rho(N, \beta) \cdot \theta$.

4. In this section we will show that in many cases, including all our applications to higher-dimensional links and some classical links (see Proposition (III.2.2))), $\rho(M, \alpha)$ has a singular locus Σ which is a special subvariety. When this is the case, $\rho(M, \alpha) \mid \Sigma$ gives no information about the homology bordism class of (M, α). In other words, as long as $\rho(M, \alpha) = \rho(N, \beta)$ on the complement of Σ, there is no way to use the results of the previous section to show that (M, α) and (N, β) are not homology bordant. (We will, however, give some examples, in (III.4), in the context of classical links, where $\rho(M, \alpha)$ and $\rho(N, \beta)$ coincide on any domain of continuity but differ in any large subset of $R_k(G)$ – and so, by Corollary 3.3, are not homology bordant.)

We need a preliminary definition. Denote by $R_k^0(G)$ the set of all $\theta \in R_k(G)$ such that there exists a common non-zero fixed vector $v \in \mathbb{C}^k$ for every $\theta(g), g \in G$. In other words $R_k^0(G)$ is the conjugacy class of the subvariety $R_{k-1}(G) \subseteq R_k(G)$. This inclusion is defined by $\theta \mapsto \bar{\theta}$ where $\bar{\theta}(g) \cdot (z_1, \ldots, z_k) = (\theta(g) \cdot (z_1, \ldots, z_{k-1}), z_k)$. Let $\tilde{R}_k(G)$ denote the complement of $R_k^0(G)$.

THEOREM (4.1). *Let (M, α) be an odd-dimensional connected oriented G-manifold which satisfies*:

(i) $H_i(M) = 0$ *for* $1 < i < n - 1$ $(n = \dim M)$

(ii) $H_1(M) \xrightarrow{\alpha_*} H_1(G)$ *is an isomorphism, and* $H_1(G)$ *is torsion-free*

(iii) *If* $n = 3$, *then* α *factors through a finitely-presented group* $\pi : \pi_1(M) \xrightarrow{\alpha'} \pi \to G$, *with* $H_1(\pi) \approx H_1(G)$ *and* $H_2(\pi) = 0$.

Then, for some large subset L of $R_k(G)$, $L \cap \tilde{R}_k(G)$ is contained in some domain of continuity for (M, α).

Remark. (a) If $n > 3$, (iii) is automatically satisfied for $\pi = \pi_1(M)$.

(b) I do not know whether $\rho(M, \alpha)$ is continuous on some large subset.

Before proving Theorem (4.1) we point out a corollary.

COROLLARY (4.2). *Suppose (M, α) and (N, β) are G-manifolds of the same dimension satisfying* (i)–(iii) *in Theorem* (4.1). *Suppose that, for every domain of continuity D in any $R_k(G)$, there exists a large subset L such that $\rho(M, \alpha)$ and $\rho(N, \beta)$ agree on $D \cap L$. Then for every k there exists a large subset L_k of $R_k(G)$ such that* $\rho(M, \alpha) \mid L_k = \rho(N, \beta) \mid L_k$.

In other words, if ρ can detect that (M, α) and (N, β) are not homology cobordant, then it can, in fact, detect it in some domain of continuity.

Proof of Corollary. Set $A_k = \{\theta \in R_k(G) : \rho(M, \alpha) \cdot \theta \neq \rho(N, \beta) \cdot \theta\}$. We show, by induction on k, that A_k is contained in a special subvariety of $R_k(G)$. By the theorem $A_k \subseteq \Sigma \cup R_k^0(G)$, for some special subvariety Σ. By induction $A_{k-1} = A_k \cap R_{k-1}(G)$ is contained in a special subvariety Σ' of $R_{k-1}(G)$. It is an immediate consequence of the definition of special subvariety that $\Sigma' = R_{k-1}(G) \cap \Sigma''$, for some special subvariety Σ'' of $R_k(G)$. Since $R_k^0(G)$ is the conjugacy class of $R_{k-1}(G)$ and Σ'' is invariant under conjugation, we have $A_k \cap R_k^0(G) \subseteq \Sigma''$, and so $A_k \subseteq \Sigma \cup \Sigma''$. Since $\Sigma \cup \Sigma''$ is special, the proof is complete.

Proof of Theorem (4.1). By (i), (ii) we can choose $X \subseteq M$, X a one-point union of circles, so that $H_i(M, X) = 0$ for $i < n - 1$. Thus, by Proposition (3.2), there is a special subvariety Σ of $R_k(G)$ so that $H_i(M, X; \theta\alpha) = 0$ for $\theta \notin \Sigma$, $i < n-1$. So $H_i(M; \theta\alpha) \approx H_i(X; \theta\alpha_j)$ for $i < n-2, \theta \in \Sigma$, where $j : \pi_1 X \to \pi_1 M$ is induced by inclusion. If $n \geq 5$, then $H_{n-2}(M; \theta\alpha) = 0$ for $\theta \in \Sigma$, since $H_{n-2}(M; \theta\alpha) = H_2(M; \theta\alpha)$, by duality, and $2 < n-2$.

Consider the continuity stratification $\{\Sigma_i\}$ constructed in the proof of Theorem (2.1): $\Sigma_d = \{\theta : \dim_{\mathbb{C}} H_*(M, \theta\alpha) \geq d\}$. By the above considerations and duality, we have, for $n \geq 5$:

$$(1) \quad \dim_{\mathbb{C}} H_*(M; \theta\alpha) = 2(\dim_{\mathbb{C}} H_0(X; \theta\alpha) + \dim_{\mathbb{C}} H_1(X; \theta\alpha)) \qquad \text{for } \theta \notin \Sigma.$$

Furthermore

$$\dim_{\mathbb{C}} H_1(X; \theta\alpha) = \dim_{\mathbb{C}} H_0(X; \theta\alpha) - k\chi(X) = \dim_{\mathbb{C}} H_0(X; \theta\alpha) + k(m-1);$$
$$m = \operatorname{rank} H_1(X).$$

So (1) becomes $\dim_{\mathbb{C}} H_*(M; \theta\alpha) = 4 \dim H_0(X; \theta\alpha) + 2k(m-1)$, if $\theta \in \Sigma$. But a simple computation shows that $H_0(X; \theta\alpha) = 0$ exactly when $\theta \in R_k^0(G)$. Thus we have shown that, for $d = 2k(m-1)$, $R_k(G) = \Sigma_d \cup \Sigma$ and $\Sigma_{d+1} \subseteq R_k^0(G) \cup \Sigma$. The theorem now follows, for $n \geq 5$, using the continuity stratification:

$$R_k(G) \supseteq \Sigma_{d+1} \cup \Sigma \supseteq \Sigma_{d+2} \cup \Sigma \supseteq \cdots \supseteq \Sigma \supseteq \Sigma \cap \Sigma_1 \supseteq \Sigma \cap \Sigma_2 \supseteq \cdots$$

We now look at $n = 3$. By duality:

$$(2) \quad \dim_{\mathbb{C}} H_*(M; \theta\alpha) = 2(\dim H_1(M; \theta\alpha) + \dim H_0(M; \theta\alpha))$$

We will also use the following exact homology sequences:

$$(3) \quad 0 \to H_1(M; \theta\alpha) \to H_1(M, *; \theta\alpha) \to H_0(*; \theta\alpha) \to H_0(M; \theta\alpha) \to 0$$

$$(4) \quad H_2(M, X; \theta\alpha) \xrightarrow{\partial} H_1(X, *; \theta\alpha_j) \to H_1(M, *; \theta\alpha) \to H_1(M, X; \theta\alpha) \to 0.$$

Note that $H_0(*; \theta\alpha) = \mathbb{C}^k$ for all θ, $H_1(X, *; \theta\alpha_j) = \mathbb{C}^{mk}$ for all θ, $H_0(M; \theta\alpha) = 0$ if $\theta \in R_k^0(G)$, and $H_1(M, X; \theta\alpha) = 0$ if $\theta \notin \Sigma$.

From (iii) we can construct maps:

$M \to B\Pi \to BG$, whose composition induces α.

This enables us to factor ∂, in (4), as a composition:

$$H_2(M, X; \theta\alpha) \to H_2(B\pi, X; \theta\alpha) \to H_1(X, *; \theta\alpha_j).$$

But $H_2(\pi) = 0$ implies $H_2(B\pi, X) = 0$ and so, by Prop. 3.2, we can choose a special subvariety Σ' of $R_k(G)$ so that $H_2(B\pi, X; \theta\alpha) = 0$ if $\theta \notin \Sigma'$. Note that $H_i(B\pi, X) = 0$ if $i \le 1$ and, since π is finitely-presented, the chain complex $\{C_q(B\pi, X; \mathbb{Z}G)\}$ is of finite type for $i \le 2$.

Let $d = 2k(m-1)$ again. Then we have $R_k(G) = \Sigma_d \cup \Sigma \cup \Sigma'$, since if $\theta \notin \Sigma \cup \Sigma'$ we have, by (4), that $\dim_{\mathbb{C}} H_1(M, *; \theta\alpha) = mk$, using $\partial = 0$, and thus, by (3), that $\dim_{\mathbb{C}} H_1(M; \theta\alpha) \ge k(m-1)$. By (2), $\theta \in \Sigma_d$. We also see that $\Sigma_{d+1} \subseteq R_k^0(G) \cup \Sigma \cup \Sigma'$, since if $\theta \notin R_k^0(G) \cup \Sigma \cup \Sigma'$ we have from the preceding argument and (3) that $\dim_{\mathbb{C}} H_1(M; \theta\alpha) = k(m-1)$. Thus, by (2), $\dim_{\mathbb{C}} H_*(M; \theta\alpha) = 2k(m-1)$ since $\theta \notin R_k^0(G)$.

