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L^p-pinching and the geometry of compact Riemannian manifolds

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Abstract. We prove a Harnack-type inequality $\inf|S|/\sup|S| > 1 - \varepsilon(W, M, V)$ satisfied by the sections of a Riemannian vector bundle W lying in the kernel of a Schrödinger operator $\nabla^*\nabla + V$ under L^p-pinching assumptions on the potential V and derive various topological and geometric consequences.

For instance, we prove a fibration theorem which gives a classification of almost non-negatively curved compact manifolds by the first Betti number. In the case of almost non-positively curved compact manifolds, we prove that the minimal volume must vanish whenever the isometry group is not finite and give conditions implying that it is abelian.

1. Introduction

The Gauß-Bonnet formula $2\pi\chi = \int K\,dv$ which expresses the Euler characteristic of a closed surface as the integral of the curvature has been the starting point of an intense field of investigations of the interactions between the local properties of the metric of a Riemannian manifold (such as curvature bounds) and its global topology and/or geometry (e.g. Betti numbers, fundamental group, isometry group etc...).

One way to do this is embodied in the celebrated Bochner method. Initiated by S. Bochner in 1946 ([B1], [B2] and [B–Y]), this method is by now classical in differential geometry.

In its essence, it can be summarized as follows: we express a topological or geometric invariant through the kernel $\mathscr{S}$ of a Schrödinger operator.

$$\mathscr{S} = \{S \in \Gamma(W) \mid \nabla^*\nabla S + VS = 0\}$$

where S is a section of a Riemannian connected vector bundle $W \to M$ over a compact manifold and the potential V is a field of symmetric endomorphisms of the fibre given by the geometry of the situation and in most cases computable in terms of the curvature.

Bochner's method then reads:

(1) $V \geq 0$ implies that $\dim \mathcal{S} \leq \operatorname{rk} W$ and that any section $S \in \mathcal{S}$ is parallel so that in particular it never vanishes.
(2) If $V \geq 0$ and if there exists $x \in M$ such that $V_x > 0$, then $\mathcal{S} = \{0\}$.

1.1. *Examples*

The Hodge Laplacian $(d + d^*)^2$, acting on p-forms, i.e. sections of the bundle $\bigwedge^p T^*M \to M$, is a Schrödinger operator, since the Bochner-Weitzenböck formulas give an expression of the previous type for this operator. The field V of endomorphisms is given by the curvature tensor of M. For $p = 1$, for instance, V is the Ricci tensor of M. The Hodge-De Rham decomposition theorem states that the kernel of this operator (the space of harmonic p-forms) is isomorphic to the p-th De Rham cohomology group $\mathscr{H}^p_{DR}(M)$.

Through the Eells-Sampson formula, the differential of a harmonic map can be viewed as a solution of a Schrödinger equation, thus giving the non-existence of non trivial harmonic maps when the source has non negative Ricci curvature and the target has negative sectional curvature.

A last example is given by Lie groups of transformations preserving some structure. For instance, the Killing vector fields of a Riemannian manifold and the holomorphic vector fields of a Kählerian manifold both lie in the kernel of $\nabla^*\nabla - \mathrm{ric}$, where ric denotes the ricci curvature, acting on sections of the tangent bundle.

1.2. *Statement of results*

One of the limitations of Bochner's method lies in the fact that the positivity assumption on the potential is strong, so that it would be desirable to weaken it. Indeed it seems likely that some topological properties holding for the class of manifolds for which the potential is nonnegative will persist if it is allowed to take small negative values.

The best one can hope for, considering the counterexamples given in the appendix of [Ga2], is to prove pinching and vanishing theorems while keeping control only on an L^p-norm of the potential. We show that this is indeed possible under mild extra hypotheses which allow for an infinite number of topologies.

More precisely, we prove a Harnack-type inequality which shows that a section lying in the kernel of $\nabla^*\nabla + V$, while it cannot be expected to be parallel, retains the property to be nowhere vanishing. We thus obtain a topological classification similar to the geometric one given above by describing the behaviour of the elements of $\mathrm{Ker}(\nabla^*\nabla + V)$ while earlier results of P. Li [Li], S. Gallot [Ga1] [Ga2], P. Berard and G. Besson [B–B] gave a-priori estimates of its dimension.

Notice that our assumptions allow $\mathrm{Ker}(\nabla^*\nabla + V)$ to be non trivial, while results of K. D. Elworthy and S. Rosenberg [E–R] rely on assumptions implying that this kernel vanishes.

Before we state the main result of this paper, let us fix the notations.

1.3. *Notations*

Throughout this paper, $W \to M$ will denote a Riemannian connected vector bundle, endowed with a metric $\langle . \,|\, . \rangle_W$ and a compatible Riemannian connexion ∇. The basis (M, g) is a compact n-dimensional Riemannian manifold. We denote by R^M and R^W the curvature tensors of M and W respectively and ric the ricci curvature tensor of M.

The rough Laplacian acting on sections of W will be denoted indifferently by $\nabla^*\nabla$ and $\bar{\Delta}$.

Given a field V of symmetric endomorphisms of the fiber, we denote by $\underline{V}(m)$ (resp. $\bar{V}(m)$) the lowest (resp. highest) eigenvalue of V at the point $m \in M$.

We also define for any function $f\colon M \to \mathbb{R}$ its positive part $f^+(m) = \sup(0, f(m))$ and its negative part $f^-(m) = \sup(0, -f(m))$.

