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Mass of rays in Alexandrov spaces of nonnegative curvature

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§0. Introduction

The *limit cone* $(\mathrm{Con}_\infty M, o_\infty)$ *of a metric space* M is defined to be the Hausdorff limit of the pointed space $(\frac{1}{t}M, o)$ as $t \to +\infty$, where $\frac{1}{t}M$ denotes the space M equipped with metric multiplied by $\frac{1}{t}$-times, and $o \in M$ is a fixed point. If the limit cone of a metric space M exists, i.e., $(\frac{1}{t}M, o)$ converges as $t \to +\infty$, then $\mathrm{Con}_\infty M$ has a scale-invariant metric and is isometric to the cone over some metric space $M(\infty)$, which is called the *ideal boundary of* M. Note that the limit cone and the ideal boundary are both independent of the point $o \in M$. The concept of the limit cone is very useful to study the global behavior of Riemannian manifolds in various classes, such as Hadamard manifolds and complete open Riemannian manifolds with nonnegative sectional curvature, for both which the limit cones exit (see [BGS]). It is also known that there exists the limit cone of every finitely connected, complete, noncompact surface M admitting (finite or infinite) total curvature, and that the one-dimensional Hausdorff measure $\mathscr{H}^1(M(\infty))$ of its ideal boundary $M(\infty)$ satisfies $\mathscr{H}^1(M(\infty)) = 2\pi\chi(M) - \int_M K_M d\mathrm{vol}_M$ (see [Sy2–4]), where $\chi(M)$ is the Euler characteristic number, K_M the sectional curvature, and $d\mathrm{vol}_M$ the volume element. A standard way to prove the existence of the limit cone of a Riemannian manifold (and a metric space) is that we first define the ideal boundary $M(\infty)$ as the equivalence classes of rays with respect to some equivalence relation, and then prove $(\frac{1}{t}M, o)$ to tend to $(K(M(\infty)), o)$ as $t \to +\infty$, where $K(X)$ denotes the cone over a metric space X and $o \in K(X)$ its vertex.

In this paper, our concern is to study the relation between mass of rays and the limit cone. Let M be a complete noncompact Riemannian manifold. A ray in M is defined to be a unit-speed curve $\gamma : [0, +\infty) \to M$ any subarc of which is a minimal segment. Denote by A_p the set of unit vectors at a point $p \in M$ tangent to rays. Since A_p is a compact subset of the unit tangent sphere S_pM, so is measurable with respect to the Lebesgue measure m on S_pM. We call $m(A_p)$ the *measure of rays from* p. The study on mass of rays was originally begun by Maeda (see [Md1–2]), and after that there were many generalizations (see [Og, Sg1–2, Shm, SST, Sy1]). One of the final forms of the results on mass of rays is stated as follows: if M is a finitely connected, complete, open surface admitting total curvature and with unique end e, then

$$\lim_{M \ni p \to e} m(A_p) = \min\{\mathscr{H}^1(M(\infty)), 2\pi\},$$

(see [SST, Syl]). Note that all the known results on mass of rays are only in the 2-dimensional case except obvious things. The purpose of this paper is to estimate the mass of rays in nonnegatively curved Alexandrov spaces of any dimension, where an *Alexandrov space* is defined to be a finite-dimensional, complete, locally compact length space of curvature bounded from below in the sense of the Alexandrov convexity (see [BGP]). Assume that M is a noncompact n-dimensional Alexandrov space of nonnegative curvature. Then the limit cone $\mathrm{Con}_\infty M$ exits. Rays and the set A_p for $p \in M$ are defined in the same manner, where A_p is a subset of the space of directions Σ_p at p. We have a natural distance nonincreasing map from A_p for any $p \in M$ to $M(\infty)$ (see §1), so that in particular,

$$\mathscr{H}^{n-1}(A_p) \geq \mathscr{H}^{n-1}(M(\infty)). \tag{$*$}$$

However, except in the 2-dimensional case, this inequality is not optimal even in the case where M is a cone itself (see for detail §1).

Let us now give some notations to state our main theorems. For $t > 0$, we find a $\theta_M(\frac{1}{t})$-pointed Hausdorff approximation $f_t : (\frac{1}{t}M, o) \to (\mathrm{Con}_\infty M, o_\infty)$, where $\theta_\alpha(\epsilon)$ is a function depending only on α and satisfying $\lim_{\epsilon \to 0} \theta_\alpha(\epsilon) = 0$. Let $p_t \in M$ and $p \in \mathrm{Con}_\infty M$ be any points such that $\lim_{t \to +\infty} f_t(p_t) = p$. Set $\delta(p) := d_H(\Sigma_p, S^{n-1})$, the Hausdorff distance between the space of directions Σ_p at p and the standard $(n-1)$-dimensional sphere S^{n-1}. One of our main theorems is stated as follows.

THEOREM A. *We have*

$$d_H(A_{p_t}, A_p) < \theta_n(\delta(p)) + \theta_M(1/t).$$

COROLLARY TO THEOREM A. *The measure of rays from p_t satisfies*

$$\mathscr{H}^{n-1}(A_{p_t}) < \mathscr{H}^{n-1}(A_p) + \theta_n(\delta(p)) + \theta_M(1/t).$$

For the proof of Theorem A, we shall define a map $\Sigma_p g : \Sigma_p \to \Sigma_q$, $q := g(p)$, for $p \in \mathrm{Con}_\infty M$ and for a Hausdorff approximation $g : (\mathrm{Con}_\infty M, o_\infty) \to (\frac{1}{t}M, o)$. Supposing $\delta(p)$ is small enough, $\Sigma_p g$ almost preserves distance and maps A_p to a subset Hausdorff-close to A_q, where it should be noticed that this closeness will be estimated independently of the point p. In this way Theorem A will be proved. In fact, in §2 we shall have more general discussion about $\Sigma_p g$.

In order to express the mass of rays in M, we consider averaging the measure of rays. For a fixed point $o \in M$ and any $t > 0$, let

$$\rho(t) := \frac{\int_{B(o,t) \ni x} \mathscr{H}^{n-1}(A_x)\, d\mathscr{H}^n}{\mathscr{H}^n(B(o, t))}.$$

Note that, since the function $M \ni x \mapsto \mathscr{H}^{n-1}(A_x) \in \mathbf{R}$ is upper semi-continuous on the set of nonsingular points in M (i.e., $\limsup_{x \to y} \mathscr{H}^{n-1}(A_x) \leq \mathscr{H}^{n-1}(A_y)$ for any $y \in M$ with $\delta(y) = 0$), and the set of singular points, $\{x \in M \mid \delta(x) \neq 0\}$, is of measure zero (see [BGP, 10.6] or [OS, Theorem A]), the above integration has a meaning. We call the two quantities

$$\underline{\rho}(M, o) := \liminf_{t \to +\infty} \rho(t) \quad \text{and} \quad \bar{\rho}(M, o) := \limsup_{t \to +\infty} \rho(t)$$

the *lower and upper mean measures of rays in M*. When $\underline{\rho}(M, o) = \bar{\rho}(M, o)$, this value is called the *mean measure of rays in M* and is denoted by $\rho(M, o)$.