Now the conclusion of Theorem (4.1) follows using the continuity stratification:

$$R_k(G) \supseteq \Sigma_{d+1} \cup \Sigma \cup \Sigma' \supseteq \Sigma_{d+2} \cup \Sigma \cup \Sigma' \supseteq \cdots \supseteq \Sigma \cup \Sigma' \supseteq (\Sigma \cup \Sigma') \cap \Sigma_1 \supseteq \cdots \quad \square$$

An examination of the proof of Theorem (4.1) shows that condition (iii) is required only to assure that, for some special subvariety Σ in $R_k(G)$, $\dim_{\mathbb{C}} H_1(M, *; \theta\alpha) = mk$ if $\theta \notin \Sigma$. Thus we can look for substitutes – for example:

ADDENDUM TO THEOREM (4.1). *The Theorem holds if (iii) is replaced by:*
(iii)′ *G is free abelian and the $\mathbb{Z}G$-module $H_1(\tilde{M}, \tilde{*})$ has rank m.*

Proof. Since $H_1(\tilde{M}, \tilde{X}) \otimes_{\mathbb{Z}G} \mathbb{Z} \approx H_1(M, X) = 0$ we can apply Nakayama's lemma to construct $\Delta \in \mathbb{Z}G$, with $\epsilon(\Delta) = 1$, so that $\Delta H_1(\tilde{M}, \tilde{X}) = 0$. Thus, if $\Lambda = \mathbb{Z}G[1/\Delta]$, then $H_1(X, *; \Lambda) \to H_1(M, *; \Lambda)$ is onto. Since $H_1(X, *; \Lambda)$ is free (over Λ) of rank m and $H_1(M, *; \Lambda) \approx H_1(\tilde{M}, \tilde{*}) \otimes_{\mathbb{Z}G} \Lambda$ still has rank m, we conclude that $H_1(M, *; \Lambda)$ is free of rank m. Now let $\Sigma = \{\theta : \det \theta(\Delta) = 0\}$ a special subvariety. If $\theta \notin \Sigma$, then we can extend θ to a representation of Λ, $\theta' : \Lambda \to M(k, \mathbb{C})$, by defining $\theta'(1/\Delta) = \theta(\Delta)^{-1}$. For such θ we have $H_1(M, *; \theta\alpha) \approx H_1(\tilde{M}, \tilde{*}) \otimes_\theta \mathbb{C}^k \approx H_1(M, *; \Lambda) \otimes_{\theta'} \mathbb{C}^k$ which is clearly of complex dimension mk.

Chapter III: Application to links

1. We now apply the invariants $\rho(M, \alpha)$, defined in Chapter II, to obtain several invariants of links. We first fix terminology and notation. By an *n-link* we will mean a smooth imbedding $f : S_1^n + \cdots + S_m^n \to S^{n+2}$, where $\{S_i^n)$ are m copies of S^n; m is the *multiplicity* of f. $L_i(f) = f(S_i^n)$ is the *i-th component* of f. $L(f) = \bigcup_i L_i(f)$ admits a unique (up to homotopy) trivialization of its normal bundle (*0-framing* of f) agreeing with the orientation induced by the natural orientations of S^{n+2} and $\{S_i^n\}$ and satisfying the extra condition, if $n = 1$, that the translate $L_i'(f)$ of $L_i(f)$

along either normal field in the 0-framing has zero linking number with $L_i(f)$. We will also only consider links f with the property that the linking number of any two components of f is zero.

We recall the notions of meridian and, for $n = 1$, longitude. Given an n-link f, let

$$F : S_1^n \times D^2 + \cdots + S_m^n \times D^2 \to S^{n+2}$$

be an imbedding such that $F \mid S_i^n \times 0 = f \mid S_i^n$, for all i, and such that the associated trivialization of the normal bundle of $L(f)$ agrees with the 0-framing. Choose $x_i \in S_i^n$ and $x \in S^1$ and let $m_i = F \mid x_i \times S^1$ and, if $n = 1$, $l_i = F \mid S_i^n \times x$. Choose γ_i from the base-point of $S^{n+2} - L(f)$ to $F(x_i, x)$. Then $\gamma_i \cdot m_i \cdot \gamma_i^{-1}$ and $\gamma_i \cdot l_i \cdot \gamma_i^{-1}$ define elements $\mu_i, \lambda_i \in \pi_1(S^{n+2} - L(f))$ which depend only on f and the choice of γ_i. If we make another choice of $\{\gamma_i\}$ then we obtain $\xi_i \mu_i \xi_i^{-1}$, $\xi_i \lambda_i \xi_i^{-1}$ for some $\xi_i \in \pi_1(S^{n+2} - L(f))$ and, conversely, for any ξ_i we can choose a corresponding $\{\gamma_i\}$. The set $\{\mu_i, \lambda_i\}$ is a *meridian-longitude pair* – μ_i is a meridian, λ_i a longitude (for $n = 1$).

The surgery manifold $M(f)$ is defined to be:

$$M(f) = \overline{S^{n+2} - \text{Image}\,(F)} \cup_F \coprod_{i=1}^{m} D_i^{n+1} \times S^1.$$

Clearly

$$\pi_1(M(f)) = \begin{cases} \pi_1(S^{n+2} - L(f)) & \text{if } n > 1 \\ \pi_1(S^{n+2} - L(f))/\langle \lambda_1, \ldots, \lambda_m \rangle & \text{if } n = 1 \end{cases}.$$

There are four ways in which we can consider $M(f)$ as a G-manifold, for different choices of G and with varying restrictions on f. Three will depend on a choice of meridans.

(a) $G = \mathbb{Z}^m$ (free abelian group of rank m), $\alpha : \pi_1(M(f)) \to \mathbb{Z}^m$ defined by $\alpha(\xi) = (l(\xi, L_i(f))$, where l denotes linking number, or, alternatively, by the Hurewicz homomorphism followed by the identification of $H_1(M(f)) \approx \mathbb{Z}^m$ defined by $\Sigma_{i=1}^m n_i[\mu_i] \leftrightarrow (n_1, \ldots, n_m)$. This does not depend on a particular choice of meridians.

(b) Suppose that f is a *boundary link* (see e.g. [Gu], [CS]). This means that the $L_i(f)$ bound disjoint submanifolds (SEIFERT "surfaces") of S^{n+2} or, more algebraically, that for some choice of meridians, the homomorphism $\mu : F^m \to \pi_1(S^{n+2} - L(f))$, where F^m is the free group with basis $\{x_1, \ldots, x_m\}$, defined by $\mu(x_i) = \mu_i$, admits a left inverse α':

$\pi_1(S^{n+2} - L(f)) \to F^m$ – i.e. $\alpha' \circ \mu =$ identity. Note α' induces $\alpha : \pi_1(M(f)) \to F^m$ since it is easy to see that $\alpha'(\lambda_i) = 1$. An *F^m-structure* on f is a choice of such α'. The F^m-structure is determined by the choice of meridians since μ induces an isomorphism $F^m \approx \pi_1(S^{n+2} - L(f))/(\pi_1(S^{n+2} - L(f))_\omega$ where, for any group G, G_ω is the intersection of all terms in the lower central series of G (see [Gu]). Not every choice of meridians for a boundary link will determine an F^m-structure, though, since not every set of conjugates of a basis of F^m is again a basis. Two different F^m-structures on a boundary link differ by a *special automorphism* of F^m i.e. one which sends each x_i to a conjugate of itself. The structure of the group of special automorphisms of F^m is known – see [Ko].

(c) Suppose that $n > 1$, or $n = 1$ and the $\bar{\mu}$-invariants of f vanish. This means that $\{\lambda_i\} \subseteq \pi_1(S^3 - L(f))_\omega$, or, alternatively by [M], that *any* meridian choice $\mu : F^m \to \pi_1(S^3 - L(f))$ induces an isomorphism $\tilde{F}^m \approx \pi_1(S^3 - L(f))$, where $\tilde{G}$ denotes the nilpotent completion of G. It is then also true that μ induces isomorphism $\overline{F^m} \approx \overline{\pi_1(S^3 - L(f))} \approx \overline{\pi_1(M(f))}$, where $\bar{G}$ denotes the residually nilpotent algebraic closure of G (see (I.2)). An *$\bar{F}^m$-structure* on f will mean any homomorphism $\alpha : \pi_1(M(f)) \to \bar{F}^m$ such that $\alpha(\mu_i)$ is a conjugate of x_i, for each meridian of f. Such α induces an isomorphism $\pi_1(M(f)) \approx \bar{F}^m$ but it is not necessarily true that $\alpha(\mu_i) = x_i$ for some choice of meridians. It is true, however, that any link with an $\bar{F}^m$-structure has vanishing $\bar{\mu}$-invariants. Any two $\bar{F}^m$-structures on f differ by a *special* automorphism of $\bar{F}^m$ i.e. an automorphism which sends x_i to a conjugate of x_i, for every i. Also note that, for any sequence of elements $g_1, \ldots, g_m \in \bar{F}^m$, there is a unique automorphism of $\bar{F}^m$ defined by $x_i \mapsto g_i x_i g_i^{-1}$ (see [L1]). Since the centralizer of x_i in $\bar{F}^m$ is the cyclic group generated by x_i, it is easy to describe the group of special automorphisms of $\bar{F}^m$.

(d) A refinement of (c) is possible if $n > 1$ or, for $n = 1$, when f has vanishing $\hat{\mu}$-invariants. This means that the longitudes $\{\lambda_i\}$ lie in the kernel of the map $\pi_1(S^3 - L(f)) \to \widehat{\pi_1(S^3 - L(f))}$ to the algebraic closure (see (I.2)). This is equivalent to requiring that the map $\mu : F^m \to \pi_1(S^3 - L(f))$ defined by any choice of meridians induces an isomorphism $\hat{F}^m \approx \pi_1\widehat{(S^3 - L(f))} \approx \widehat{\pi_1(M(f))}$. An *$\hat{F}^m$-structure* of f is a homomorphism $\alpha' : \pi_1(S^{n+2} - L(f)) \to \hat{F}^m$ (and, therefore, inducing $\alpha : \pi_1(M(f)) \to \hat{F}^m$) such that, for any choice of meridians, $\alpha'(\mu_i)$ is conjugate to x_i, for all i. As in (c), there may not exist a meridian choice so that $\alpha'(\mu_i) = x_i$. If $n > 1$, or if f is a *sublink* of a *homology boundary* link (see [C], [L1]), f admits an $\hat{F}^m$-structure. This may, in fact, be true for every link with vanishing $\bar{\mu}$-invariants – it may even be true that $\hat{F}^m = \bar{F}^m$. Again any two $\hat{F}^m$-structures differ by a special automorphism of $\hat{F}^m$ (each x_i is sent to a conjugate of itself). As in (c) for any

$\{g_i\} \subseteq \hat{F}^m$, $x_i \mapsto g_i x_i g_i^{-1}$ defines a unique special automorphism of $\hat{F}^m$, but it is not known whether the centralizer of x_i consists only of powers of x_i. Thus the group of special automorphisms is not completely known.

Two links f, f' are concordant if there exists a proper smooth imbedding:

$$F : I \times (S_1^n + \cdots + S_m^n) \to I \times S^{n+2}$$

with $F(t \times S_i^n) \subseteq t \times S^{n+2}$, for $t = 0, 1$ and $F \mid 0 \times S_i^n = f \mid S_i^n$ and $F \mid 1 \times S_i^n = f' \mid S_i^n$. If f, f' are links with a G-structure ($G = F^m$, $\bar{F}^m$ or $\hat{F}^m$), then F is a *G-concordance* if it is equipped with a homomorphism $\breve{\alpha} : \pi_1((I \times S^{n+2}) - \text{Image } F) \to G$ which restricts to the given G-structures of f, f' up to an inner automorphism of G. (We will generally identify G-structures which differ by an inner automorphism.) When $G = \mathbb{Z}^m$ it is clear that any concordance admits a unique $\breve{\alpha}$.