The L^s-norms are taken with respect to the probability measure $dv_g/\mathrm{Vol}(M,g)$:

$$\|f\|_s = \left(\int_M |f|^s \frac{dv_g}{\mathrm{Vol}(M, g)}\right)^{\frac{1}{s}}$$

1.4. *Main theorem*

1.4.1. THEOREM. *Let (M^n, g) be a compact Riemannian manifold and $W \to M$ a vector bundle endowed with a metric $\langle . \,|\, . \rangle_W$ and a compatible connection ∇. For any field V of symmetric endomorphisms of the fiber, the Harnack-type inequality*:

$$\frac{\inf_{m \in M} |S(m)|}{\sup_{m \in M} |S(m)|} \geq 1 - A(n, p) e^{2B(\frac{n+p}{2})\sqrt{R}D} \left(\frac{\|\underline{V}^-\|_1}{R}\right)^{\frac{\alpha}{2}} \times \left(1 + \frac{\|V\|_{\frac{p}{2}} + \|R^W\|_{\frac{p}{2}} + \|\underline{\mathrm{ric}}^-\|_{\frac{p}{2}}}{R}\right)^{1-\alpha} \tag{1.4.2}$$

holds for all sections S solution of the Schrödinger equation $\nabla^*\nabla S + VS = 0$, where p is any number strictly greater than n, $\alpha = 2(p-n)/(p^2+np-4n) < 1$, D is any upper bound of the diameter of (M, g), and $-R < 0$ is any "almost lower bound" of the ricci curvature of $M(*)$.

$A(n, p)$ is a universal constant explicitly computed in the proof and

$$B(q) = \left(\frac{2(q-1)}{q}\right)^{\frac{1}{2}} (n-1)^{1-\frac{1}{q}} \left(\frac{q-2}{q-n}\right)^{\frac{1}{2}-\frac{1}{q}}$$

1.5. *Remarks*

It should be noted that the strength of the result lies in the universality of the constant in the theorem.

Observe that the class of manifolds satisfying the bounds of theorem 1.4.1 contains an infinite number of topologies and geometries (as well as singular spaces, although we do not concentrate on this aspect).

Also, no assumption being made on the injectivity radius, "collapsing" à la Cheeger-Gromov-Fukaya is allowed.

In case the bundle $W \to M$ is the trivial one-dimensional bundle $M \times \mathbb{R}$, the main theorem amounts roughly to the computation of an explicit upper bound of the $L^{\frac{p}{p-2}}$ norm of the ad hoc Green kernel G of M solution of $\Delta G(x_0, \cdot) = \delta_{x_0} - 1/\mathrm{Vol}(M)$ satisfying $\inf_{x \in M} G(x_0, x) = 0$: indeed, this follows from the fact that

$$\begin{aligned}\sup f - \inf f \leq 2\|f - \bar{f}\|_\infty &= 2 \sup_{x \in M} \left| \int_M G(x, y) V(y) f(y) \frac{dv_g(y)}{\mathrm{Vol}(M)} \right| \\ &\leq 2 \sup_{x \in M} \|G(x, \cdot)\|_{\frac{p}{p-2}} \|V\|_{\frac{p}{2}} \|f\|_\infty\end{aligned}$$

where

$$\bar{f} = \int_M f(x) \frac{dv_g(x)}{\mathrm{Vol}(M)}$$

is the mean value of f.

(*) *This means that there exists a non-negative function $\delta\colon M \to \mathbb{R}$ such that* $\mathrm{ric}_m \geq -(R + \delta(m))g_m$ *and that* $\|\delta\|_{q/2} \leq R/(2^{2/q}(e^{B(q)\sqrt{RD}} - 1)^{2/q})$, $q = (n+p)/2$.

Even in this simple case, it is not obvious how to compute an upper bound of $\|G(x, \cdot)\|_{\frac{p}{p-2}}$ depending only on the geometric bounds involved in the inequality of the main theorem.

In the case of a general bundle, Kato's inequality provides the estimate:

$$-|S|\,|V| - |S| \left|\nabla\left(\frac{S}{|S|}\right)\right|^2 \leq \Delta(|S|) \leq \underline{V}^{-}|S| \tag{1.5.1}$$

at any point where $|S| \neq 0$. In (1.5.1), the assumption that the right-hand side is small is not sufficient to prove that S never vanishes: a counterexample to this is the function f_ε defined by

$$f_\varepsilon(x) = \sup(0, 1 - \varepsilon G(x_0, x))$$

which can be smoothed while keeping the two properties $\Delta f_\varepsilon \leq \varepsilon/\mathrm{Vol}(M)$ and "f_ε vanishes somewhere".

Consequently, a control of the left-hand side of (1.5.1) is necessary. This means a control of $\nabla(S/|S|)$, i.e. of the rotation of S with respect to a parallel frame. This accounts for the curvature terms in (1.4.2).

1.6. *An important corollary*

The main theorem will mostly be used through the following corollary:

1.6.1. THEOREM. *Under the hypotheses of theorem* 1.4.1, *if*

$$\frac{\|\underline{V}^{-}\|_1}{R} < A(n,p)^{-2/\alpha}\, e^{-4B\left(\frac{n+p}{2}\right)\sqrt{R}D/\alpha}\left(1 + \frac{\|V\|_{\frac{p}{2}} + \|R^W\|_{\frac{p}{2}} + \|\underline{\mathrm{ric}}^{-}\|_{\frac{p}{2}}}{R}\right)^{2\left(1-\frac{1}{\alpha}\right)} \tag{1.6.2}$$

then every non trivial solution of the Schrödinger equation $\nabla^*\nabla S + VS = 0$ *never vanishes. In particular, this implies that all Stieffel-Whitney classes of* W *vanish from order* $\mathrm{rk}(W) - \dim \mathrm{Ker}(\nabla^*\nabla + V) + 1$ *to order* $\mathrm{rk}(W)$.