THEOREM B. *The upper mean measure of rays in M satisfies*

$$\bar{\rho}(M, o) \leq \rho(\mathrm{Con}_\infty M, o_\infty).$$

To obtain Theorem B, it is important that the $\theta_n(\cdot)$ and $\theta_M(\cdot)$ in Corollary to Theorem A are both independent of $\{p_t\}$ and p, since $\mathrm{Con}_\infty M \ni p \mapsto \mathscr{H}^{n-1}(A_p)$ is not necessarily semi-continuous on the set of singular points. In order to estimate the mean measure of rays, we need to investigate the relation between the Hausdorff measures on M and $\mathrm{Con}_\infty M$, which was independently studied in [BGP, 10.8], but our claim, Theorem 3.1, is stronger than [BGP, 10.8]. This investigation produces a corollary as stated that the n-dimensional Hausdorff measure $\mathscr{H}^n$ is a continuous function on the set $\mathscr{A}(n, k, D)$ of n-dimensional Alexandrov spaces of curvature $\geq k$ and of diameter $\leq D$ with respect to d_H.

Remarks.

(1) If $\dim \mathrm{Con}_\infty M \leq n - 1$, we have $\bar{\rho}(M, o) = 0$ (see the proof of Theorem B).

(2) It follows from Bishop-Gromov's volume comparison theorem that $\underline{\rho}(M, o)$ and $\bar{\rho}(M, o)$ are both independent of the point o (cf. the proof of Lemma 3.3 (1)).

(3) Of course $\rho(\mathrm{Con}_\infty M, o_\infty)$ is explicitly determined by $M(\infty)$ (see for detail §1). In the case where $n = 2$, we have $\rho(\mathrm{Con}_\infty M, o_\infty) = \mathscr{H}^1(M(\infty))$, which together with $(*)$ and Theorem B implies $\rho(M, o) = \rho(\mathrm{Con}_\infty M, o_\infty) = \mathscr{H}^1(M(\infty))$.

Problem. If $\dim \mathrm{Con}_\infty M = n$, does $\rho(M, o) = \rho(\mathrm{Con}_\infty M, o_\infty)$ always hold?

Notations and convention

For the basic notion used in this paper, we refer to [GLP, Fk, BGP]. Denote by $|pq|$ (or $|p, q|$) the distance between two points p and q in a metric space X. The open metric ball centered at $p \in X$ and of radius $r > 0$ is denoted by $B(p, r)$ or $B(p, r; X)$. We sometimes write $B(r)$ or $B(r; X)$ instead, when X is a complete simply connected space form. Let d_H^Z denote the Hausdorff distance between subsets of a metric space Z, and d_H (resp. $d_{p.H}$) denote the Hausdorff distance on the set of metric spaces (resp. pointed metric spaces). Denote by $\mathscr{H}^n$ (resp. V_r^n) for $n \geq 0$ the n-dimensional Hausdorff measure (resp. rough volume), where the rough volume is described in the following. An *ϵ-discrete net*, $\epsilon > 0$, *of a metric space X* is defined to be a discrete subset $N \subset X$ such that $|pq| \geq \epsilon$ for any pair of different points $P, b \in N$. The *n-dimensional rough volume of X* is, by definition

$$V_r^n(X) := \limsup_{\epsilon \to 0} \epsilon^n \beta_X(\epsilon),$$

where $\beta_X(\epsilon)$ is the number of a maximal ϵ-discrete net of X. Note that, when X is not precompact, $\beta_X(\epsilon) = V_r^n(X) = +\infty$, and that the rough volume is not completely additive and does not measure X in general since the rough volume of a countable set is not necessarily zero. For a metric space X of diameter $\leq \pi$, we denote by $K(X)$ (resp. $\Sigma(X)$) the cone (resp. spherical suspension) over X and by $\pi_X : K(X) - \{o\} \to X$ (resp. $\Sigma(X) - \{N, S\} \to X$) the projection, where $o \in K(X)$ is the vertex and $N, S \in \Sigma(X)$ the north and south pole.

Assume now that X is an n-dimensional Alexandrov space of curvature $\geq k$ (i.e., a complete FDSCBB in the sense of [BGP]). The space of directions (resp. the tangent cone) at $p \in X$ is denoted by Σ_p or $\Sigma_p X$ (resp. K_p or $K_p X$). For $p, q \in X$, the symbol pq indicates a minimal segment from p to q, and v_{pq} the direction in Σ_p corresponding to pq. Let $p, q, r \in X$. The angle between two minimal segments pq and pr at p is denoted by $\angle pqr$, i.e., $\angle pqr := |v_{pq} v_{pr}|$. Let $M^n(k)$ denote the complete simply connected n-dimensional space form of constant curvature k and set $\tilde{\angle} pqr := \angle \tilde{p}\tilde{q}\tilde{r}$, where $\tilde{p}$, $\tilde{q}$, and $\tilde{r}$ are points in $M^2(k)$ such that $|pq| = |\tilde{p}\tilde{q}|$, $|qr| = |\tilde{q}\tilde{r}|$, and $|rp| = |\tilde{r}\tilde{p}|$. It follows that $\angle pqr \geq \tilde{\angle} pqr$, which we call the *Toponogov convex-*

ity. An *(m, δ)-strainer at $p \in X$* (resp. an *(m, δ)-strainer of X*), where $m \geq 1, \delta \geq 0$, is defined to be a sequence $\{p_i\}_{i=1}^{2m} \subset X$ satisfying that $\tilde{\angle} p_i p p_{m+1} > \pi - \delta$ (resp. $|p_i p_{m+i}| > \pi - \delta$) for any $i = 1, \ldots, m$ and that $\tilde{\angle} p_i p p_j > \pi/2 - \delta$ (resp. $|p_i p_j| > \pi/2 - \delta$) for any $i, j = 1, \ldots, 2m$ with $i \neq j$. For Alexandrov spaces we have *Bishop's* (resp. *Bishop-Gromov's*) *volume comparison theorem* to be true (see [Ymg, Appendix]), i.e., for any $p \in X$ and $R > r > 0$,

$$\mathscr{H}^n(B(p, r; X)) \leq \operatorname{vol} B(r; M^n(k)),$$

(resp.)

$$\frac{\mathscr{H}^n(B(p, R; X))}{\mathscr{H}^n(B(p, r; X))} \leq \frac{\operatorname{vol} B(R; M^n(k))}{\operatorname{vol} B(r; M^n(k))}.$$

We denote by "$\text{const}_{\alpha_1, \alpha_1, \ldots}$" a constant depending only on $\alpha_1, \alpha_2, \ldots$, and by "const" a universal constant. Denote by $\theta_{\alpha_1, \alpha_2, \ldots}(\epsilon_1, \ldots, \epsilon_k)$ a function depending only on $\alpha_1, \alpha_2, \ldots$ such that

$$\lim_{\epsilon_1 \to 0} \cdots \lim_{\epsilon_k \to 0} \theta_{\alpha_1, \alpha_2, \ldots}(\epsilon, \ldots, \epsilon_k) = 0,$$

and by $\theta_*(\epsilon_1, \ldots, \epsilon_k)$ a function depending possibly on all objects set up there (except $\epsilon_1, \ldots, \epsilon_k$) and satisfying the similar formula as above.