PROPOSITION (1.1). *If f, f' are G-concordant G-links, then $M(f)$ and $M(f')$ are homology G-bordant. Moreover, there exists a homology G-bordism $(V, \breve{\alpha})$ such that the inclusions induce homomorphisms $\pi_1(M(f)) \to \pi_1(V)$, $\pi_1(M(f')) \to \pi_1(V)$ which are normally surjective.*

Proof. If F is a G-concordance, then we can extend F to an imbedding $F' : I \times (S_1^n \times D^2 + \cdots + S_m^n \times D^2) \to I \times S^{n+2}$ and then we define $V = \overline{(I \times S^{n+2}) - \text{Image } F'} \cup_{F'} (I \times (D_1^{n+1} \times S^1 + \cdots + D_m^{n+1} \times S^1))$. If $\breve{\alpha}$ is induced from the G-structure on F, then the assertions of the Proposition follow easily.

PROPOSITION (1.2). *Suppose f is a G-link and f is a concordant to another link f' – if $G = F^m$, suppose f' is a boundary link and f is boundary concordant to f'. Then f' admits a G-structure so that f and f' are G-concordant.*

A *boundary concordance* between two boundary links is a concordance $F : I \times (S_1^n + \cdots + S_m^n) \to S^{n+2}$ such that some choice of meridians $\mu : F^m \to \pi_1(S^{n+2} - \text{Image } F)$ admits a left inverse $\breve{\alpha} : \pi_1(S^{n+2} - \text{Image } F) \to F^m$ i.e. $\breve{\alpha}\mu = 1$.

Proof. Consider the homomorphism

$$\pi = \pi_1(S^{n+2} - L(f)) \to \pi_1(I \times S^{n+2} - \text{Im } F) = \theta$$

induced by inclusion. It follows from [L1] that the induced map $\bar{\pi} \to \bar{\theta}$ and $\hat{\pi} \to \hat{\theta}$ are isomorphisms. When $G = F^m$, f is a boundary link and F a boundary concordance, then $\pi/\pi_\omega \to \theta/\theta_\omega$ is an isomorphism. In fact μ induces isomorphisms $F^m \approx \pi/\pi_\omega$ and

$F^m \approx \theta/\theta_\omega$ by Stallings Theorem. Since the G-structure on f gives an identification of π/π_ω, $\bar{\pi}$ or $\hat{\pi}$ with G, we obtain, via the inclusion, an identification of θ/θ_ω, $\bar{\theta}$ or $\hat{\theta}$ with G. By the same argument we obtain from the G-structure on F an identification of π'/π'_ω or $\bar{\pi}'$ or $\hat{\pi}'$ with G, where $\pi' = \pi_1(S^{n+2} - L(f'))$. To see that this identification defines a G-structure on f', which then is clearly G-concordant to f, we only need note that any meridian choices for f' are conjugate, in θ, to meridians of f. When $G = F^m$, we note that any meridian choice for F corresponds, under $\theta/\theta_\omega \approx \pi'/\pi'_\omega$, to a meridian choice for f'.

2. Suppose f is an n-link with a G-structure, where $G = \mathbb{Z}^m$, F^m, $\bar{F}^m$ or $\hat{F}^m$ (a $\mathbb{Z}^m$-structure means no extra structure). We will use the G-structure α', or rather the induced $\alpha : \pi_1(M(f)) \to G$, to define an invariant for f from $\rho(M(f), \alpha)$: $R_k(G) \to \mathbb{R}$. We will denote this invariant $\underline{\sigma(f)}$, $\underline{\sigma_b(f)}$, $\underline{\bar{\sigma}(f)}$ or $\underline{\hat{\sigma}(f)}$, respectively, when $G = \mathbb{Z}^m$, F^m, $\bar{F}^m$ and $\hat{F}^m$.

Although these invariants take values in $\mathbb{R}$, the non-integral part is determined in most – and probably all – cases by a classical signature invariant of the component knots. For any link f let $s_i(f)$ denote the signature of $L_i(f)$, i.e. the signature of any Seifert surface bounded by $L_i(f)$ – thus $s_i(f) = 0$ if $n \not\equiv 3 \bmod 4$. For a complex number z, with $|z| = 1$, we define $\arg z \in \mathbb{R}/\mathbb{Z}$ by $z = e^{2\pi i \arg z}$.

THEOREM (2.1). *Let f be any $(2q-1)$-link. Then*:

(a) $$\sigma(f) \cdot \theta \equiv (-1)^{q+1} 2 \sum_{i=1}^{m} s_i(f) \arg \theta(x_i) \mod \mathbb{Z}$$

for any $\theta \in \mathscr{R}_1(\mathbb{Z}^m)$.

(b) *If f is also a sublink of a homology boundary link with an $\bar{F}^m$-structure, then*:

$$\bar{\sigma}(f) \cdot \theta \equiv (-1)^{q+1} 2 \sum_{i=1}^{m} s_i(f) \arg \det \theta(x_i) \mod \mathbb{Z}$$

for any $\theta \in \mathscr{R}_k(\bar{F}_m)$.

In (a), (b), *$\{x_i\}$ are the standard generators of $\mathbb{Z}^m$ or $F^m \subseteq \bar{F}^m$.*

Proof. (a) If we define $V(f) = D^{n+3} \cup_F (D_1^{n+1} \times D^2 + \cdots + D_m^n + D^2)$ then $M(f) = \partial V(f)$. By pushing the interior of a Seifert surface for $L_i(f)$ into D^{n+3} and attaching to its boundary $D^{n+1} \times 0$, we obtain a closed oriented $(n+1)$-manifold $V_i \subseteq \operatorname{int} V(f)$. We can arrange that the $\{V_i\}$ are mutually disjoint by choosing the pushoffs carefully. Now remove the tubular neighborhoods of $\{V_i\}$ from $V(f)$ to obtain a cobordism V between $M(f)$ and $S^1 \times V_1 + \cdots + S^1 \times V_m$. Note that

$H_1(M(f)) \approx H_1(V)$ with a basis represented by $\{S^1 \times x_i\}$, $x_i \in V_i$. By the index theorem we have:

$$\sigma(f) = \sum_i \rho(S^1 \times V_i, \alpha_i) \quad \text{mod } \mathbb{Z}$$

where $\alpha_1 : \pi_1(S^1 \times V_i) \to \mathbb{Z}^m$ is induced by the inclusion $S^1 \times V_i \subseteq V$ and the identification $H_1(V) \approx H_1(M(f)) \approx \mathbb{Z}^m$ defined by the ordering and orientation of the components of f. Note that α_i, on $\pi_1(S^1 \times V_i) \approx \pi_1(S^1) \times \pi_1(V_i)$, is of the form $e_i \times \beta_i$, where $e_i : \pi_1(S^1) \to \mathbb{Z}^m$ is an isomorphism onto the i-th summand in $\mathbb{Z}^m$ and $\beta_i : \pi_1(V_i) \to \mathbb{Z}^m$ is induced by the inclusion $V_i \subseteq V$. According to [APS II] – see also [N, Th. 1.2], we have the formula:

$$\rho(S^1 \times V_i, a_i) \cdot \theta = \begin{cases} (-1)^q \operatorname{sign}(V_i, \theta\beta_i) \cdot (1 - 2 \arg \theta(x_i)) & \text{if } \theta(x_i) \neq 1 \\ 0 & \text{if } \theta(x_i) = 1 \end{cases}.$$

The desired formula will follow if we show that $\operatorname{sign}(V_i, \theta\beta_i) = \operatorname{sign} V_i$.

First note that $\operatorname{sign}(V_i, \theta\beta_i)$, considered as a function $\mathscr{R}_1(\mathbb{Z}^m) \to \mathbb{Z}$, depends only on the bordism class of (V_i, β_i) in $\Omega^{n+1}(\mathbb{Z}^m)$ and so defines an additive function on $\Omega^{n+1}(\mathbb{Z}^m)$. Now consider the familiar isomorphism:

$$\Omega^q(\mathbb{Z}^m) \approx \Omega^q(\mathbb{Z}^{m-1}) \oplus \Omega^{q-1}(\mathbb{Z}^{m-1})$$

where $[M^q, \beta] \in \Omega^q(\mathbb{Z}^{m-1})$ corresponds to $[M^q, \beta \times o] \in \Omega^q(\mathbb{Z}^m)$ and $[M^{q-1}, \beta] \in \Omega^{q-1}(\mathbb{Z}^{m-1})$ corresponds to $[S^1 \times M^{q-1}, 1 \times \beta] \in \Omega^q(\mathbb{Z}^m)$. But $\operatorname{sign}(S_1 \times M^{q-1}, \rho) = 0$, for any ρ, since intersection numbers are all zero in $S^1 \times M = \mathbb{R} \times \tilde{M}$, and, as a result, we see that $\operatorname{sign}(M^q, \beta) = \operatorname{sign}(M^q, p\beta)$, if $\beta \in \pi_1(M^q) \to \mathbb{Z}^m$ and $p : \mathbb{Z}^m \to \mathbb{Z}^{m-1}$ is projection on the first $(m - 1)$ factors. The result now follows by induction on m.

(b) Suppose f is a homology boundary link, i.e. we have an epimorphism $\phi : \pi_1(M(f)) \to F^m$ so that $\phi(\mu_i) \equiv x_i \bmod [F^m, F^m]$, for any i-th meridian μ_i. Applying the Pontriagin construction to ϕ yields disjoint "singular" Seifert surfaces, i.e. closed oriented disjoint submanifolds $M_i \subseteq M(f)$. We can make the $\{M_i\}$ connected by a simple surgery argument using the fact that ϕ is an epimorphism. Two different components of M_i can be joined by a path whose image under ϕ, a closed path in $K(F^m, 1)$, is null-homotopic – such a path can be used to form a connected sum of the two components. By pushing the $\{M_i\}$ slightly into int $V(f)$ we obtain $\{V_i\}$ and define V to be the complement in $V(f)$ of the union of disjoint tubular neighborhoods of the $\{V_i\}$. Now $\pi_1(V) \approx F^m$ with a basis $\{x_i\}$ consisting of meridians of the $\{V_i\}$ – in fact, there is a standard construction of the universal cover of V with fundamental domain $V(f)$, attached together along copies of $\{I \times V_i\}$, similar to the construction of Viro [Vi] for finite branched covers. Imbed $[-1, 1] \times M_i$ into $M(f)$ disjointly, so that $0 \times M_i$ is identified with $M_i \subseteq M(f)$. For each $w \in F^m$ define $D(w)$ to be a copy of $V(f)$. Attach $D(w)$ to $D(x_i^\epsilon w)$, $\epsilon = \pm 1$ by the attaching diffeomorphism $(t, x) \leftrightarrow (-t, x)$, $x \in M_i$, $\frac{1}{2} \leq \epsilon t \leq 1$. Then $V(f) = \bigcup_{w \in F^m} D(w)$ is simply-connected and a free cover of $V(f)$.