Proof. The hypotheses of theorem 1.6.1 imply that

$$A(n,p)e^{2B\left(\frac{n+p}{2}\right)\sqrt{R}D}\left(\frac{\|\underline{V}^{-}\|_1}{R}\right)^{\frac{\alpha}{2}}\left(1 + \frac{\|V\|_{\frac{p}{2}} + \|R^W\|_{\frac{p}{2}} + \|\underline{\mathrm{ric}}^{-}\|_{\frac{p}{2}}}{R}\right)^{1-\alpha} < 1$$

and the conclusion of the main theorem implies that every non trivial solution of the Schrödinger equation $\nabla^*\nabla S + VS = 0$ satisfies $\inf(|S|)/\sup(|S|) > 1 - 1 = 0$, i.e. that it never vanishes. □

1.7. *Acknowledgments*

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2. Applications

Before the proof, we give some applications:

2.1. *Harmonic 1-forms*

As we mentioned in the introduction, the Hodge Laplacian is an example of Schrödinger operator acting on p-forms. This was one of the first fields of application of the Bochner technique, particularly in the case of harmonic 1-forms, for which case the first result is due to Bochner himself in [B–Y]:

2.1.1. THEOREM. *If (M, g) is a Riemannian manifold with non-negative Ricci curvature, then its Albanese map is a totally geodesic submersion.*

Recall that the Albanese torus of M is defined as the quotient of the dual of the first De Rham cohomology group $\mathscr{H}^1_{DR}(M)^*$ by the lattice Γ obtained as the image of the torsion free part of $\mathscr{H}_1(M, \mathbb{Z})$ under the De Rham isomorphism between $\mathscr{H}_1(M, \mathbb{R})$ and $\mathscr{H}^1_{DR}(M)^*$.

Let $\pi : \tilde{M} \to M$ be the universal cover of M and fix $\tilde{x}_0 \in \tilde{M}$. We define $\tilde{\mathscr{A}} : \tilde{M} \to \mathscr{H}^1_{DR}(M)^*$ by $\tilde{\mathscr{A}}(\tilde{x}).\alpha = \int_{\tilde{x}_0}^{\tilde{x}} \pi^*\alpha$. Since the Hurewicz homorphism sends $\Pi_1(M)$ to $\mathscr{H}_1(M, \mathbb{Z})$, $\tilde{\mathscr{A}}$ projects to a harmonic map $\mathscr{A} : M \to \mathscr{H}^1_{DR}(M)^*/\Gamma$, called the Albanese map of M. It satisfies the

2.1.2. PROPOSITION. *For each harmonic map f from M to a flat torus $\mathbb{T}^k$, there exists a linear mapping Φ from $\mathbb{T}^{b_1(M)}$ to $\mathbb{T}^k$ satisfying $f = \Phi \circ \mathscr{A}$.*

Sketch of Proof. This follows from the fact that the coordinates of $\tilde{f} : \tilde{M} \to \mathbb{R}^k$ are harmonic functions on $\tilde{M}$, thus giving an homomorphism ${}^t\tilde{\Phi}$ from $\mathbb{R}^{k*}$ to $\mathscr{H}^1_{DR}(M) = \mathbb{R}^{b_1(M)*}$. $\tilde{\Phi}$ projects to a linear mapping $\Phi : \mathbb{T}^{b_1(M)} \to \mathbb{T}^k$ and it is easy to check that $f = \Phi \circ \mathscr{A}$. □

Theorem 2.1.1 given above follows then from the fact that harmonic 1-forms are parallel whenever the ricci curvature is non-negative. □

One may expect that part of this result still holds when the negative part of the ricci curvature is supposed to be small. Indeed, M. Gromov in [Gr1] using geometric arguments and S. Gallot in [Ga1] by an analytical method proved the following results:

2.1.3. THEOREM. *There exists* $\varepsilon(n) > 0$ *such that if* (M, g) *is an n-dimensional Riemannian manifold with* $\operatorname{diam}(M)^2\|\underline{\mathrm{ric}}^-\|_\infty$ *less than* $\varepsilon(n)$, *then the first Betti number of M is not greater than n.*

The conclusion of theorem 2.1.3 is very much weaker than that of theorem 2.1.1, so that one may wonder whether it would not be possible to retain part of the conclusion of theorem 2.1.1 under the hypotheses of theorem 2.1.3. Indeed, M. Gromov in [Gr1] made the following conjecture:

2.1.4. CONJECTURE. *There exists* $\varepsilon(n) > 0$ *such that if* (M, g) *is an n-dimensional Riemannian manifold with* $\operatorname{diam}(M)^2\|\underline{\mathrm{ric}}^-\|_\infty$ *less than* $\varepsilon(n)$ *and such that* $b_1(M) = n$, *then M is homeomorphic to a torus.*

A partial answer to this conjecture has been given by T. Yamaguchi in [Y1] where he proves the following:

2.1.5. THEOREM. *There exists a theoretical function* $\varepsilon(\bullet, \bullet) > 0$ *such that, if a compact Riemannian manifold* M^n *satisfies the inequality*

$$\operatorname{diam}(M)^2\|\underline{\mathrm{ric}}^-\|_\infty \leq \varepsilon(n, \operatorname{diam}(M)^2\|R^M\|_\infty) \tag{2.1.6}$$

then its Albanese map is a harmonic fibration.

He further obtains in [Y2] the following

2.1.7. PINCHING THEOREM. *There exists a positive number* $\varepsilon(n)$ *depending only on n such that, if the (sectional) curvature* σ *and the diameter of a compact Riemannian n-manifold M satisfy*:

$$\operatorname{diam}(M)^2\|\sigma^-\|_\infty \leq \varepsilon(n)$$

then the following hold:

(a) *A finite covering of M fibers over a $b_1(M)$-torus.*

(b) *If $b_1(M) = n$, then M is diffeomorphic to a torus.*

Observe that this result involves no assumption on the positive part of σ, but the assumption on the negative part is more stringent, since it is not only supposed to be bounded, but also needs to be small.