§1. Limit cone and ideal boundary

Existence of the limit cones of nonnegatively curved Alexandrov spaces

Let M be a noncompact n-dimensional Alexandrov space of nonnegative curvature, and $o \in M$ a fixed point. We assign to any pair of rays σ and γ from o in M the number

$$\angle_\infty(\sigma, \gamma) := \lim_{\substack{s \to +\infty \\ t \to +\infty}} \tilde{\angle} \sigma(s) o \tau(t),$$

where the limit always exists since $\tilde{\angle} \sigma(s) o \gamma(t)$ is monotone increasing in s and t. Obviously we have $\angle_\infty(\sigma, \gamma) \leq |\dot{\sigma}(0)\dot{\gamma}(0)|$. It can be easily checked that $\angle_\infty$ is a pseudo-distance between rays from o. The *ideal boundary $M(\infty)$ of M* is defined to be the quotient metric space of the set of rays from o modulo the equivalence relation $\angle_\infty(\cdot, \cdot) = 0$. We denote by $\gamma(\infty) \in M(\infty)$ the equivalence class represented by a ray γ from o. Since a limit of a sequence of rays is a ray, the ideal boundary $M(\infty)$ is a compact metric space, so that the cone $K(M(\infty))$ over $M(\infty)$ is a complete metric space.

PROPOSITION 1.1 (Gromov [BGS, §4]). *The pointed space* $(\frac{1}{t}M, o)$ *tends to* $(K(M(\infty)), o)$ *as* $t \to +\infty$ *with respect to* $d_{p.H}$.

Since the proof of this proposition is nowhere published, we shall give that in §4. Although $\lim_{t\to+\infty}(\frac{1}{t}M, o)$ is isometric to the cone $(K(M(\infty)), o)$, letting

$$(\mathrm{Con}_\infty M, o_\infty) := \lim_{t\to+\infty}\left(\frac{1}{t}M, o\right)$$

we notionally distinguish the limit $\mathrm{Con}_\infty M$ from the cone $K(M(\infty))$ over $M(\infty)$. The limit cone $\mathrm{Con}_\infty M$ of M is an Alexandrov space of dimension $\le n$ and of curvature ≥ 0, and hence, by [BGP, 4.2], the ideal boundary $M(\infty)$ is a compact Alexandrov space of dimension $\le n-1$ and of curvature ≥ 1. Note that the limit cone $\mathrm{Con}_\infty M$ is independent of the base point $o \in M$ and so is the ideal boundary $M(\infty)$. Attaching $M(\infty)$ to M we have a compactification $\bar{M} := M \cup M(\infty)$ (disjoint union), which has a natural topology satisfying that a sequence $\{p_i\}$ of points in M converges to a point $p \in M(\infty)$ if and only if

$$\lim_{i\to+\infty}|op_i| = +\infty \quad \text{and} \quad \pi_{M(\infty)}(\lim_{i\to+\infty} f_{|op_i|}(p_i)) = p,$$

where $f_t : (\frac{1}{t}M, o) \to (\mathrm{Con}_\infty M, o_\infty)$ is a $\theta_M(1/t)$-pointed Hausdorff approximation.

Mass of rays in a cone

Assume in this section that M is the cone $K(X)$ over an Alexandrov space X of curvature ≥ 1, and $n := \dim M$. Then, we find the correspondence between rays in M and minimal segments in X as follows. For $x \in M$, $a > 0$, $p := (x, a) \in M = K(X)$, we have

(1) for any ray $\gamma : [0, +\infty) \to M$ from p, the closure $\overline{\pi_X \circ \gamma}$ of $\pi_X \circ \gamma$ is a minimal segment from x to $\lim_{t\to+\infty} \pi_X \circ \gamma(t)$,
(2) conversely, for any minimal segment $\sigma : [0, \ell] \to X$ with $\sigma(0) = x$,

$$[0, \ell) \ni t \mapsto \left(\sigma(t), \frac{a}{\cos t - \operatorname{cotan} \ell \sin t}\right) \in M$$

is a (not constant speed) ray from p.

For any $x \in M(\infty)$ let $\Sigma(x) := \Sigma(\Sigma_x M(\infty))$ ($= \Sigma_p M$ for any $p \in \pi^{-1}_{\bar{M}(\infty)}(x) - o$), let $N(x) \in \Sigma(x)$ be the north pole, and let $I_x : K_x M(\infty) \to K_{N(x)}\Sigma(x)$ be the natural isometry. Then, by (1) and (2), for any $v \in A_p, p \in \pi^{-1}_{\bar{M}(\infty)}(x) - o$, we have

$$\gamma_v(\infty) = \exp_x \circ I_x^{-1} \circ \exp^{-1}_{N(x)} v$$

and hence

$$A_p = \exp_{N(x)} \circ I_x(\bar{U}_x),$$

where $U_x \subset K_x X$ is the interior of the tangential cut locus at x. Therefore,

$$\rho(M, o) = \int_{X \ni x} \mathscr{H}^{n-1}(\exp_{N(x)} \circ I_x(\bar{U}_x))\, d\mathscr{H}^{n-1}.$$

§2. Asymptotic behavior of the directions of rays

We first consider a general situation to clarify discussion. Let X and Y be compact Alexandrov spaces such that $n := \dim X \le \dim Y$, and assume there exists an ϵ-Hausdorff approximation $f : X \to Y$, $\epsilon > 0$. Let $p \in X$, $q := f(p)$, and $r > 0$. We can find an (n, μ)-strainer $\{p_i\}_{i=1}^{2n}$ at p such that $|pp_i| = r$, where $\mu := \theta(\delta(p)) + \theta_X(r)$. Setting $q_i := f(p_i)$, we have the (n, ν)-strainer $\{q_i\}_{i=1}^{2n}$ at q, where $\nu := \mu + \theta(\epsilon/r) = \theta(\delta(p)) + \theta_X(r, \epsilon)$. Since $\{v_{pp_i}\}$ (resp. $\{v_{qq_i}\}$) is an (n, μ)- (resp. (n, ν)-) strainer of Σ_p (resp. Σ_q), there exists a $\theta(\nu)$-Hausdorff approximation $\Sigma_p f : \Sigma_p \to \Sigma \subset \Sigma_q$ such that $\Sigma_p f(v_{pp_i}) = v_{qq_i}$ for any $i = 1, \dots, 2n$ (cf. [BGP, §9]). Note that, when $\dim X = \dim Y$, the map $\Sigma_p f$ is a $\theta_n(\nu)$-almost isometry from Σ_p to Σ_q. The following theorem plays an important role in the proof of Theorem A.

THEOREM 2.1 (Naturality for $\Sigma_p f$). *Any minimal segment σ (resp. τ) from p (resp. q) of length $= r$ satisfies*

$$|\dot{\tau}(0), \Sigma_p f(\dot{\sigma}(0))| < \theta_n(\delta(p)) + \theta_X(r, \epsilon + |\tau(r), f(\sigma(r))|).$$

Recall here that $\Sigma_p f$ depends on r. In order to prove this theorem we need some lemmas.