The inclusion $M(f) \subseteq V$ induces the homomorphism ϕ under this identification $\pi_1(V) \approx F^m$. Since $\partial V = M(f) - (S^1 \times V_1 + \cdots + S^1 \times V_m)$, the Index Theorem tells us that:

$$\rho(M(f), \phi) = \sum_{i=1}^{m} \rho(S^1 \times V_i, \phi_i)$$

where ϕ_i, induced by the inclusion $S^1 \times V_i \subseteq V$, is easily seen to be projection onto $\pi_1(S^1)$ followed by the inclusion $e_i : \pi_1(S^1) \to \mathbb{Z}^m$ defined by $e_i(t) = x_i$, where t is the appropriate generator. The product formula of [N, Th. 1.2) gives $\rho(S^1 \times V_i, \phi_i) = (-1)^q \operatorname{sign}(V_i)\rho(S^1, e_i)$. If $\{e^{2\pi i a_j}\}$ are the eigenvalues of $\theta(x_i)$ for $\theta \in R_k(F^m) - 0 \leq a_j \leq 1$ – then, by [APS, II]:

$$\rho(S^1, e_i) \cdot \theta = \sum_{a_j \neq 0} (1 - 2a_j) = -2 \arg\det \theta(x_i) \mod \mathbb{Z}.$$

Since $x_i = \phi(\mu_i) \mod [F^m, F^m]$ and the determinant of a commutator is 1, we have:

$$(1) \quad \rho(M(f), \phi) \cdot \theta \equiv (-1)^{q+1} 2 \sum_i \operatorname{sign}(V_i) \arg\det \theta\phi(\mu_i) \mod \mathbb{Z}.$$

Now suppose f is a sublink of a homology boundary link g. Suppose, in addition, that f is equipped with an $\bar{F}^m$-structure ψ and g has an F^n-structure ϕ as above ($n \geq m$). Let $x_1, \ldots, x_m$ be a basis of F^m and $y_1, \ldots, y_n$ a basis of F^n. We construct a commutative diagram:

$$(2) \quad \begin{tikzcd} \pi_1(M(f)) \arrow[r, "\psi"] \arrow[d, "i_*"'] & \bar{F}^m \arrow[dd, "e"] \\ \pi_1(W) \arrow[ur, "\psi'"'] & \\ \pi_1(M(g)) \arrow[u, "i'_*"] \arrow[r, "\phi"] & F^n \end{tikzcd}$$

W is the manifold obtained from $I \times M(f)$ by adding handles along the components of g not in f; thus $\partial W = M(f) - M(g)$. Note that i_* and i'_*, induced by inclusions, are both onto – we can, in effect, identify base points of these three spaces by choosing them on an arc $I \times x_0$ for some $x_0 \in M(f)$ away from the components of g. Choose meridians $\mu_1, \ldots, \mu_m \in \pi_1(M(f))$ and $\mu'_1, \ldots, \mu'_n \in \pi_1(M(g))$ so that $i_*(\mu_i) = i'_*(\mu'_i)$ for $i \leq m$. We show that there is a unique homomorphism $e : F^n \to \bar{F}^m$ satisfying:

$$(3) \quad e(\phi(\mu'_i)) = \begin{cases} \psi(\mu_i) & i \leq m \\ 1 & i > m \end{cases}.$$

Write $y_i = w_i(\phi(\mu'_i), \ldots, \phi(\mu'_n); y_1, \ldots, y_n)$, where $w_i(z_1, \ldots, z_n; y_1, \ldots, y_n)$ is a product of conjugates of the $\{z_i\}$ as a word in the free group on $\{z_i, y_i\}$. Thus the system of equations:

$$(4) \quad u_i = w_i(\psi(\mu_1), \ldots, \psi(\mu_m), 1, \ldots, 1; u_1, \ldots, u_n)$$

is contractible over $\bar{F}^m$ and so has a unique solution $\{u_i\}$ in $\bar{F}^m$. We define e by $e(y_i) = u_i$ for $1 \leq i \leq n$. We now show that e satisfies (3). By (4), e extends to a homomorphism: $e' : G \to \bar{F}^m$, where G is the group with presentation $\{y_1, \ldots, y_n, z_1, \ldots, z_n : y_i = w_i(z_1, \ldots, z_n, y_1, \ldots, y_n)\}$, by defining

$$e'(z_i) = \begin{cases} \psi(\mu_i) & i \leq m \\ 1 & i > m \end{cases}.$$

Let $\gamma : G \to F^n$ be the epimorphism defined by $\gamma(y_i) = y_i$, $\gamma(z_i) = \phi(\mu'_i)$. We need to show that $e' = e \circ \gamma$. But this will follow from two observations:

(1) $e'(y_i) = e \circ \gamma(y_i)$, and

(2) Kernel $\gamma = G_\omega$ (the intersection of the lower central series), by STALLINGS theorem [St], and $\bar{F}^m$ is residually nilpotent.

Since Kernel $i'_* = \langle \mu'_{m+1}, \ldots, \mu'_n \rangle$, then, by (3), ψ' is defined by the requirement that the bottom square of (2) commute. It remains to show that $\psi' \circ i_* = \psi$. But $\psi' \circ i_*(\mu_i) = \psi' i'_*(\mu'_i) = e\phi(\mu'_i) = \psi(\mu_i)$, if $i \leq m$. Moreover π/π_q is generated by $\{\mu_i\}$ for every q, where $\pi = \pi_1(M(f))$, since f has vanishing $\bar{\mu}$-invariants if $n = 1$ (see [M]) or by [St] if $n > 1$. Thus ψ and $\psi' \circ i_*$ induce the same homomorphism on every nilpotent quotient. Since $\bar{F}^m$ is residually nilpotent $\psi = \psi' \circ i_*$.

We now apply the Index Theorem and (2) to conclude that:

$$\bar{\sigma}(f) = \rho(M(f), \psi) \equiv \rho(M(g), e \circ \phi) = e^* \rho(M(g), \phi) \mod \mathbb{Z}.$$

Thus, by (1) and (3) we have:

$$\bar{\sigma}(f) \cdot \theta \equiv \rho(M(g), \phi) \cdot \theta e \equiv (-1)^{q+1} 2 \sum_{i=1}^{m} \operatorname{sign}(V_i) \arg \det \theta\psi(\mu_i) \mod \mathbb{Z}.$$

Since $\psi(\mu_i)$ is conjugate to x_i, it only remains to check that $\operatorname{sign}(V_i) = s_i(f)$. Now $s_i(f)$ is defined to be the signature of any Seifert surface for $L_i(f) = L_i(g)$. We may add a disk to its boundary to obtain a closed submanifold V_i' of $M(f)$ and, furthermore, since the linking numbers of the components of g are all zero, we may assume V_i' misses the other components of g and so $V_i' \subseteq M(g)$. It now suffices to observe that V_i and V_i' are homologous in $M(g)$, since this means that they determine homotopic mappings $M(g) \to S^1$ via the Pontriagin construction, and so are (oriented) cobordant.

This completes the proof of Theorem (2.1).

There is a completely analogous result for F^m-links but, in fact, this is already contained in (b) as a consequence of the result of Vogel [V1] that every unitary representation of F^m extends to one of $\bar{F}^m$.

Some unsolved questions are:

(i) Does (b) hold for $\hat{F}^m$ in place of $\bar{F}^m$? The proof uses residual nilpotence of $\bar{F}^m$. It is open whether $\hat{F}^m = \bar{F}^m$ and, of course, whether every unitary representation of $\hat{F}^m$ induces one of $\bar{F}^m$.

(ii) Is (b) true for every $\bar{F}^m$-link? It is open whether every $\bar{F}^m$-link is a sublink of a homology boundary link. There are precise homotopy and group-theoretic obstructions to an $\hat{F}^m$-link being concordant to a sublink of a homology boundary link (see [L1], [Le], [LMO]). For example if $H_3(\hat{F}^m) = 0$ (for $n = 1$) or if the "Vogel localization" of the m-fold wedge of circles is aspherical (for $n \geq 4$) then every $\hat{F}^m$-link is concordant to a sublink of a homology boundary link.

We conclude this section by observing that for many links f the invariants $\sigma(f)$, $\sigma_b(f)$, $\bar{\sigma}(f)$ or $\hat{\sigma}(f)$ satisfy the conclusion of Theorem (II.4.1), i.e. they are continuous on $L \cap \tilde{R}_k(G)$, for some large subset L of $R_k(G)$.

PROPOSITION (2.2). (a) *If f is any link of (odd) dimension $n > 1$, or an $\hat{F}^m$-link with $n = 1$ (or a boundary link), then $\hat{\sigma}(f)$, $\bar{\sigma}(f)$ and $\sigma(f)$ (or $\sigma_b(f)$) have a domain of continuity in $R_k(G)$ which contains $L \cap \tilde{R}_k(G)$ for some large subset L of $R_k(G)$.*

(b) *If f is a one-dimensional link with Alexander invariant of rank $m-1$, then $\sigma(f)$ has a large domain of continuity.*

The Alexander invariant is $H_1(\tilde{X})$, where $\tilde{X}$ is the universal abelian covering of $S^3 - L(f)$, regarded as a module over $\mathbb{Z}[\mathbb{Z}^m]$. If $m = 2$, for example, the condition in (b) means the Alexander polynomial is zero.

Proof. (a) is an immediate consequence of Theorem (II.4.1) for $n > 1$ $(H_1(G) \approx \mathbb{Z}^m)$. For $n = 1$, recall from [Le] that, for an $\hat{F}^m$-link, the $\hat{F}^m$-structure

$\pi_1(S^3 - L(f)) \to \hat{F}^m$ factors through a finitely-presented group satisfying (iii). To prove (b), we apply the Addendum to Theorem (4.1).

Putting Proposition (2.2) together with Corollary (II.4.2) we conclude that, for the class of links described in the Proposition, no additional concordance information can be obtained from $\hat{\sigma}$, $\bar{\sigma}$, σ, σ_b on any singular locus that cannot be already obtained on a domain of continuity. On the other hand we will show by examples that, for some one-dimensional links f, $\sigma(f)$ can detect concordance on a singular locus when it is useless on the domains of continuity.

3. We will now concern ourselves with some general methods of constructing links to display the possible values of these signature invariants. We present two such realization theorems.