Unfortunately, since the proofs use convergence methods, it is impossible to have an explicit value for these ε. In particular, for a given manifold (M, g), it is impossible to tell if it satisfies the hypotheses of the theorems. One has to compare it with all the other potential candidates.

The value of ε in (2.1.6) depends on a bound on the absolute value of the sectional curvature σ. In view of the conjecture, this assumption may look two drastic. However, it is impossible to do without any assumption on σ, as shown by M. Anderson in [An], where he gives explicit counterexamples:

2.1.8. THEOREM. *Given any $n \geq 4$, $k \leq n-1$ and $\varepsilon > 0$, there are compact n-manifolds M^n satisfying* $\operatorname{diam}(M)^2\|\operatorname{ric}\|_\infty \leq \varepsilon$, *and $b_1(M) = k$ such that no cover of M^n fibers over $\mathbb{S}^1$. In particular, any harmonic 1-form on M^n must vanish somewhere.*

We prove that it is possible to weaken the L^∞ bound of theorem 2.1.5 on the sectional curvature to an $L^{\frac{p}{2}}$ bound and to give an explicit value for ε:

2.1.9. THEOREM. *There exists a function $\zeta(n, p)$ such that, if a compact Riemannian manifold M of dimension n satisfies the inequality*

$$\operatorname{diam}(M)^2\|\underline{\operatorname{ric}}^-\|_{\frac{p}{2}} \leq \zeta(n,p)[1 + \operatorname{diam}(M)^2\|R^M\|_{\frac{p}{2}}]^{-\beta} \tag{2.1.10}$$

for at least one $p > n$ and $\beta = (p+n)(p-2)/(p-n)$, then its Albanese map is a harmonic fibration.

The value of $\zeta(n, p)$ is

$$\zeta(n,p) = \inf[(2(e^{B(\frac{n+p}{2})} - 1))^{-4/(n+p)}, A(n,p)^{-(\beta+2)}\, e^{-2(\beta+2)B(\frac{n+p}{2})}(2n-1)^{-\beta}]$$

Proof. It is enough to show that (2.1.10) implies that a non-trivial harmonic 1-form never vanishes, since then the Albanese map will be a submersion. For this, we need only check that (2.1.10) ensures that the hypotheses of theorem 1.6.1 are satisfied for $W = T^*M$ and $V = \operatorname{ric}$.

Set $D = \operatorname{diam}(M)$, $q = (n+p)/2$ and $R = 1/D^2$. Since

$$\frac{\zeta(n,p)}{D^2} \leq \frac{R}{2^{2/q}(e^{B(q)\sqrt{R}D} - 1)^{2/q}},$$

the choice $\delta = \underline{\mathrm{ric}}^-$ ensures that $-R$ is an almost lower bound for ric.

Since $-\beta = 2(1 - 1/\alpha)$, we can rewrite (2.1.10):

$$\frac{\|\underline{\mathrm{ric}}^-\|_1}{R} \leq \frac{\|\underline{\mathrm{ric}}^-\|_{\frac{p}{2}}}{R} \leq A(n,p)^{-2/\alpha} e^{-4B(\frac{n+p}{2})/\alpha}(2n-1)^{-\beta}\left[1 + \frac{\|R^M\|_{\frac{p}{2}}}{R}\right]^{-\beta}$$

Therefore, (1.6.2) follows from the fact that $\|\mathrm{ric}\|_{\frac{p}{2}} \leq (n-1)\|R^M\|_{\frac{p}{2}}$. □

2.2. *Harmonic maps into flat tori*

As we have seen in proposition 2.1.2, harmonic maps into flat tori are classified by the Albanese map. Consequently theorem 2.1.9 admits the following corollary:

2.2.1. THEOREM. *Under the hypothesis of theorem* 2.1.9, *every harmonic map f from M into a flat torus $\mathbb{T}^k$ is a submersion onto a totally goedesic sub-torus of $\mathbb{T}^k$ of dimension less than or equal to $b_1(M)$.*

Proof. As $\mathscr{A}$ is a submersion, the lift $\tilde{f}$ of f is a submersion onto the image of the linear mapping defined in proposition 2.1.2 which is a vector subspace A of dimension less than or equal to $b_1(M)$, hence $\tilde{f}(\tilde{M})$ is an open subset of A.

$f(M)$ is then an open subset of $A/(\Gamma \cap A)$, which is connected.

$f(M)$ being compact in $\mathbb{T}^k$, $f(M) = A/(\Gamma \cap A)$ is a totally geodesic torus in $\mathbb{T}^k$. □

2.3. *Killing vector fields*

In [B1], S. Bochner proves the following theorem:

2.3.1. THEOREM. *If (M, g) is a Riemannian manifold with non-positive Ricci curvature, then every Killing vector field is parallel.*

We obtain an extension of this result in the following theorem:

2.3.2. THEOREM. *There exists an explicit function* $\eta(n, p, \lambda)$ *such that if a compact Riemannian manifold* (M, g) *of dimension n satisfies the inequality*

$$\frac{\|\overline{\mathrm{ric}}^{+}\|_1}{R} \le \eta(n, p, \sqrt{R}D)\left[1 + \frac{\|R^M\|_{\frac{p}{2}}}{R}\right]^{-\beta} \tag{2.3.3}$$

for at least one $p > n$ *and* $\beta = (p+n)(p-2)/(p-n)$, *where D is an upper bound of* diam(M) *and* $-R$ *an almost lower bound of* ric (*cf. theorem* 1.4.1), *then its isometry group G acts locally freely, furthermore the mapping* $g \mapsto g(x)$ *is a finite covering of each orbit by G.*