LEMMA 2.2. *For any $p, x \in X$ there exists a $(\theta_n(\delta(p)) + \theta_{n,p}(|px|))$-almost isometry $T^p_x : \Sigma_x \to \Sigma_p$ having the following property: if $\{\sigma_x\}_{x \in X}$ is a family of* *minimal segments σ_x from x for which σ_x tends to σ_p as $x \to p$, then*

$$\limsup_{x \to p} |\dot\sigma_p(0), T^p_x(\dot\sigma_x(0))| < \theta_n(\delta(p)).$$

Proof. Take a $\theta(\delta(p))$-strainer $\{p_i\}_{i=1}^{2n}$ at p such that pp_i is a unique minimal segment joining p and p_i for each i. Since $\{p_i\}_{i=1}^{2n}$ is a $(\theta(\delta(p)) + \theta(|px|/\min_i |pp_i|))$-strainer at x, there exists a $(\theta_n(\delta(p)) + \theta_n(|px|/\min_i |pp_i|))$-almost isometry $T^p_x : \Sigma_x \to \Sigma_p$ such that $T^p_x(v_{xp_i}) = v_{pp_i}$ for any i. It follows that $\theta_n(|px|/\min_i |pp_i|)$ is reduced to $\theta_{n,p}(|px|)$. For any $i = 1, \ldots, n$,

(i) $\quad |\dot\sigma_p(0), v_{pp_i}| + |\dot\sigma_p(0), v_{pp_{i+n}}| > \pi - \theta(\delta(p)),$

and, since $\liminf_{x \to p} \angle p_i x p_{i+n} \geq \tilde\angle p_i p p_{i+n} > \pi - \theta(\delta(p))$,

(ii) $\quad \limsup_{x \to p} (|\dot\sigma_x(0), v_{xp_i}| + |\dot\sigma_x(0), v_{xp_{i+n}}|) < \pi + \theta(\delta(p)).$

Lemma on the Limit Angle (see [BGP, 2.8.3]) implies

(iii) $\quad \liminf_{x \to p} |\dot\sigma_x(0), v_{xp_i}| \geq |\dot\sigma_p(0), v_{pp_i}|$

for any $i = 1, \ldots, 2n$. Combining (i), (ii), and (iii) yields

$$\limsup_{x \to p} \big| |\dot\sigma_p(0), v_{pp_i}| - |\dot\sigma_x(0), v_{xp_i}| \big| < \theta(\delta(p)),$$

which completes the proof (cf. [BGP, §9]). □

For $p \in X$, $0 < \rho < 1$, and $r > 0$, we set

$$h_\rho(p, r) := \sup\{\angle xpy - \tilde\angle xpy \mid \rho r \leq |px|, |py| \leq r,\ x, y \in X\}.$$

It is known that $h_\rho(p, r) < \theta_{\rho,p}(r)$ (see [BGP, 11.2]). We now prove

LEMMA 2.3. $h_\rho(p, r) < \theta_n(\delta(p)) + \theta_{X,\rho}(r)$.

Proof. Fix any $0 < \rho < 1$, and take any $p_i, x_i, y_i \in X$ for each $i = 1, 2, \ldots$ and $r_i \to 0$ such that $\rho r_i \leq |p_i x_i|, |p_i y_i| \leq r_i$. By the compactness of X, replacing a subsequence we assume that p_i, x_i, y_i all tend to a point $p \in X$ as $i \to +\infty$. By taking more subsequence if necessarily, the directions $T^p_{p_i}(v_{p_i x_i})$ and $T^p_{p_i}(v_{p_i y_i})$ in Σ_p are assumed to respectively tend to some u and v in Σ_p. For any $\epsilon > 0$ there exist $x, y \in X$ such that

(i) both px and py are unique minimal segments joining their terminal points,

(ii) $|uv_{px}|, |vv_{py}| < \epsilon$,

(iii) $|\angle xpy - \tilde{\angle} xpy| < \epsilon$.

Then, by Lemma 2.2,

$$\limsup_{i\to+\infty} |v_{px}, T^p_{p_i}(v_{p_ix})|, \limsup_{i\to+\infty} |v_{py}, T^p_{p_i}(v_{p_iy})| < \theta_n(\delta(p)),$$

and hence,

$$\limsup_{i\to+\infty} \angle xp_ix_i, \limsup_{i\to+\infty} \angle yp_iy_i < \theta_n(\delta(p)) + \epsilon. \tag{$*$}$$

From (ii) and (iii),

$$\begin{aligned}\angle x_ip_iy_i &= |uv| + \theta_n(\delta(p)) + \theta_*(1/i)\\ &= \tilde{\angle} xpy + \theta_n(\delta(p)) + \theta*(1/i + \epsilon)\\ &= \tilde{\angle} xp_iy + \theta_n(\delta(p)) + \theta_*(1/i + \epsilon).\end{aligned}$$

Since $\angle xp_iy \geq \tilde{\angle} x_i'p_iy_i' \geq \tilde{\angle} xp_iy$, where x_i' (resp. y_i') denotes the point on p_ix (resp. p_iy) with $|p_ix_i'| = |p_ix_i|$ (resp. $|p_iy_i'| = |p_iy_i|$), we have

$$\angle xp_iy - \tilde{\angle} x_i'p_iy_i' < \theta_n(\sigma(p)) + \theta_*(1/i + \epsilon),$$

which together with ($*$) implies

$$\limsup_{i\to+\infty} (\angle x_ip_iy_i - \tilde{\angle} x_ip_iy_i) < \theta_n(\delta(p)).$$

This completes the proof. □

Proof of Theorem 2.1. Lemma 2.3 implies that

$$\angle\sigma(r)pp_i - \tilde{\angle}\sigma(r)pp_i < \theta_n(\delta(p)) + \theta_X(r)$$

for any $i = 1, \dots, 2n$, and hence, by setting $\theta := \theta_n(\delta(p)) + \theta_X(r, \epsilon + |\tau(r), f(\sigma(r))|)$,

$$\angle\tau(r)qq_i \geq \tilde{\angle}\tau(r)qq_i > \tilde{\angle}\sigma(r)pp_i - \theta\left(\frac{\epsilon + |\tau(r), f(\sigma(r))|}{r}\right) > \angle\sigma(r)pp_i - \theta.$$

Therefore, for any $i = 1, \ldots, n$,

$$\pi + \theta(v) > \angle\tau(r)qq_i + \angle\tau(r)qq_{i+n} > \angle\sigma(r)pp_i + \angle\sigma(r)pp_{i+n} - \theta > \pi - \theta,$$

where $\theta(\theta)$ is reduced to θ. Thus we obtain

$$|\angle\sigma(r)pp_i - \angle\tau(r)qq_i| < \theta. \qquad \square$$

For the M as in Theorem A, we embed $\frac{1}{t}M$ for all $t > 0$ and $\mathrm{Con}_\infty M$ into some metric space Z isometrically such that, in Z, $(\frac{1}{t}M, o)$ tends to $(\mathrm{Con}_\infty M, o_\infty)$ as $t \to +\infty$. This naturally induces an embedding of $(\frac{1}{t}M)(\infty)$ and $(\mathrm{Con}_\infty M)(\infty)$ into a metric space $Z(\infty)$ such that

(1) $\lim_{t\to+\infty} d_H^{Z(\infty)}((\frac{1}{t}M)(\infty), (\mathrm{Con}_\infty M)(\infty)) = 0$,
(2) if a sequence $\{p_{t,i} \in \frac{1}{t}M\}_{t>0, i=1,2,\ldots}$ tends to a point $p_{t,\infty} \in (\frac{1}{t}M)(\infty)$ as $i \to +\infty$ for each $t > 0$, and to a point $p_{\infty,i} \in \mathrm{Con}_\infty M$ as $t \to +\infty$ for each i, then

$$\lim_{t\to+\infty} p_{t,\infty} = \lim_{i\to+\infty} p_{\infty,i}.$$

Recall here that $(\frac{1}{t}M)(\infty)$ and $(\mathrm{Con}_\infty M)(\infty)$ are both identified with $M(\infty)$.