THEOREM (3.1). *Let G be a finitely-generated group with a set of normal generators $\{g_1, \ldots, g_m\}$ and $\lambda = (\lambda_{ij})$ a $(-1)^{q+1}$-Hermitian matrix over $\mathbb{Z}G$ – i.e. $\lambda_{ij} = (-1)^{q+1}\bar{\lambda}_{ji}$, where $\lambda \mapsto \bar{\lambda}$ is the anti-involution of $\mathbb{Z}G$ defined by $g \mapsto g^{-1}$ for every $g \in G$ – satisfying:*

(i) *$H_1(G)$ is free abelian of rank m*

(ii) *$\epsilon(\lambda)$ is non-singular and, in addition, $\epsilon(\lambda_{ij}) = \pm\delta_{ij}$ for all i, j if $q = 1$, and $\epsilon(\lambda_{ii})$ is even for all i, if $q \neq 1$, 3, or 7. ($\epsilon : \mathbb{Z}G \to \mathbb{Z}$ is the usual augmentation.)*

(iii) *The coefficients, in any λ_{ii}, of all elements of order 2 in G are even.*

Then there exists a $(2q-1)$-dimensional sublink f of a homology boundary link, and a G-structure α on $M(f)$ such that $\alpha(\mu_i) = g_i$, for some set $\{\mu_i\}$ of meridians of f and such that:

$$(1) \quad \rho(M(f), \alpha) \cdot \theta = k \operatorname{sign} \epsilon(\lambda) - \operatorname{sign} \theta(\lambda)$$

for all $\theta \in R_k(G)$ if $q > 1$, or for all θ in some large subset of $R_k(G)$ if $q = 1$.

Note that $(M(f), \alpha)$ has a large domain of continuity, namely the set of all θ with $\det \theta(\lambda) \neq 0$. To obtain examples of links f where $\sigma(f)$ does not have a large domain of continuity we will use:

THEOREM (3.2). *Let $\lambda = (\lambda_{ij})$ be a Hermitian matrix over $\mathbb{Z}[\mathbb{Z}^2]$ satisfying:*

(i) *$\epsilon(\lambda_{ij}) = \pm\delta_{ij}$ for $i, j \geq 2$*

(ii) *$\lambda_{11} = 0$.*

Then there exists a one-dimensional 2-component link f such that:

$$(2) \quad \sigma(f) \cdot \theta = \operatorname{sign} \epsilon(\lambda) - \operatorname{sign} \theta(\lambda), \qquad \textit{for all } \theta \in R_1(\mathbb{Z}^2).$$

Proof of Theorem (3.1). Let f_0 be the trivial m-component link in S^{2q+1} and so $L(f_0)$ bounds the trivial disk link Δ_0 in D^{2q+2}. Choose a set of generators $h_1, \dots, h_n$ for G. By assumption there exist words $w_i(x_1, \dots, x_n, y_1, \dots, y_m)$, for $1 \le i \le n$, satisfying $w_i(x_1, \dots, x_n, 1, \dots, 1) = 1$ such that $h_i = w_i(h_1, \dots, h_n, g_1, \dots, g_m)$. Let E be the "finite E-group" (see [C]) with generators: $x_1, \dots, x_n, y_1, \dots, y_m$ and relations $x_i = w_i(x_1, \dots, x_n, y_1, \dots, y_m)$ $(1 \le i \le n)$. Clearly E is normally generated by $y_1, \dots, y_m$ and $H_1(E) \approx \mathbb{Z}^m$. Let $\phi : E \to G$ be the epimorphism defined by $\phi(x_i) = h_i$, $\phi(y_i) = g_i$.

We start by constructing a slice link f_1 with an E-structure on $M(f_1)$. Attach n 1-handles to D^{2q+2} along $S^{2q+1} - L(f_0)$ to produce X_0 so that $\pi_1(X_0 - \Delta_0)$ is free on generators $\{x_1, \dots, x_n, y_1, \dots, y_m\}$ where $\{x_i\}$ are the classes of the cores of the 1-handles and $\{y_i\}$ are meridians of f_0. Now attach n 2-handles to X_0 along normally framed smooth curves in $\partial X_0 - L(f_0)$ representing $w_i(x_1, \dots, x_n, y_1, \dots, y_m)x_i^{-1} \epsilon \pi_1(X_0 - \Delta_0)$ to obtain X. If $n = 1$ we may choose these curves to be isotopic in ∂X_0 to the curves which go once around the 1-handles, since $w_i(x_1, \dots, x_n, 1, \dots, 1) = 1$ (if $n > 1$, this is automatic) – see [L2] for example. Thus we may choose the normal framings so that these 2-handles cancel the 1-handles on D^{2q+2} – and so $X \approx D^{2q+2}$. Let f_1 be the slice link defined by f_0 in ∂X and Δ_1, in D^{2q+2}, the slice disk defined by $\Delta_0 \subseteq X$. Note that $D^{2q+2} - \Delta_1 = X - \Delta_0 \simeq K$ the standard 2-complex associated to the presentation: $\{x_1, \dots, x_n, y_1, \dots, y_m : x_i = w_i(x_1, \dots, x_n, y_1, \dots, y_m)\}$ of E. Thus $D^{2q+2} - \Delta_1 \simeq K$ and $M(f_1) \subseteq D^{2q+2} - \Delta_1$ are G-manifolds via $\phi : E \to G$. If $q > 1$, then $H_{q+1}(K; \theta) = 0$ for all θ. If $q = 1$, we can apply Proposition (II.3.2) to conclude that $H_2(K; \theta) = 0$ for all θ in some large subset of $R_k(G)$.

We also note that f_1 is a sublink of a homology boundary link. Let $\bar{f}_1$ be the link obtained by adding to f_1 the transverse spheres of the 2-handles used to construct X. $M(\bar{f}_1)$ is the boundary of the manifold Y obtained by removing from X the transverse disks of the 2-handles and $\pi_1(Y) \approx \pi_1(X_0 - \Delta_0)$ is the free group on $\{x_1, \dots, x_n, y_1, \dots, y_m\}$. Under the induced homomorphism $\pi_1(M(\bar{f}_1)) \to \pi_1(Y)$, meridians map to $\{y_1, \dots, y_m, x_i^{-1}w_i(x_1, \dots, x_n, y_1, \dots, y_m)\}$ which normally generate $\pi_1(Y)$. Thus $\bar{f}_1$ is a homology boundary link.

We now produce the desired link f from f_1 by adding handles of index $q + 1$ to D^{2q+2} along $S^{2q+1} - L(f_1)$. Suppose $\{S_i'\}$ is a collection of k disjoint q-spheres, with normal framings $\{t_i'\}$, in $S^{2q+1} - L(\bar{f}_1)$ satisfying:

(a) S_i' is null-homotopic in $S^{2q+1} - L(\bar{f}_1)$

(b) the linking numbers $l(S_i', S_j') = \epsilon(\lambda_{ij})$ if $i \neq j$

(c) if $q = 1$, $\{S_i'\}$ is the trivial link in S^{2q+1}

(d) if t_i' differs from the standard normal framing on S_i' by $\alpha_i \in \pi_q(SO_{q+1})$ – note that S_i' is unknotted in S^{2q+1} by (c) or a classical theorem of WHITNEY if $q > 1$ – then $h(\alpha_i) = \epsilon(\lambda_{ii})$ where $h : \pi_q(SO_{q+1}) \to \pi_q(S_q) \approx \mathbb{Z}$

is the standard evaluation map. Recall that h is onto for $q = 1, 3, 7$ and onto $2\mathbb{Z}$ for all other odd q. (Also note that S_i' is oriented by t_i'.)

Let W' be the manifold produced by surgery on S^{2q+1} along $\{S_i', t_i'\}$. Then W' is homeomorphic to S^{2q+1} by (b) and (d) if $q > 1$ since $(\epsilon(\lambda_{ij}))$ is non-singular, and by (c) and (d) if $q = 1$, since $\epsilon(\lambda_{ii}) = \pm 1$ (see [L5], [L6]). If f' is the link defined by f_1 in W', then $M(f') = \partial X'$, where X' is produced from $D^{2q+1} - \Delta_1$ by adding handles along $\{S_i', t_i'\}$. By (a), $X' \simeq K \vee S_i^{q+1} \vee \cdots \vee S_k^{q+1}$ and so $M(f')$ and X' are G-manifolds via ϕ and the cores of the handles represent a basis $\{\alpha_i'\}$ of a free summand of the $\mathbb{Z}G$-module $H_{q+1}(\tilde{X}')$, where $\tilde{X}'$ is the G-covering of X'. Then $H_{q+1}(X'; \theta) \approx \mathbb{C}^n \otimes_\theta H_{q+1}(\tilde{X}') \approx \mathbb{C}^{nk}$ for any $\theta \in R_n(G)$ if $q > 1$, or for θ in some large subset L of $R_n(G)$ if $q = 1$. The intersection pairing in $H_{q+1}(\tilde{X}')$ is represented, via the basis $\{\alpha_i'\}$, by a matrix $\{\lambda_{ij}'\}$ over $\mathbb{Z}R$ – by (b) and (d), $\epsilon(\lambda_{ij}') = \epsilon(\lambda_{ij})$. We conclude, from the Index theorem, that:

$$\rho(M(f'), \theta) = n \operatorname{sign}(\epsilon(\lambda_{ij})) - \operatorname{sign}(\theta(\lambda_{ij}'))$$

if $q > 1$, or for $\theta \in L$ if $q = 1$.

One way to construct a family of $\{S_i', t_i'\}$ satisfying (a)–(d) is to choose a ball $B \subseteq S^{2q+1} - L(\tilde{f}_1)$ and construct $\{S_i'\} \subseteq B$. In fact such $\{S_i'\}$ is completely determined by (b)–(d) and automatically satisfies (a) – note that $\lambda_{ij}' = \epsilon(\lambda_{ij})$. Our goal is then to modify $\{S_i', t_i'\}$, without disturbing (a)–(d), but changing λ_{ij}' to the desired λ_{ij}. This can be done in almost the identical manner as in the argument in [L3: appendix]. If $q > 1$, then one can change λ_{ij}' to $\lambda_{ij}' \pm g$ if $i \neq j$, or to $\lambda_{ii}' \pm (g + (-1)^{q+1} g^{-1})$ if $i = j$, for some $g \neq 1$ in G and particular values of i, j, by changing S_i' to $S_i' \# s$, a connected sum of S_i' with s a small sphere linking S_j', along an arc which, when lifted to $\tilde{X}'$, connects a lift $\tilde{B}$ of B to $g\tilde{B}$. Such a change does not affect property (a). A sequence of such changes will realize λ_{ij} – at each stage $l(S_i', S_j') = \epsilon(\lambda_{ij}')$ so (b) will hold at the end. To achieve (d) we simply construct t_i as stipulated. For $q = 1$, we must be more careful in the construction in order to preserve property (c). We follow the argument in [L3] more closely. Note that the change from λ_{ij}' to λ_{ij} can be broken up into a sequence of elementary changes of the form

$$\lambda_{ij}' \to \begin{cases} \lambda_{ij} \pm (g - 1) & i = a, j = b, a \neq b \\ \lambda_{ij}' \pm (g^{-1} - 1) & i = b, j = a, a \neq b \\ \lambda_{ij} \pm [g + g^{-1} - 2] & i = j = a = b \\ \lambda_{ij}' & (i, j) \neq (a, b) \text{ or } (b, a) \end{cases}$$

for some $g \in G$ and $1 \leq a, b \leq k$. To effect this change we replace S_a' by $S_a' \# S_0 \# S_1$, where S_0, S_1 are two small circles in B linking S_b'. The arc γ_0

connecting S'_a to S_0 is inside B, while the arc γ_1 connecting S'_a to S_1 represents a product of conjugates of meridians in $\pi_1(S^3 - L(f_1))$ which maps to $g \in G$. To preserve property (c) it suffices to choose γ_0 and γ_1 so that they are isotopic in $S^3 - \bigcup_i S'_i$. The method for choosing γ_1 is indicatd in Figure 1. We use the well known fact that $\pi_1(S^3 - L(f_1))$ is generated by conjugates of the meridians.