In case $\dim(G) = n$, *this implies that G acts transitively on M and induces a finite cover of M.*

The value of $\eta(n, p, \lambda)$ is

$$\eta(n, p, \lambda) = A(n, p)^{-(\beta+2)} e^{-2(\beta+2)B(\frac{n+p}{2})\lambda}(2n-1)^{-\beta}$$

Proof. Recall that a Killing vector field X satisfies the equation $\nabla^*\nabla X - \mathrm{ric}\, X = 0$ (cf. for instance [Be], p. 41), so that under the hypotheses of theorem 2.3.2, a simple application of theorem 1.6.1 shows that a non-trivial Killing vector field never vanishes. It is then easy to see that this implies that the action of the isometry group on M is locally free. □

In case the isometry group is not finite, theorem 2.3.2 gives topological information on the manifold. Indeed, R. Bott proved in [Bo] that the existence of a non vanishing Killing vector field implies that all Pontryagin numbers of M vanish. Moreover, since G contains a subgroup isomorphic to $\mathbb{S}^1$, M admits a locally free action of $\mathbb{S}^1$. This implies that Gromov's minimal volume of the manifold is zero [Gr2].

2.3.4. COROLLARY. *If M has non-zero minimal volume, and if g is a Riemannian metric on M satisfying the hypotheses of theorem* 2.3.2, *then its isometry group is finite.*

Remarks. Sufficient topological conditions for M to have non-zero minimal volume are given in [Gr2], for instance if M has one non-zero characteristic class or if its simplicial volume is non-zero.

Let G_0 be the connected component of the identity in G. The L^2 scalar product acting on Killing vector fields defines a bi-invariant Riemannian metric on G_0. We have the following:

2.3.5. THEOREM. *If a compact Riemannian manifold* (M, g) *of dimension* n *satisfies the inequality*

$$\operatorname{diam}(G_0)^2\|\overline{\mathrm{ric}}^+\|_1 < \left(1-\frac{4}{p+n}\right)^{\frac{p(p+n)(p+n-4)}{4(p-n)}} \times \left[1+C_2(n,p)\, e^{B\left(\frac{n+p}{2}\right)\sqrt{R}D}\left(\frac{\|\overline{\mathrm{ric}}^+\|_{\frac{p}{2}}}{R}\right)^{\frac{1}{2}}\right]^{-\frac{2p(p+n)}{p-n}}$$

for at least one $p > n$, *where* D *is an upper bound of* $\operatorname{diam}(M)$ *and* $-R$ *an almost lower bound of* ric (*cf. theorem* 1.4.1), *then* G_0 *is abelian.*

Proof. Since the L^2 scalar product on Killing vector fields induces a biinvariant metric on the Lie algebra $\mathscr{G}$ of G_0, the adjoint action is a morphism from G_0 to $\mathcal{O}(\mathscr{G})$. Since every non trivial one parameter subgroup of $\mathcal{O}(\mathscr{G})$ contains an element with eigenvalue -1, we will be finished if we prove $\|\mathrm{Ad}_g(X) - X\|_2 < 2\|X\|_2$ for all $g \in G_0$ and $X \in \mathscr{G}$, which prohibits the eigenvalue -1.

Since the Lie bracket is the differential of the adjoint action, a simple integration proves that this follows from:

$$\|[X, Y]\|_2 < \frac{2}{\operatorname{diam}(G_0)}\|X\|_2\|Y\|_2 \tag{2.3.6}$$

In order to establish (2.3.6), we recall that $[X, Y] = \nabla_X Y - \nabla_Y X$ and that (since X and Y are Killing vector fields on M)

$$\|\nabla_X Y\|_2 \leq \|X\|_\infty \|\nabla Y\|_2 \leq \|\overline{\mathrm{ric}}^+\|_1^{\frac{1}{2}}\|X\|_\infty\|Y\|_\infty$$

Therefore we will be done if we can control $\|X\|_\infty$ with $\|X\|_2$. Since every Killing vector field X satisfies $\Delta|X| \leq \overline{\mathrm{ric}}^+|X|$, and since the Sobolev constant $K_2(M,g)$ for the inclusion of $H^{1,2}(M)$ in $L^{\frac{2q}{q-2}}(M)$ satisfies the inequality

$$K_2(M, g) \leq C_2(n, p)\frac{e^{B\left(\frac{n+p}{2}\right)\sqrt{R}D}}{\sqrt{R}}$$

derived from theorem 6 of [Ga2], the result is a consequence of the following lemma, where we set $q = (n+p)/2$.

2.3.7. LEMMA. *If a positive function* f *on a compact Riemannian manifold* (M, g) *satisfies* $\Delta f \leq \lambda f$, *where* λ *is any positive function, then we have*

$$\|f\|_\infty \leq [1 + K_2(M, g)\|\lambda\|_{p/2}^{1/2}]^{\frac{pq}{2(p-q)}}\left[\frac{q}{q-2}\right]^{\frac{pq(q-2)}{8(p-q)}}\|f\|_2 \tag{2.3.8}$$

for all p and q such that $n \leq q < p$, *where* $K_2(M, g)$ *is the Sobolev constant for the inclusion of* $H^{1,2}(M)$ *in* $L^{\frac{2q}{q-2}}(M)$.