Applying Theorem 2.1 yields the following proposition, which is a part of Theorem A.

PROPOSITION 2.4. *If a sequence $\{p_t\}_{t>0}$ of points in Z with $p_t \in \frac{1}{t}M$ converges to a point $p \in \mathrm{Con}_\infty M$ as $t \to +\infty$, then*

$$\inf_{X \subset A_p} d_H(A_{p_t}, X) < \theta_n(\delta(p)) + \theta_M(1/t).$$

Thus, the proof of Theorem A is completed by

LEMMA 2.5. *For any ray γ in $\mathrm{Con}_\infty M$ there exist rays γ_t in $\frac{1}{t}M$ for $t > 0$ tending to γ as $t \to +\infty$.*

Proof. Take a sequence $\{p_t\}_{t>0}$ of points $p_t \in \frac{1}{t}M$ tending to $p := \gamma(0)$. If $\gamma(\infty) \in M(\infty)$ is not a cut point to $\pi_{M(\infty)}(p)$, then any ray σ in $\mathrm{Con}_\infty M$ from p with $\sigma(\infty) = \gamma(\infty)$ is just the γ, so that we take the desired γ_t as a ray from p_t such that, in $Z(\infty)$, $\gamma_t(\infty) \in (\frac{1}{t}M)(\infty)$ tends to $\gamma(\infty) \in (\mathrm{Con}_\infty M)(\infty)$ as $t \to +\infty$.

In the case where $\gamma(\infty)$ is a cut point to $\pi_{M(\infty)}(p)$, for each $r>0$ a ray γ^r from p with $\gamma^r(\infty)=\pi_{M(\infty)}\circ\gamma(r)$ has the property that $\gamma^r(\infty)$ is not a cut point to $\pi_{M(\infty)}(p)$. Hence, for each $r>0$ there exist rays γ_t^r from p_t for $t>0$ tending to γ^r as $t\to+\infty$. Since γ^r tends to γ as $r\to+\infty$, we can find $r(t)\nearrow+\infty$ such that $\gamma_t^{r(t)}$ tends to γ as $t\to+\infty$. □

Proof of Corollary to Theorem A. When $\dim \mathrm{Con}_\infty M = n$ ($:=\dim M$), the corollary is obvious because $\Sigma_p f$ is an almost isometry.

When $\dim \mathrm{Con}_\infty M \le n-1$, it follows from the proof of Theorem 2.1 that

$$A_{p_t}\subset\left\{v\in\Sigma_{p_t}\,\middle|\,\sum_{i=1}^{n}\cos^2|v,v_{p_tp_i}|>1-\theta_n(\delta(p))-\theta_M(1/t)\right\},$$

which completes the proof. □

§3. Estimate of the mean measure of rays

Let X be an n-dimensional Alexandrov space of curvature $\ge k$, and set

$$S_\delta:=S_\delta X:=\{p\in X\mid \mathrm{vol}\,S^{n-1}-\mathscr{H}^{n-1}(\Sigma_p)\ge\delta\}\quad\text{for }\delta>0.$$

Note that $p\in S_\delta$ if and only if $\delta(p)\le\theta_n(\delta)$. It follows from $\liminf_{x\to p}\Sigma_x\ge\Sigma_p$ (see [BGP, 7.14]) that S_δ is a closed subset of X. By [BGP, 10.6] or [OS, Theorem A], the Hausdorff dimension of S_δ for any $\delta>0$ is not greater than $n-1$. We have

THEOREM 3.1. *Let X and Y be n-dimensional compact Alexandrov spaces of curvature $\ge k$, and $\delta, r>0$. Assume that an ϵ-Hausdorff approximation $g: X\to Y$ exists. If $\epsilon>0$ is small enough against X,δ,r, there exists a* $(\theta_n(\delta)+\theta_{X,\delta,r}(\epsilon))$-almost isometric *map $f: X-B(S_\delta,r)\to Y$ with* $|f,g|:=\sup\{|f(x),g(x)|\mid x\in X-B(S_\delta,r)\}<\theta_n(\delta)+\theta_{X,\delta,r}(\epsilon)$.

Remarks.

(1) The above $\theta_n(\delta)$ cannot be equal to zero. In other words, denoting by d_L the Lipschitz distance, we have $d_L(X,Y)<\theta_X(d_H(X,Y))$ only when X has no singular points (i.e., $S_\delta X=\varnothing$ for any $\delta>0$).

(2) In [Ymg], Yamaguchi treated the case where $\dim X\le\dim Y$ and $S_\delta X=\varnothing$, and proved that there exists an almost Riemannian submersion from X to Y, which becomes an almost isometry in the case where $\dim X=\dim Y$. The approaches of the proofs of that and Theorem 3.1 are different.

Proof of Theorem 3.1. We can find a $\theta(\delta)$-strainer $\{p_{x,j}\}_{j=1}^{2n}$ at every $x \in X - B(S_\delta, r)$ such that $\ell = |p_{x,j}x|$ for any $j = 1, \ldots, 2n$, $x \in X - B(S_\delta, r)$, and some constant $\ell > 0$, where we note that $\ell = \mathrm{const}_{X,\delta,r}$. According to [BGP, 9.4], the map $\varphi_x := (|p_{x,j}, \cdot|)_{j=1,\ldots,n} : B(x, t) \to \mathbf{R}^n$ for any $t > 0$ is $(\theta_n(\delta) + \theta(t/\ell))$-almost isometric. For a $0 < t \ll \ell$ take a maximal t-discrete net $\{x_i\}_{i=1}^m$ of $X - B(S_\delta, r)$ and, for simplicity, set $\varphi_i := \varphi_{x_i}$, $p_{ij} := p_{x_i,j}$, $q_{ij} := g(p_{ij})$, and $y_i := g(x_i)$. Then, $\{q_{ij}\}_{j=1}^{2n}$ is a $(\theta(\delta) + \theta(\epsilon/\ell))$-strainer at y_i. Letting $\psi_i := (|q_{ij}, \cdot|)_{j=1,\ldots,n}$, $\theta := \theta_n(\delta) + \theta((\varepsilon + t)/\ell)$, $U_i := B(x_i, 2t)$, $U_i' := B(x_i, t)$, and $V_i := B(y_i, 5t)$, we have that $\{U_i'\}_{i=1}^m$ covers $X - B(S_\delta, r)$, and that both $\varphi_i : U_i \to \mathbf{R}^n$ and $\psi_i : V_i \to \mathbf{R}^n$ are θ-almost isometric and hence θt-Hausdorff approximations. Since $||p_{ij}x| - |q_{ij}g(x)|| < \epsilon$ for any $x \in X$,