Finally note that f is a sublink of a homology boundary link. In fact $\tilde{f}_1$, as a link in W', is a homology boundary link.

This completes the proof of Theorem 3.1.

Proof of Theorem (3.2). We use the construction in [L3: Appendix] and point out the existence of the required 4-manifold W with $\partial W = M(f)$.

Let $M(f_0) = S^1 \times S^2 \# S^1 \times S^2$, where f_0 is the trivial 2-component link; $M(f_0) = \partial W_0$, where W_0 is the boundary connected sum $S^1 \times D^3 \perp\!\!\perp S^1 \times D^3$. Then, for the universal abelian covers, $\tilde{M}(f_0)$ and $\tilde{W}_0$, we have $H_1(\tilde{M}(f_0)) \approx H_1(\tilde{W}_0)$ the free $\mathbb{Z}[\mathbb{Z}^2]$-module of rank one generated by the element e as described in [L3]. We now add k 2-handles to W_0 along framed circles $\sigma_i \subseteq S^3 - L(f_0) \subseteq M(f_0)$, whose lifts $\tilde{\sigma}_i \subseteq \tilde{M}(f_0)$ represent $\lambda_{i1} e \in H_1(\tilde{M}(f_0))$. We can choose $\{\sigma_i\}$ so

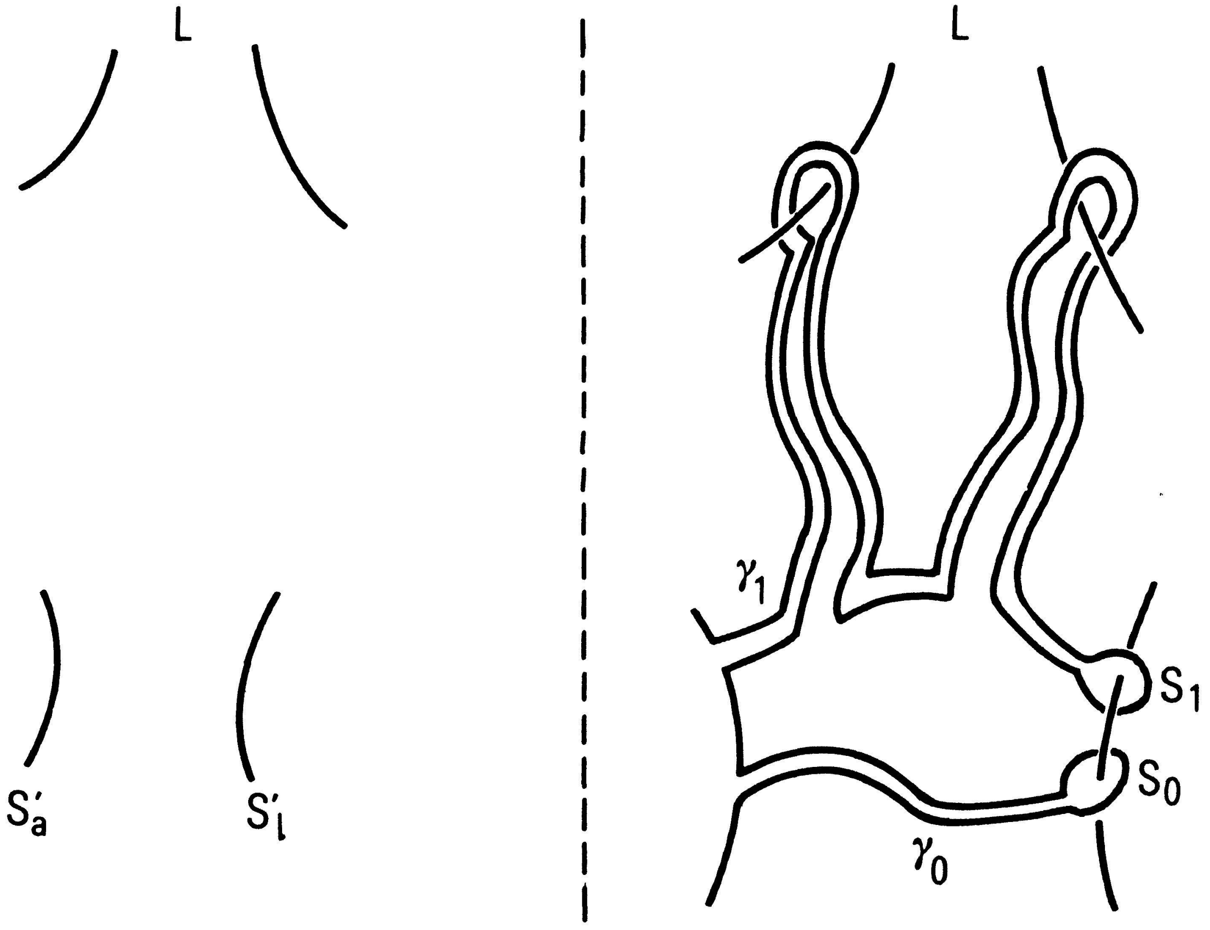

Figure 1

that they form a trivial link in S^3 and so that the framing of σ_i has winding number $\epsilon(\lambda_{ii}) = \pm 1$. The resulting 4-manifold W' has $\partial W' = M(f')$ for a new link f'. For convenience in analyzing the homology, let us further modify W' by doing surgery along an interior curve γ representing e – call the resulting manifold W''. Now $H_1(\tilde{W}'') = 0$ and $H_2(\tilde{W}'')$ is a free $\mathbb{Z}[\mathbb{Z}^2]$-module on $k+1$ generators $s_1, \ldots, s_k, s$ where s is represented by a transverse 2-sphere of the surgery and s_i has a representative cycle in $\tilde{W}''$ $\Delta_i - \lambda_{i1}\Delta + c_i$ as follows: Δ_i is the core of the (lifted) i-th 2-handle added to W_0, with boundary $\tilde{\sigma}_i$; Δ is a disk, bounding a lift of a translate γ' of γ, created by the surgery – and c_i is a homology in $\widetilde{W' - \gamma}$ between $\tilde{\sigma}_i$ and $\lambda_{i1}\tilde{\gamma}'$. We now check some intersection numbers in $\tilde{W}''$. Clearly $s \cdot \Delta = 1$ (i.e. we can so choose s) and $s \cdot \Delta_i = s \cdot c_i = 0$; therefore $s \cdot s_i = \lambda_{i1}$, and obviously $s \cdot s = 0$. We set $\lambda'_{ij} = s_i \cdot s_j$ – this is easily seen to agree with the definition of λ'_{ij} in [L3]. The modification of λ'_{ij} to achieve the desired λ_{ij} is then exactly as in [L3], as well as the proof of Theorem (3.1). Thus the intersection matrix of $\tilde{W}''$ is (λ_{ij}). Since $H_1(\tilde{W}'') = 0$ it will be true that $H_2(W''; \theta) \approx \mathbb{C} \otimes_\theta H_2(\tilde{W}'')$, unless θ is the trivial representation. This follows from a universal coefficient argument using that the group ring $\mathbb{C}[\mathbb{Z}^2]$ has homological dimension 2 and $H_2(\mathbb{Z}^2; \theta) = 0$ unless θ is the trivial representation. Formula (2) now follows from the Index Theorem.

This completes the proof of Theorem (3.2).

ADDENDUM TO THEOREM (3.2). *The Alexander polynomial of f is $(x-1)(y-1)D(x, y)$, where $D = \det \lambda$. The Alexander polynomials of the component knots of f are $\Phi(x, 1)$ and $\Phi(1, y)$, where Φ is the determinant of the matrix obtained from λ by removing the first row and column. Note that $\Phi(1, 1) = \pm 1$.*

This is all proved in [L3].

The continuity stratification of $R_1(\mathbb{Z}^2)$ for $\sigma(f)$ is easy to describe for the link constructed in Theorem (3.2). Let $\Sigma_1 \subseteq R_1(\mathbb{Z}^2) = T^2$, the 2-torus, be the zero set of D and $\Sigma_2 \subseteq T^2$ the zero set of Φ – thus Σ_2 is a special subvariety but Σ_1 is not, in general. Then $\sigma(f)$ is continuous on $T^2 - \Sigma_1$ and on $\Sigma_1 - (\Sigma_1 \cap \Sigma_2)$. If Σ_1 is not special, then we have the possibility that concordance might be detected by the values of $\sigma(f)$ on Σ_1. We will give some examples of this phenomenon in the next section.

4. We consider some examples of Theorem (3.2).

Consider the (2×2)-matrix over $\mathbb{Z}[\mathbb{Z}^2] = \mathbb{Z}[x, x^{-1}, y, y^{-1}]$:

$$\lambda = \begin{pmatrix} 0 & \rho \\ \bar{\rho} & \tau \end{pmatrix}$$

where

$$\rho(x, y) = p(x + x^{-1}) - q$$

$$\tau(x, y) = [1 - N_0(2 - x - x^{-1})(2 - y - y^{-1})][1 - N_1(2 - x - x^{-1})(2 - y - y^{-1})]$$

where p, q, N_0, N_1 are integers to be specified. Let f be a one-dimensional link of 2 components with:

$$\sigma(f) \cdot \theta = \operatorname{sign} \epsilon(\lambda) - \operatorname{sign} \theta(\lambda) \qquad \text{for any } \theta \in R_1(\mathbb{Z}^2)$$

as promised by Theorem (3.2).