Proof. As $\Delta f \leq \lambda f$, the Sobolev inequality

$$\|f^k\|_{\frac{2q}{q-2}} \leq \|f^k\|_2 + K_2(M, g)\|d(f^k)\|_2$$

together with the fact that

$$\|d(f^k)\|_2^2 = k^2 \int_M f^{2k-2}|df|^2 \frac{dv_g}{\mathrm{Vol}(M)} = \frac{k^2}{2k-1} \int_M f^{2k-1} \Delta f \frac{dv_g}{\mathrm{Vol}(M)}$$

gives the inequality

$$\|f\|^k_{\frac{2kq}{q-2}} \leq \|f\|^k_{2k} + \frac{k}{\sqrt{2k-1}} K_2(M, g)\|\lambda\|^{1/2}_{p/2}\|f\|^k_{\frac{2kp}{p-2}} \tag{2.3.9}$$

The Hölder inequality $\|f\|^{2kp/(p-2)}_{2kp/(p-2)} \leq \|f\|^{2kq/(p-2)}_{2kq/(q-2)} \|f\|^{2k(p-q)/(p-2)}_{2k}$ thus gives

$$\begin{aligned}\|f\|_{\frac{2kq}{q-2}} &\leq \left[1 + \frac{k}{\sqrt{2k-1}} K_2(M, g)\|\lambda\|^{1/2}_{p/2}\right]^{\frac{p}{k(p-q)}} \|f\|_{2k} \\ &\leq [1 + K_2(M, g)\|\lambda\|^{1/2}_{p/2}]^{\frac{p}{k(p-q)}} k^{\frac{p}{2k(p-q)}} \|f\|_{2k}\end{aligned}$$

Setting $\sigma = q/(q-2)$ and $k = \sigma^i$, this yields

$$\begin{aligned}\|f\|_\infty &\leq [1 + K_2(M, g)\|\lambda\|^{1/2}_{p/2}]^{\frac{p}{(p-q)} \sum_{i=0}^\infty \frac{1}{\sigma^i}} \exp\left[\frac{p}{2(p-q)} \sum_{i=0}^{\infty} \frac{i \log \sigma}{\sigma^i}\right] \|f\|_2 \\ &\leq [1 + K_2(M, g)\|\lambda\|^{1/2}_{p/2}]^{\frac{p}{(p-q)} \cdot \frac{q}{2}} \exp\left[\frac{p}{2(p-q)} \cdot \frac{q(q-2)}{4} \log\left(\frac{q}{q-2}\right)\right] \|f\|_2\end{aligned}$$

□

2.4. *Kähler manifolds*

In kählerian geometry, the situation is rather rigid: indeed if the ricci curvature of a Kähler manifold is definite, then M can be holomorphically embedded in a complex projective space (cf. [K]). In this context, S. Bochner proved the

2.4.1. THEOREM. *Let* (M, ω) *be a Kähler manifold.*

(1) *If the ricci curvature of* ω *is non positive, then every holomorphic vector field is parallel.*

(2) *If the ricci curvature of* ω *is non negative, then the Albanese map is a totally geodesic holomorphic fibration.*

Now in the same fashion as we did for harmonic forms and Killing vector fields, we can retain part of these properties while relaxing the assumptions on the curvature:

2.4.2. THEOREM. *Let* $\eta(n, p, \lambda)$ *be the function defined in theorem* 2.3.2; *if a compact complex manifold* (M, J) *of complex dimension n admits a kählerian metric* ω *satisfying the inequality*

$$\frac{\|\underline{\mathrm{ric}}^+\|_1}{R} \leq \eta(2n, p, \sqrt{R}D)\left[1 + \frac{\|R^M\|_{\frac{p}{2}}}{R}\right]^{-\beta} \tag{2.4.3}$$

for at least one $p > 2n$ *and* $\beta = (p + 2n)(p - 2)/(p - 2n)$, *where D is an upper bound of* diam(M) *and* $-R$ *an almost lower bound of* ric (*cf. theorem* 1.4.1), *then the group G of bi-holomorphic transformations of M acts locally freely, in particular its (real) dimension is not greater than* $2n$.

Remark. In case the group is not discrete, theorem 2.4.2 implies that all the Chern numbers of M vanish (cf. [Bo]).

2.4.4. THEOREM. *Let* $\zeta(n, p)$ *be the function defined in theorem* 2.1.9; *if a compact complex manifold* (M, J) *of complex dimension n admits a kählerian metric* ω *satisfying the inequality*

$$\mathrm{diam}(M)^2\|\underline{\mathrm{ric}}^-\|_{\frac{p}{2}} \leq \zeta(2n, p)[1 + \mathrm{diam}(M)^2\|R^M\|_{\frac{p}{2}}]^{-\beta} \tag{2.4.5}$$

for at least one $p > 2n$ *and* $\beta = (p + 2n)(p - 2)/(p - 2n)$, *then the Albanese map of M is a holomorphic fibration.*

These two theorems are proved in exactly the same way as theorem 2.3.2 and theorem 2.1.9. □

3. Proofs

3.1. *Proof of the main theorem*

Let S be a non-trivial section of W satisfying $\nabla^*\nabla S + VS = 0$. We have:

$$\|\nabla S\|_2^2 = \int_M \langle \nabla^*\nabla S \mid S\rangle_W \frac{dv_g}{\mathrm{Vol}(M)} = -\int_M \langle VS \mid S\rangle_W \frac{dv_g}{\mathrm{Vol}(M)} \leq \|\underline{V}^-\|_1 \|S\|_\infty^2$$

In order to obtain a control of $\sup|S| - \inf|S|$, we use the Sobolev inequality for the inclusion of $H^{1,p}(M)$ into $L^\infty(M)$ and Kato's inequality thus getting:

$$\sup|S| - \inf|S| \leq K_p(M,g)\,\|d(|S|)\|_p \leq K_p(M,g)\,\|\nabla S\|_p \tag{3.1.1}$$

These two inequalities will yield the result, provided we can control $K_p(M,g)\,\|\nabla S\|_p$ with $\|\nabla S\|_2$.