$$|\varphi_i, \psi_i \circ g| < \sqrt{n}\epsilon$$

for any i. Suppose that $\epsilon = t/100$ and $\theta \ll 1$. Find a C^∞-function $\rho : [0, +\infty) \to [0, 1]$ such that $\rho = 1$ on $[0, 1]$ and $\rho = 0$ on $[2, +\infty)$. We set $\chi_i := \rho(|x_i, \cdot|) : X \to [0, 1]$, so that $\chi_i = 1$ on U_i' and $\chi_i = 0$ on $X - U_i$. Let us define functions $f_i : \bigcup_{j=1}^i U_j' \to Y$ for $i = 1, \ldots, m$ inductively. Set

$$f_1 := \psi_1^{-1} \circ \varphi_1 : U_1' \to Y,$$

and for $i \geq 2$,

$$f_i := f_{i-1} \quad \text{on} \quad \bigcup_{j=1}^{i} U_j' - U_i,$$

$$f_i := \psi_i^{-1} \circ ((1 - \chi_i)\psi_i \circ f_{i-1} + \chi_i \varphi_i) \quad \text{on} \quad \bigcup_{j=1}^{i} U_j' \cap U_i.$$

Note that, although f_{i-1} is not defined on $U_i' - \bigcup_{j=1}^{i-1} U_j'$, we have $\chi_i = 1$ there and hence $f_i = \psi_i^{-1} \circ \varphi_i$ on U_i'. We now prove

Claims.

(i) f_1 is θ-almost isometric and satisfies $|f_1, g| < (1 + \theta)\sqrt{n}\epsilon$.

(ii) If f_{i-1} is α-almost isometric and satisfies $|f_{i-1}, g| < \beta$ for two positive numbers α and β, then f_i is $(\theta + \theta(\alpha + \beta + \sqrt{n}\epsilon))$-almost isometric and satisfies $|f_i, g| < \theta(\beta + \sqrt{n}\epsilon)$.

(i) is trivial. Let us show (ii). Taking any $x, y \in \bigcup_{j=1}^i U_j'$ we have the following (a), (b), and (c).

(a) When $x, y \notin U_i$, then $f_i(x) = f_{i-1}(x)$ and $f_i(y) = f_{i-1}(y)$.
(b) When $x \in U_i$ and $y \notin U_i$,

$$\psi_i \circ f_i(x) - \psi_i \circ f_i(y) = \psi_i \circ f_{i-1}(x) - \psi_i \circ f_{i-1}(y) + \chi_i(x)(\varphi_i(x) - \psi_i \circ f_{i-1}(x)).$$

(A similar equation holds when $x \notin U_i$ and $y \in U_i$.)
(c) When $x, y \in U_i$,

$$\begin{aligned}\psi_i \circ f_i(x) - \psi_i \circ f_i(y) = {} & (\chi_i(x) - \chi_i(y))(\varphi_i(x) - \psi_i \circ f_{i-1}(x)) \\ & + (1 - \chi_i(y))(\psi_i \circ f_{i-1}(x) - \psi_i \circ f_{i-1}(y)) \\ & + \chi_i(y)(\varphi_i(x) - \varphi_i(y)).\end{aligned}$$

It follows that $\chi_i(x) \le \mathrm{const}\,|xy|$ in (b) and $|\chi_i(x) - \chi_i(y)| \le \mathrm{const}\,|xy|$ in (c). Besides,

$$|\varphi_i, \psi_i \circ f_{i-1}| \le |\varphi_i, \psi_i \circ g| + |\psi_i \circ g, \psi_i \circ f_{i-1}| < \sqrt{n}\epsilon + (1+\theta)\beta.$$

Therefore, in either case of (b) or (c),

$$|f_i(x), f_i(y)| = (1 + \theta + \theta(\alpha + \beta + \sqrt{n}\epsilon))|xy|.$$

The proof of $|f_i, g| < \theta(\beta + \sqrt{n}\epsilon)$ is easier and left to the reader.

Let us define two sequences $\{\alpha_i\}$ and $\{\beta_i\}$ by

$$\alpha_1 := \theta, \qquad \beta_1 := (1+\theta)\sqrt{n}\epsilon,$$

$$\alpha_i := \theta + \theta(\alpha_{i-1} + \beta_{i-1} + \sqrt{n}\epsilon), \qquad \beta_i := \theta(\beta_{i-1} + \sqrt{n}\varepsilon) \quad \text{for } i \ge 2.$$

By remarking that $f_i(x) = f_{i-1}(x)$ for $x \notin U_i$, Claims (i) and (ii) show that there exists a neighborhood U_x of each point $x \in \bigcup_{j=1}^{i} U_j'$ such that $f_i|_{U_x}$ is $\alpha_{\nu(x)}$-almost isometric and satisfies $|f_i|_{U_x}, g|_{U_x}| < \beta_{\nu(x)}$, where $\nu(x) := \#\{j \mid U_j \ni x\}$. Since a standard discussion using Bishop's volume comparison theorem proves $\nu(x) \le \mathrm{const}_n$, $\alpha_{\nu(x)}$ and $\beta_{\nu(x)}$ are both reduced to $\theta_n(\delta) + \theta_{X,\delta,r}(\epsilon)$, so that $f := f_m : \bigcup_{j=1}^{m} U_j' \supset X - B(S_\delta, r) \to Y$ satisfies the conclusion of the theorem. $\square$

LEMMA 3.2. *Setting* $v(p) := \mathscr{H}^n(B(o, 1; K_p))$ *we have*

$$\lim_{\rho \to 0} \frac{\mathscr{H}^n(B(p, \rho))}{\rho^n} = v(p)$$

for any $p \in X$.

Proof. Bishop–Gromov's volume comparison theorem implies

$$(1 - \theta_{n,k}(\rho)) \frac{\mathscr{H}^n(B(p, \rho))}{\rho^n} \leq v(p).$$

Since $\frac{1}{\rho}B(p, \rho)$ tends to $B(o, 1; K_p)$ as $\rho \to 0$, applying Theorem 3.1 yields that there exists a θ-almost isometric map $f: B(o, 1; K_p) - B(S_\delta K_p, r) \to \frac{1}{\rho}B(p, \rho)$, where $\theta := \theta_n(\delta) + \theta_{X,\delta,r}(\rho)$. Hence,

$$\begin{aligned} &v(p) - \mathscr{H}^n(B(o, 1; K_p) \cap B(S_\delta K_p, r)) \\ &\quad = (1 + \theta)^n \mathscr{H}^n(\operatorname{Im} f) \leq (1 + \theta)^n \frac{\mathscr{H}^n(B(p, \rho))}{\rho^n}. \end{aligned}$$

Since $\lim_{r \to 0} \mathscr{H}^n(B(o, 1; K_p) \cap B(S_\delta K_p, r)) = 0$, this completes the proof. □

LEMMA 3.3. *Let* $C \subset X$ *be a compact subset and* $v_0 := \inf_{p \in C} v(p)$. *Then,*

(1) $$\frac{\mathscr{H}^n(B(p, \rho))}{\rho^n} > v_0 - \theta_X(\rho) \quad \textit{for any } p \in C \textit{ and } \rho > 0,$$

(2) $$\mathscr{H}^n(C) \geq v_0 V_r^n(C).$$

Remark. It follows in general that $\mathscr{H}^n(\cdot) \leq \text{const}_n V_r^n(\cdot)$.