If we project the torus $T^2 = R_1(\mathbb{Z}^2)$ onto the square S in $\mathbb{R}^2$ consisting of all (u, v) with $|u| \leq 1$ and $|v| \leq 1$ by setting $u = \operatorname{Re}(x)$, $v = \operatorname{Re}(y)$, then the zero sets Σ_1, Σ_2 of ρ, τ are pull-backs of the zero sets Σ_1', Σ_2' of ρ', τ', respectively, where:

$$\rho'(u, v) = 2pu - q$$

$$\tau'(u, v) = (1 - 2N_0(1 - u)(1 - v))(1 - 2N_1(1 - u)(1 - v)).$$

So Σ_1', Σ_2' consist of a straight-line and a pair of hyperbolas, respectively. See Figure 2 – we assume $|\gamma| < 2|p|$, $N_0 \neq N_1$ and $4N_i > 2p/(2p - q)$ for the curves to intersect in the manner shown.

Clearly $\sigma(f) \cdot \theta = 0$, if $\theta \in T^2 - \Sigma_1$. If $\theta \in \Sigma_1 - \Sigma_2$, then $\sigma(f) \cdot \theta = \operatorname{sign} \tau(\theta)$. Thus $\sigma(f) \cdot \theta = +1$ on points θ of Σ_1 projecting to the upper and lower segments of $\Sigma_1' - \Sigma_2'$ and $\sigma(f) \cdot \theta = -1$ on points θ of Σ_1 projecting to the middle segment of $\Sigma_1' - \Sigma_2'$.

Compare this to a link f' satisfying:

$$\sigma(f') \cdot \theta = \operatorname{sign} \epsilon(\lambda') - \operatorname{sign} \theta(\lambda') \qquad \text{for } \theta \in R_1(\mathbb{Z}^2)$$

where

$$\lambda' = \begin{pmatrix} 0 & \rho \\ \bar{\rho} & \tau^2 \end{pmatrix}$$

using the same ρ, τ as for f. Then $\sigma(f') \cdot \theta = \sigma(f) \cdot \theta$, except when θ projects to the middle segment of $\Sigma_1' - \Sigma_2'$ in which case $\sigma(f') \cdot \theta = +1$.

To show that f and f' are *not concordant* it suffices (see (II.3.3) and (III.1.1)) to show that no special subvariety can contain the entire middle segment J of $\Sigma_1 - \Sigma_2$. Suppose $\phi(x, y) \in \mathbb{Z}[\mathbb{Z}^2]$ satisfies $\phi(1, 1) = \pm 1$ and $\phi(J) = 0$ (see the discussion

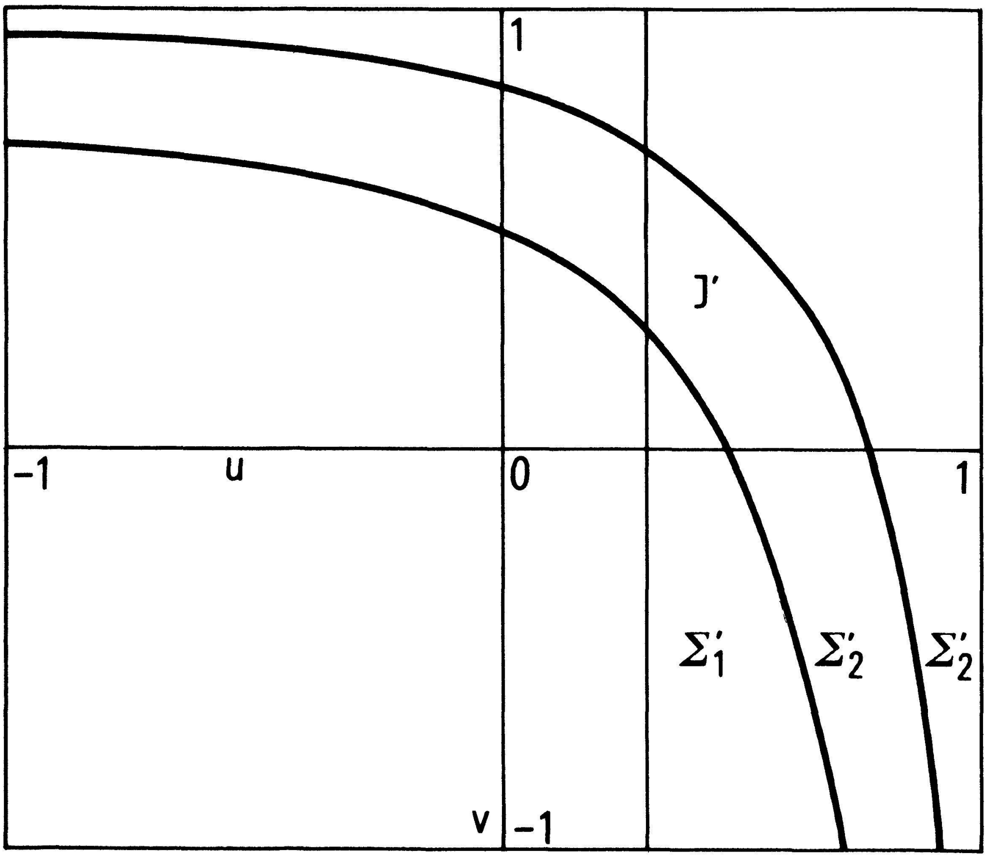

Figure 2

preceding (II.3.1)). We may assume ϕ is *symmetric* i.e. $\phi(x, y) = \phi(x^{-1}, y) = \phi(x, y^{-1})$ – e.g. replace ϕ by the product

$$\phi(x, y)\phi(x^{-1}, y)\phi(x, y^{-1})\phi(x^{-1}, y^{-1}).$$

Then we can write $\phi(x, y) = \psi(x + x^{-1}, y + y^{-1})$. The integral polynomial $\psi(2u, 2v) = \psi'(u, v)$ vanishes on J', the middle segment of Σ_1' and satisfies $\psi'(1, 1) = \pm 1$. But, since $\psi'(q/2p, v) = 0$ for a non-trivial interval of v, we have $\psi'(q/2p, v) = 0$ for all v, and so $\psi'(u, v) = (2pu - q)\psi''(u, v)$, for some rational polynomial ψ''. If $2p$ and q are assumed relatively prime, then ψ'' is integral. But now we have:

$$\pm 1 = \psi(1, 1) = (2p - q)\psi''(u, v).$$

This is impossible if $|2p - q| > 1$.

Putting all the conditions on p, q, N_0, N_1 together we have: q odd; $2p < q + 1$; p relatively prime to q; $N_0 \neq N_1$; and $4N_i > 2p/(2p - q)$. There are certainly many possibilities (an infinite number). It is also easy to see that Σ_1 contains no points whose components are both roots of unity and so the p-signatures of SMOLINSKY [S] will not detect the difference between f and f'. If $N_i > 2p^2/(2p-q)^2$, one can check that J does not intersect the diagonal of T^2 and so the TRISTRAM signatures [T] cannot detect the difference.

5. We now construct examples of links of every odd dimension which are sublinks of homology boundary links but not concordant to a boundary link. Such examples were first constructed by COCHRAN-ORR in [C-O]. We use Theorem (3.1).

Let G be the group with presentation: $\{x_1, x_2, y : y = x_1x_2yx_1y^{-1}x_2^{-1}x_1^{-2}\}$, and consider the following three matrices over $\mathbb{Z}G$:

$$\lambda = (y + y^{-1} - 1) \qquad \lambda' = \begin{pmatrix} 2(y + y^{-1} - 1) & 3 \\ 3 & 4(y + y^{-1} - 1) \end{pmatrix}$$

$$\lambda'' = \begin{pmatrix} y - y^{-1} & -1 \\ -1 & y - y^{-1} \end{pmatrix}.$$

These matrices satisfy conditions (ii) and (iii) – for λ when $q = 1, 3, 7$, for λ' when q is odd but not 1, 3 or 7 and for λ'' when q is even. By Theorem (3.1) there exists a two-component link f of dimension $(2q-1)$ with G-structure α on $M(f)$ and $\rho(M(f), \alpha)$ given by (1), substituting λ, λ' or λ'' for λ in that formula, corresponding to the values of q given above (using $x_i = g_i$).

Note that G contains the free group F generated by x_1, x_2 and, by [L1; Prop. 5], the inclusion $F \subseteq G$ extends to an isomorphism $\bar{F} = \bar{G}$. Thus α defines an $\bar{F}$-structure $\bar{\alpha} : \pi_1(M(f)) \to \bar{F}$ with $\bar{\alpha}(\mu_i) = x_i$ for some choice of meridians $\{\mu_1, \mu_2\}$.

Let π be a cyclic group of prime-power order with generator t. We can define a map $F \to \mathbb{Z}\S\pi = \mathbb{Z}\pi \times \pi$ by $x_1 \mapsto t \in \pi$ and $x_2 \mapsto 1 \in \mathbb{Z}\pi$. Since $\overline{\mathbb{Z}\S\pi} \subseteq Q\S\pi$, by Theorem (I.4.1), this map extends to $\phi : G \to \mathbb{Q}\S\pi$; so $\phi(x_1) = t$, $\phi(x_2) = 1$ and it is not hard to explicitly solve for $\phi(y)$ – e.g. $\phi(y) = \frac{1}{3}(t-1)$ if $|\pi| = 2$ and $\phi(y) = \frac{1}{2}(t-1)$ if $|\pi| = 3$.

Now recall the analytic imbedding $\check{\imath} : \mathbb{R}^k \to R_k(\mathbb{R}\S\pi) \to R_k(\overline{\mathbb{Z}\S\pi})$ from (I.1.(f)) and (I.4). In fact we can simply use $\mathring{\imath} : \mathbb{R} \to R_k(\mathbb{Z}\S\pi)$ defined by $\mathring{\imath}(t) = \check{\imath}(t, t, \ldots, t)$. Consider the function $\tau(f) : \mathbb{R} \to \mathbb{R}$ defined as the composite:

$$\mathbb{R} \xrightarrow{\mathring{\imath}} R_k(\overline{\mathbb{Z}\S\pi}) \xrightarrow{\bar{\phi}} R_k(\bar{G}) \to R_k(\bar{F}) \xrightarrow{\bar{\sigma}(f)} \mathbb{R}.$$

Suppose that this $\bar{F}$-structure on f is induced by an F-structure; then, from the discussion in (I.1(f)), we conclude that $\tau(f)$ is the lift of a function $S^1 \to \mathbb{R}$, i.e. $\tau(f)$

is periodic: $\tau(f)\cdot(s+1)=\tau(f)\cdot s$, for any $s\in\mathbb{R}$. If f is $\bar{F}$-concordant to an F-link, then $\tau(f)$ is periodic except perhaps on $\mathring{\imath}^{-1}(\Sigma)$ for some special subvariety of $R_k(\mathbb{Z}\S\pi)$; this follows from Corollary (II.3.3) and Proposition (III.1.1). Now $\mathring{\imath}^{-1}(\Sigma)$ is an analytic subvariety of $\mathbb{R}$ and so is either discrete or all of $\mathbb{R}$. But $\mathring{\imath}(0)$ is the trivial representation, which belongs to no special subvariety, and so $\mathring{\imath}^{-1}(\Sigma)$ must be a discrete set. This shows that if f is $\bar{F}$-concordant to an F-link, then $\tau(f)$ is periodic of period one except on some discrete subset of $\mathbb{R}$.