To this end, we use a bootstrapping argument analogous to the De Giorgi-Moser iteration scheme described in [M] involving the Sobolev imbedding of $H^{1,2}(M)$ in $L^{\frac{2q}{q-2}}(M)$,

$$\|u\|_{\frac{2q}{q-2}} \leq \|u\|_2 + K_2(M,g)\,\|du\|_2 \tag{3.1.2}$$

In order to obtain meaningful topological and/or geometric results, we need a uniform control of the Sobolev constants $K_2(M,g)$ and $K_p(M,g)$. Such a control has been achieved by S. Gallot in theorems 5 and 6 of [Ga2], giving the inequalities

$$\begin{aligned} K_p(M,g) &\leq \frac{C_p(n,p)}{\sqrt{R}}\, e^{B(q)\sqrt{R}D} \\ K_2(M,g) &\leq \frac{C_2(n,p)}{\sqrt{R}}\, e^{B(q)\sqrt{R}D} \end{aligned} \tag{3.1.3}$$

where D is any upper bound of the diameter of (M,g), $-R < 0$ is any "almost lower bound" of the ricci curvature of M, and $n < q < p$. This explains the choice $q = (n+p)/2$.

Now, setting $u = |\nabla S|^k$ in (3.1.2), we obtain

$$\|\nabla S\|^k_{\frac{2kq}{q-2}} \leq \|\nabla S\|^k_{2k} + K_2(M,g)\,\|d(|\nabla S|^k)\|_2 \tag{3.1.4}$$

Therefore, we need an estimate of $\|d(|\nabla S|^k)\|_2$. This is the purpose of the following lemma which is the key to the proof of the theorem:

3.1.5. LEMMA. *For all $k \geq 1$ and $s > 1$, any section S of any vector bundle W over any compact Riemannian manifold (M,g) satisfies*

$$\begin{aligned} \|d(|\nabla S|^k)\|_2^2 \leq{}& k\left(\|\underline{\mathrm{ric}}^-\|_{\frac{s}{s-1}} + n\,\|R^W\|_{\frac{s}{s-1}}\right)\|\nabla S\|^{2k}_{2ks} \\ &+ k(2k-1)\left(\|R^W\|^2_{\frac{2ks}{1+k(s-1)}}\,\|S\|^2_\infty + \|\nabla^*\nabla S\|^2_{\frac{2ks}{1+k(s-1)}}\right)\|\nabla S\|^{2k-2}_{2ks} \end{aligned} \tag{3.1.6}$$

Note that in this result, no assumption is made on the vector bundle or on the basis.

The proof is postponed to a later section.

For the sake of simplicity, we fix $\|S\|_\infty = 1$. If we observe that $\|\nabla^*\nabla S\|_{\frac{2ks}{1+k(s-1)}} \leq \|V\|_{\frac{2ks}{1+k(s-1)}}\|S\|_\infty$; injecting inequality (3.1.6) in (3.1.4) we obtain:

$$\|\nabla S\|^k_{\frac{2kq}{q-2}} \leq \|\nabla S\|^k_{2ks} + K_2(M,g)\|\nabla S\|^{k-1}_{2ks} f_{k,s}(\|\nabla S\|_{2ks}) \tag{3.1.7}$$

where

$$f_{k,s}(\|\nabla S\|_{2ks}) = \Big[k(\|\underline{\mathrm{ric}}^-\|_{\frac{s}{s-1}} + n\|R^W\|_{\frac{s}{s-1}})\|\nabla S\|^2_{2ks} + k(2k-1)(\|R^W\|^2_{\frac{2ks}{1+k(s-1)}} + \|V\|^2_{\frac{2ks}{1+k(s-1)}})\Big]^{\frac{1}{2}}$$

is a function depending on the ricci curvature of M, the curvature of the bundle and the potential V.

We then use a Hölder inequality relating the $L^{\frac{2kq}{q-2}}$, L^{2ks} and L^2 norms of $|\nabla S|$, namely:

$$\|\nabla S\|^{k+\tau(s)}_{2ks} \leq \|\nabla S\|^{\tau(s)}_2 \|\nabla S\|^k_{\frac{2kq}{q-2}}$$

where $\tau(s) = \frac{k[q-(q-2)s]}{(ks-1)q}$. Now setting $s = \frac{p}{p-2}$ and $k = \frac{p-2}{2}$ so that $2ks = p$, $\frac{s}{s-1} = \frac{2ks}{1+k(s-1)} = \frac{p}{2}$ and $\tau = \frac{2(p-q)}{q(p-2)}$, we obtain

$$\|\nabla S\|^{k+\tau}_p \leq \|\nabla S\|^\tau_2 \|\nabla S\|^{k-1}_p [\|\nabla S\|_p + K_2(M,g) f(\|\nabla S\|_p)] \tag{3.1.8}$$

where

$$f(\|\nabla S\|_p) = \sqrt{\frac{p-2}{2}(\|\underline{\mathrm{ric}}^-\|_{\frac{p}{2}} + n\|R^W\|_{\frac{p}{2}})\|\nabla S\|^2_p + \frac{(p-2)(p-3)}{2}(\|R^W\|^2_{\frac{p}{2}} + \|V\|^2_{\frac{p}{2}})}$$

This can be reformulated as

$$\|\nabla S\|^\tau_2 \geq \frac{\|\nabla S\|^{1+\tau}_p}{\|\nabla S\|_p + K_2(M,g) f(\|\nabla S\|_p)} \tag{3.1.9}$$