Proof. (1): Suppose the contrary, so that there exist $c > 0$, $\{p_i\}_{i=1,2,\ldots} \subset C$, and $\rho_i \to 0$ such that

$$\frac{\mathscr{H}^n(B(p_i, \rho_i))}{\rho_i^n} \leq v_0 - c.$$

We may assume that p_i tends to a point $p \in C$ as $i \to +\infty$. By Lemma 3.2,

$$\frac{\mathscr{H}^n(B(p, \rho))}{\rho^n} > v_0 - \frac{c}{2}$$

for all sufficiently large $\rho > 0$. Bishop–Gromov's volume comparison theorem implies that

$$(1-\theta_{n,k}(\rho))\frac{\mathscr{H}^n(B(p_i,\rho))}{\rho^n} \le \frac{\mathscr{H}^n(B(p_i,\rho_i))}{\rho_i^n} \le v_0 - c$$

for $\rho \ge \rho_i$. On the other hand, setting $A(x, r_1, r_2) := B(x, r_2) - B(x, r_1)$ we have

$$\begin{aligned}\mathscr{H}^n(B(p_i,\rho)) - \mathscr{H}^n(B(p,\rho))| &\le \mathscr{H}^n(A(p_i,\rho,\rho+|pp_i|)) + \mathscr{H}^n(A(p,\rho,\rho+|pp_i|)) \\ &\le 2\,\mathrm{vol}\,A(\rho,\rho+|pp_i|; M^n(k)) = \theta_{n,k,\rho}(|pp_i|),\end{aligned}$$

where $A(r_1, r_2; M^n(k))$ denotes a concentric annulus of radii $r_i < r_2$ in $M^n(k)$. Thus

$$(1-\theta_{n,k}(\rho))\left(v_0 - \frac{c}{2} - \frac{\theta_{n,k,\rho}(|pp_i|)}{\rho^n}\right) \le v_0 - c,$$

which is a contradiction.

(2): Finding a maximal ρ-discrete net $\{p_i\}$ of C, we have, by (1),

$$\mathscr{H}^n(B(C,\rho/2)) \ge \sum_i \mathscr{H}^n(B(p_i,\rho/2)) > \beta_C(\rho)\rho^n(v_0 - \theta_X(\rho)).$$

By taking $\rho \to 0$, this completes the proof. □

LEMMA 3.4. *In Theorem* 3.1 we have

(1) $Y - \mathrm{Im} f \subset B(g(S_\delta), r + 2|fg| + 4\varepsilon)$,
(2) $\mathscr{H}^n(Y - \mathrm{Im} f) < \theta_{x,\delta}(r,\epsilon)$.

Proof. (1): Take any $y \in Y - B(g(S_\delta), r + 2|fg| + 4\epsilon)$. Since $g(X)$ is ϵ-dense, we can find $x \in X$ such that $|g(x), y| < \epsilon$. It follows that

$$|g(x), g(S_\delta)| > r + 2|fg| + 3\epsilon$$

and hence

$$B(x, 2|fg| + 2\epsilon) \subset X - B(S_\delta, r).$$

Thus, supposing $\theta \le 1/2$ we have

$$f(X - B(S_\delta, v)) \supset f(B(x, 2|fg| + 2\epsilon)) \supset B(f(x), 2(1-\theta)(|fg| + \epsilon))$$
$$\supset B(g(x), \epsilon) \ni y.$$

(2): Let $\{p_i\}$ be a maximal ρ-discrete net of S_δ. It follows from $S_\delta \subset \bigcup_i B(p_i, \rho)$ that $g(S_\delta) \le \bigcup_i B(g(p_i), \rho + \epsilon)$, and therefore, by (1),

$$Y - \operatorname{Im} f \subset \bigcup_i B(g(p_i), R + \rho),$$

where $R := r + 2|fg| + 5\epsilon$. Hence, by Bishop's volume comparison theorem,

$$\mathscr{H}^n(Y - \operatorname{Im} f) \le \beta_{S_\delta}(\rho) \operatorname{vol} B(R + \rho; M^n(k)).$$

Since Lemma 3.3 (2) and $\mathscr{H}^n(S_\delta) = 0$ together imply $V_r^n(S_\delta) = 0$, setting for instance $\rho := R$ we have

$$\mathscr{H}^n(Y \to \operatorname{Im} f) \le \theta_{X,\delta}(R),$$

where we note that $\theta_{n,k,X,\delta}(R)$ is reduced to $\theta_{X,\delta}(R)$. Since $R = r + \theta_{X,\delta,r}(\epsilon)$, this completes the proof. □

Denoting by $\mathscr{A}(n, k, D)$ the set of n-dimensional Alexandrov spaces of curvature $\ge k$ and of diameter $\le D$, we have the following corollary as a direct consequence of Theorem 3.1 and Lemma 3.4 (2).

COROLLARY 3.5. *The n-dimensional Hausforff measure* $\mathscr{H}^n : \mathscr{A}(n, k, D) \to \mathbf{R}$ *is a continuous function with respect to* d_H.

Proof of Theorem B. By Theorem 3.1, for any $\delta, r, t > 0$ there exists a $(\theta_n(\delta) + \theta_{M,\delta,r}(1/t))$-almost isometric map $f_t: B(o_\infty, 1; \operatorname{Con}_\infty M) - B(S_\delta, r) \to B(o, 1; \frac{1}{t}M)$. Corollary to Theorem A states

$$\mathscr{H}^{n-1}(A_{f_t(p)}) < \mathscr{H}^{n-1}(A_p) + \theta_n(\delta) + \theta_M(1/t)$$

for any $p \in B(o_\infty, 1; \operatorname{Con}_\infty M) - B(S_\delta, r)$. If $\dim \operatorname{Con}_\infty M \le n - 1$, since $\mathscr{H}^{n-1}(A_p) = 0$ for any $p \in \operatorname{Con}_\infty M$, the above inequality implies the conclusion.

Assume that $\dim \mathrm{Con}_\infty M = n$. By the above inequality,

$$\int_{\mathrm{Dom} f_t \ni p} \mathscr{H}^{n-1}(A_p)\, d\mathscr{H}^n > (1 - \theta_n(\delta) - \theta_{M,\delta,r}(1/t)) \int_{\mathrm{Im} f_t \ni q} \mathscr{H}^{n-1}(A_q)\, d\mathscr{H}^n$$
$$- (\theta_n(\delta) + \theta_M(1/t))\mathscr{H}^n(B(o_\infty, 1; \mathrm{Con}_\infty M)).$$

Here it follows that

$$\mathscr{H}^n(B(S_\delta, r)) < \theta_{M,\delta}(r) \quad \text{and} \quad \mathscr{H}^n(B(o, 1, \tfrac{1}{t}M) - \mathrm{Im} f_t) < \theta_{M,\delta}(r, 1/t).$$