We now compute $\tau(f)$ in the cases of q odd, $|\pi|=2$ and q even, $|\pi|=3$. First note that

$$\mathring{\imath}(s)\cdot y=\begin{cases}\begin{bmatrix}\exp\left(\dfrac{-2\pi is}{3}\right) & 0\\ 0 & \exp\left(\dfrac{2\pi is}{3}\right)\end{bmatrix} & \text{if } |\pi|=2\\[2ex] \begin{pmatrix}\exp(-\pi is) & 0 & \\ 0 & \exp(\pi is) & \\ 0 & 0 & 1\end{pmatrix} & \text{if } |\pi|=3.\end{cases}$$

This uses the solutions for $\phi(y)$ mentioned above. Set:

$$A_s=\begin{bmatrix}\cos\left(\dfrac{-2\pi s}{3}\right) & 0\\ 0 & \cos\left(\dfrac{2\pi s}{3}\right)\end{bmatrix}$$

$$B_s=\begin{pmatrix}\sin(-\pi s) & 0 & 0\\ 0 & \sin(\pi s) & 0\\ 0 & 0 & 0\end{pmatrix}.$$

Then it is immediate that:

(i) $\mathring{\imath}(s)\cdot\lambda=2A_s-I$ for $|\pi|=2$

(ii) $\mathring{\imath}(s)\cdot\lambda'=\begin{pmatrix}4(A_s-I) & 3I\\ 3I & 2(A_s-I)\end{pmatrix}$ for $|\pi|=2$

(iii) $\mathring{\imath}(s)\cdot\lambda''=\begin{pmatrix}2iB_s & I\\ -I & 2iB_s\end{pmatrix}$ for $|\pi|=3$

and so $\tau(f)\cdot s$ is given by the signature of these matrices, respectively, in the cases

(i) $q = 1, 3, 7$ and $|\pi| = 2$; (ii) q odd $\neq 1, 3$, or 7 and $|\pi| = 2$; (iii) q even and $|\pi| = 3$. Moreover $\tau(f) \cdot s$ is locally constant with jumps at the values of s for which the respective matrices are singular. Thus, if $\tau(f) \cdot s_0 \neq \tau(f) \cdot (s_0 + 1)$ and the matrices are non-singular when $s = s_0$ and $s_0 + 1$, we conclude that $\tau(f) \cdot s \neq \tau(f) \cdot (s + 1)$ for all s sufficiently close to s_0. It is therefore impossible for $\tau(f)$ to be periodic of period one except on a discrete set and we conclude that f is not $\bar{F}$-concordant to an F-link. But it is a straightforward computation that this is the case for (i) and (ii) with $s_0 = 0$, and for (iii) with $s_0 = \frac{1}{2}$.

To show that f is not concordant to a boundary link, it now suffices by Prop. (1.2), to show that changing the $\bar{F}$-structure on f does not change $\tau(f)$ – and so $\tau(f)$ represents a concordance invariant of f. This will follow immediately from:

PROPOSITION (5.1). *Let $\psi : \bar{F} \to \mathbb{Q}\S\pi$ be a homomorphism with $\psi(x) = t \in \pi \subseteq \mathbb{Q}\S\pi$ and $\psi(x) = \lambda \in \mathbb{Z}\pi \subseteq \mathbb{Q}\pi \subseteq \mathbb{Q}\S\pi$. If π is a p-group and t is in the center of π, then for any special automorphism α of F, there exists an inner automorphism α' of $\mathbb{Q}\S\pi$ so that $\alpha' \circ \psi = \psi \circ \alpha$.*

Proof. Suppose $\alpha(x_1) = g x_1 g^{-1}$, $\alpha(x_2) = h x_2 h^{-1}$ and $\psi(g) = \xi u$, $\psi(h) = v\eta$, where $u, v \in \pi$ and $\xi, \eta \in \mathbb{Q}\pi$. Set $\gamma = \xi v$; then we have:

$$\gamma\psi(x_1)\gamma^{-1} = \gamma t \gamma^{-1} = \xi v t v^{-1} \xi^{-1} = \xi t \xi^{-1}$$

$$\psi(g x_1 g^{-1}) = \xi u t u^{-1} \xi^{-1} = \xi t \xi^{-1}$$

$$\gamma\psi(x_2)\gamma^{-1} = \gamma\lambda\gamma^{-1} = \xi v \lambda v^{-1} \xi^{-1} = \xi + v \cdot \lambda - \xi = v \cdot \eta \in \mathbb{Q}\pi$$

$$\psi(h x_2 h^{-1}) = v\eta\lambda\eta^{-1}v^{-1} = v \cdot (\eta + \lambda - \eta) = v \cdot \lambda$$

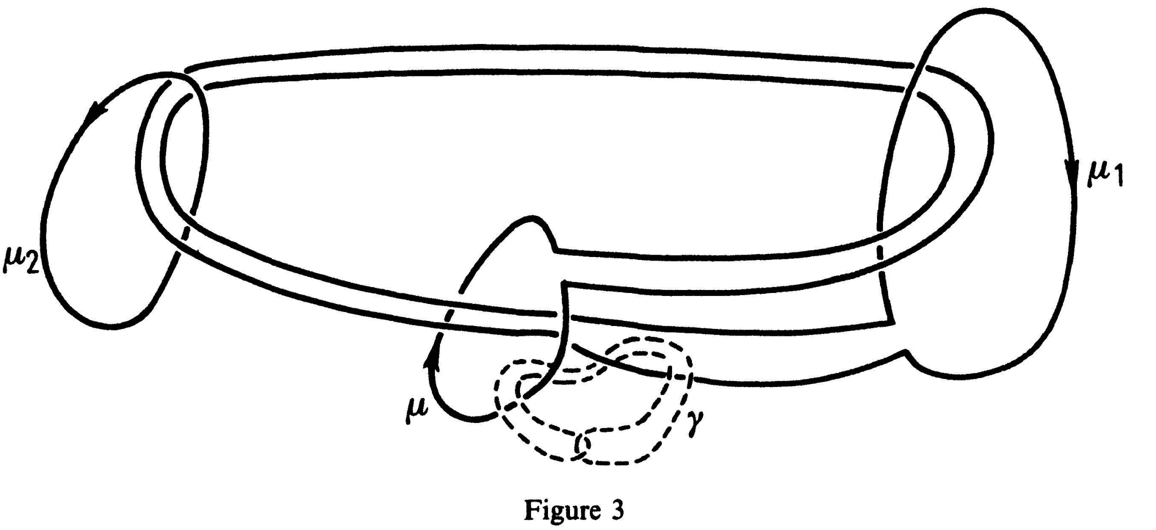

Figure 3

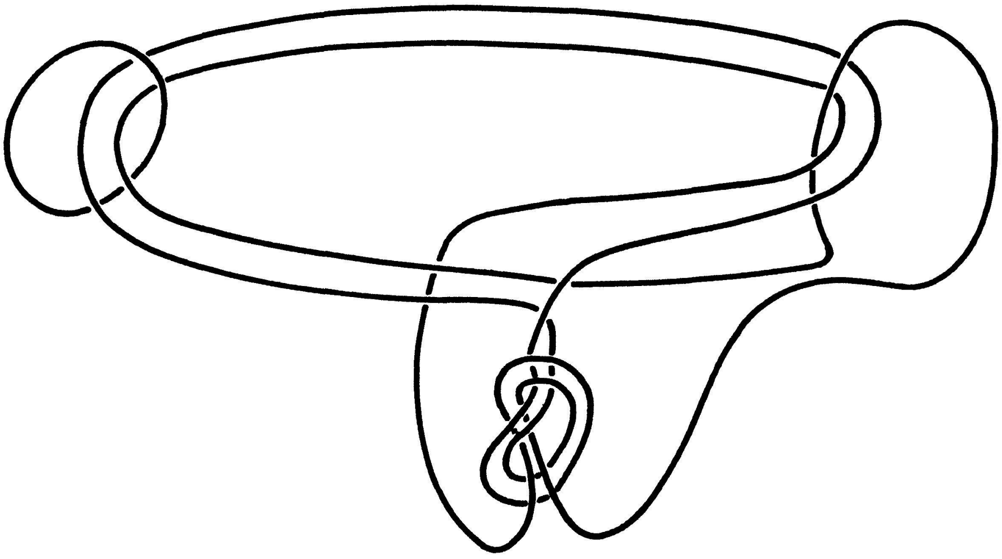

Figure 4

Thus the inner automorphism α' defined by conjugation by γ satisfies the equation $\alpha' \circ \psi = \psi \circ \alpha$ on x_1 and x_2. Since $\psi(F) \subseteq \mathbb{Z}\S\pi$, we have $\psi(\bar{F}) \subseteq \overline{\mathbb{Z}\S\pi}$ and $\psi(g) \in \overline{\mathbb{Z}\S\pi}$. Thus $\gamma = \psi(g)u^{-1}v$ also belongs to $\overline{\mathbb{Z}\S\pi}$. As a result we see that $\alpha' \circ \psi$ and $\psi \circ \alpha$ are homomorphisms $\bar{F} \to \overline{\mathbb{Z}\S\pi}$ which agree on F. But then they induce the same homomorphisms on the nilpotent completions: $\tilde{F} \to \mathbb{Z}\S\pi$. Since $\bar{F} \subseteq \tilde{F}$ and $\overline{\mathbb{Z}\S\pi} \subseteq \mathbb{Z}\S\pi$, we conclude that $\alpha' \circ \psi = \psi \circ \alpha$ on all of $\bar{F}$.

When $q = 1$, we can draw a picture of a link corresponding to this example. First note that the ribbon link f_0 of Figure 3 admits an epimorphism α from its group to G. The meridians in Figure 3, which generate the group of f_0, map to G as follows: $\mu_1 \mapsto x_1$, $\mu_2 \mapsto x_2$, $\mu \mapsto yx_1$. Now a $+$ 1-surgery on the complement of f_0, along the curve γ in Figure 3 will produce a new link f such that, according to the proof of Theorem 3.1 and preceding discussion:

$$\tau(f) \cdot s - \tau(f_0) \cdot s = \operatorname{sign}(2A_s - I).$$

Since f_0 is slice, $\tau(f_0)$ is periodic of period one (except on a discrete set) and so $\tau(f)$ cannot be. The link f is given in Figure 4.

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Brandeis University
Department of Mathematics
Waltham, MA 02254-9110
USA

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