Moreover, $\mathscr{H}(B(o, 1; \frac{1}{t}M))$ tends to $\mathscr{H}^n(B(o_\infty, 1; \mathrm{Con}_\infty M))$ as $t \to +\infty$ and, by the assumption, $\mathscr{H}^n(B(o_\infty, 1; \mathrm{Con}_\infty M)) > 0$. Thus we obtain

$$\rho(\mathrm{Con}_\infty M, o_\infty) > (1 - \theta_n(\delta))(\bar{\rho}(M, o) - \theta_{M,\delta}(r)) - \theta_M(\delta, r),$$

which completes the proof. □

§4. Appendix

PROOF OF PROPOSITION 1.1. Let $R_o := \{\gamma(t) \mid \gamma \text{ is a ray from } o \text{ and } t \geq 0\}$. It suffices to show

(i) $\lim_{t \to +\infty} d_{p.H}((\frac{1}{t}R_o, o), (K(M(\infty)), o)) = 0,$

(ii) $\lim_{t \to +\infty} d_{p.H}((\frac{1}{t}R_o, o), (\frac{1}{t}M, o)) = 0.$

To prove (i), we define the surjective map $f_t : \frac{1}{t}R_o \to K(M(\infty))$, $\gamma(at) \mapsto (\gamma(\infty), a)$ for $t > 0$, where γ is any ray in M from o and $a \geq 0$. Let us prove that f_t is a $\theta_M(1/t)$-pointed Hausdorff approximation. If not, there exist a const > 0, rays σ_i, γ_i from o, and $s_i, t_i \nearrow +\infty$ for $i = 1, 2, \ldots$ such that

$$\tilde{\angle} \sigma_i(s_i) o \gamma_i(t_i) \geq \angle_\infty(\sigma_i, \gamma_i) + \text{const}.$$

Taking a subsequence we may assume that σ_i and γ_i tend to some rays σ and γ from o respectively. Then, $\angle_\infty(\sigma_i, \gamma_i)$ tends to $\angle_\infty(\sigma, \gamma)$. Moreover, for any fixed $s, t > 0$, the Alexandrov convexity implies that

$$\tilde{\angle} \sigma_i(s_i) o \gamma_i(t_i) \leq \tilde{\angle} \sigma_i(s) o \gamma_i(t)$$

for any i with $s_i \geq s$ and $t_i \geq t$. Here the right-hand side of the above inequality tends to $\tilde{\angle}\sigma(s)o\gamma(t)$ as $i \to \infty$. Thus,

$$\tilde{\angle}\sigma(s)o\gamma(t) \geq \angle_\infty(\sigma, \gamma) + \text{const},$$

which is a contradiction.

To prove (ii), it suffices to show that any sequence $\{p_i\}$ of points in M with $|op_i| =: t_i \to +\infty$ satisfies

$$\lim_{i \to \infty} \frac{|p_i R_o|}{t_i} = 0.$$

Suppose the contrary, i.e., there exists a sequence $\{p_i\}$ in M such that

$$|op_i| =: t_i \to +\infty \quad \text{and} \quad \frac{|p_i R_0|}{t_i} \geq \text{const} > 0.$$

Taking a subsequence we may assume that $\gamma_i := op_i$ tends to a ray γ from o. The Toponogov convexity implies

$$\tilde{\angle}\gamma(t_i)op_i \leq |\dot{\gamma}_i(0), \dot{\gamma}(0)|,$$

the right-hand side of which tends to zero as $i \to +\infty$. This is a contradiction. □

Souls and the limit cones of complete open Riemannian manifolds of nonnegative sectional curvature. Assume now that M is a complete open Riemannian manifold of nonnegative sectional curvature and Soul_M a soul of M produced by the basic construction introduced in [CG]. Recall that Soul_M is a closed totally convex submanifold of M over which the normal bundle is diffeomorphic to M (see [CG, Shr]), and that any soul of M is unique up to isometry (see [Ym1]). In this short section, we shall investigate the relation between the souls and the limit cone. Denote by Φ_{Soul_M} the normal holonomy group along Soul_M, (which is a compact Lie group fibre-wise acting the unit normal bundle over Soul_M). We have

PROPOSITION 4.1. *If M is a complete open Riemannian manifold of nonnegative sectional curvature, then*

$$\dim \mathrm{Con}_\infty M \leq \operatorname{codim} \mathrm{Soul}_M - \dim \Phi_{\mathrm{Soul}_M}.$$

Proof. Let $\pi: SN(\mathrm{Soul}_M) \to \mathrm{Soul}_M$ be the unit normal bundle over Soul_M, and $SN_p(\mathrm{Soul}_M)$ its fibre at $p \in \mathrm{Soul}_M$. Set

$$A(\mathrm{Soul}_M) := \{v \in SN(\mathrm{Soul}_M) \mid \gamma_v(t) := \exp tv \text{ for } t \geq 0 \text{ is a ray}\}.$$

We first claim that for any $v \in A(\mathrm{Soul}_M)$ and any curve $c: [0, 1] \to \mathrm{Soul}_M$ with $c(0) = \pi(v)$, the parallel translation $P_c(v) \in SN_{c(1)}(\mathrm{Soul}_M)$ of v along c is contained in $A(\mathrm{Soul}_M)$ and satisfies $\gamma_v(\infty) = \gamma_{P_c(v)}(\infty)$. In fact, in the case where c is a geodesic, applying Rauch's comparison theorem yields that

$$\{\exp tP_{c\,|\,[0,s]}(v) \mid t \geq 0, 0 \leq s \leq 1\}$$

is a totally geodesic flat half strip in M and that $\gamma_{P_{c\,|\,[0,s]}}(v)$ for each $0 \leq s \leq 1$ is a ray (cf. (a) in the proof of [CG, Theorem 1.10]). Therefore the claim in this case follows. Since any curve is approximated by a broken geodesic, the proof of the claim is completed.

For $p \in \mathrm{Soul}_M$ let $A_p(\mathrm{Soul}_M) := \{v \in A(\mathrm{Soul}_M) \mid \pi(v) = p\}$. By the claim, the map

$$A_p(\mathrm{Soul}_M)/\Phi_{\mathrm{Soul}_M} \ni [v] \mapsto \gamma_v(\infty) \in M(\infty)$$

is well-defined. Since this is a surjective and distance nonincreasing map,

$$\dim SN_p(\mathrm{Soul}_M) - \dim \Phi_{\mathrm{Soul}_M} \geq \dim M(\infty),$$

which completes the proof of the proposition. □

COROLLARY 4.2. *If in the above proposition* $\dim \mathrm{Con}_\infty M = \mathrm{codim}\, \mathrm{Soul}_M$ *holds, then a finite cover* $\hat{M}$ *of* M *is isometric to the Riemannian product* $N \times \mathrm{Soul}_{\hat{M}}$, *where* N *is a complete manifold diffeomorphic to* $\mathbf{R}^k$, $k := \mathrm{codim}\, \mathrm{Soul}_M$.

Proof. Proposition 4.1 and the assumption together imply that $\dim \Phi_{\mathrm{Soul}_M} = 0$, i.e., Φ_{Soul_M} is a discrete group and thus isomorphic to a finite subgroup G of the fundamental group $\pi_1(\mathrm{Soul}_M)$ $(\approx \pi_1(M))$ of Soul_M. The normal holonomy group $\Phi_{\mathrm{Soul}_{\hat{M}}}$ of the finite cover $\hat{M}$ of M with $\hat{M}/G = M$ is trivial. Therefore Yim's theorem (see [Ym2, Corollary 3.10]) completes the proof. □

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