As the right-hand side is a non-decreasing function of $\|\nabla S\|_p$, (3.1.1) gives

$$K_p(M,g)^\tau \|\underline{V}^-\|^{\tau/2}_1 \geq K_p(M,g)^\tau \|\nabla S\|^\tau_2$$
$$\geq \frac{(\sup|S| - \inf|S|)^{1+\tau}}{(\sup|S| - \inf|S|) + K_2(M,g)K_p(M,g) f\left(\frac{\sup|S| - \inf|S|}{K_p(M,g)}\right)}$$

Since $\sup|S| - \inf|S|$ is at most 1, and the denominator of the right-hand side of (3.1.9) is also a non-decreasing function of $\|\nabla S\|_p$, we obtain:

$$K_p(M,g)^\tau \|\underline{V}^-\|_1^{\tau/2} \geq \frac{(\sup|S| - \inf|S|)^{1+\tau}}{1 + K_2(M,g)K_p(M,g)f(\frac{1}{K_p(M,g)})} \tag{3.1.10}$$

Now, setting $\alpha = \tau/(\tau+1)$ and $\|f\|_{L^s_{R,D}} = e^{2B(\frac{n+p}{2})\sqrt{RD}}(\|f\|_s/R)$, (3.1.3) gives

$$\sup\|S\| - \inf\|S\| \leq C_p(n,p)^\alpha \|\underline{V}^-\|_{L^1_{R,D}}^{\alpha/2} \left[1 + K_2(M,g)K_p(M,g)f\left(\frac{1}{K_p(M,g)}\right)\right]^{1-\alpha} \tag{3.1.11}$$

and

$$K_2(M,g)K_p(M,g)f\left(\frac{1}{K_p(M,g)}\right)$$
$$\leq \left[\frac{p-2}{2} C_2(n,p)^2(\|\underline{\mathrm{ric}}^-\|_{L^{p/2}_{R,D}} + n\|R^W\|_{L^{p/2}_{R,D}})\right.$$
$$\left. + C_2(n,p)^2 C_p(n,p)^2 \frac{(p-2)(p-3)}{2}(\|R^W\|^2_{L^{p/2}_{R,D}} + \|V\|^2_{L^{p/2}_{R,D}})\right]^{\frac{1}{2}}$$

Therefore, we get:

$$1 + K_2(M,g)K_p(M,g)f\left(\frac{1}{K_p(M,g)}\right)$$
$$\leq F(n,p)(1 + \|V\|_{L^{p/2}_{R,D}} + \|R^W\|_{L^{p/2}_{R,D}} + \|\underline{\mathrm{ric}}^-\|_{L^{p/2}_{R,D}})$$

where

$$F(n,p) = \sup\left[C_2(n,p)C_p(n,p)\sqrt{\frac{(p-2)(p-3)}{2}}, \frac{p-2}{4} C_2(n,p)^2\right]$$

This gives the result if we set $A(n,p) = C_p(n,p)^\alpha F(n,p)^{1-\alpha}$. □

3.2. *Proof of lemma* 3.1.5

From now on, the rough Laplacian $\nabla^*\nabla$ will be denoted by $\bar{\Delta}$.
We know

$$\|d(|\nabla S|^k)\|_2^2 = \|k|\nabla S|^{k-1}d(|\nabla S|)\|_2^2$$
$$= k^2 \int_M |\nabla S|^{2k-2}|d(|\nabla S|)|^2$$

Using Kato's inequality $|\nabla S|\Delta(|\nabla S|) \le \langle \nabla S \mid \bar{\Delta}\nabla S\rangle$ we obtain

$$\begin{aligned}(2k-1)\int_M |\nabla S|^{2k-2}|d(|\nabla S|)|^2 &= \int_M \langle d(|\nabla S|^{2k-1}) \mid d(|\nabla S|)\rangle \\ &= \int_M |\nabla S|^{2k-1}\Delta(|\nabla S|) \qquad (3.2.1) \\ &\le \int_M |\nabla S|^{2k-2}\langle \nabla S \mid \bar{\Delta}\nabla S\rangle\end{aligned}$$

3.3. *Commutation formula*

In order to use the information we have on $\bar{\Delta}S$, we establish the

3.3.1. LEMMA. *Denoting by $\bar{\Delta}$ the rough Laplacian $\nabla^*\nabla$ acting on sections of W, we have the equality*:

$$(\bar{\Delta}\nabla S - \nabla\bar{\Delta}S)(X) = -\mathrm{Tr}_Y[\nabla_Y(R^W_{(Y,X)}S) + R^W_{(Y,X)}\nabla_Y S] - \nabla_{\mathrm{ric}(X)}S$$

where $\mathrm{Tr}_Y[A(Y, Y, \ldots)]$ *denotes the tensor obtained by tracing at the place indicated by Y with respect to the Riemannian metric g.*

Proof. This follows from

$$\begin{aligned}(\bar{\Delta}\nabla S - \nabla\bar{\Delta}S)(X) &= -\mathrm{Tr}_Y(\nabla_Y\nabla_Y\nabla_X S - \nabla_X\nabla_Y\nabla_Y S) \\ &= -\mathrm{Tr}_Y[\nabla_Y(\nabla_Y\nabla_X - \nabla_X\nabla_Y)S + (\nabla_Y\nabla_X - \nabla_X\nabla_Y)\nabla_Y S] \\ &= -\mathrm{Tr}_Y[\nabla_Y R^W_{(Y,X)}S + R^W_{(Y,X)}\nabla_Y S - \nabla_{R^M_{(Y,X)}Y}S]\end{aligned}$$

and $\mathrm{ric}(X) = -\mathrm{Tr}_Y R^M_{(Y,X)}Y$. □

Using this formula we get

$$\begin{aligned}\int_M |\nabla S|^{2k-2}\langle \nabla S \mid \bar{\Delta}\nabla S\rangle = \int_M |\nabla S|^{2k-2}\,\mathrm{Tr}_X\langle \nabla_X \bar{\Delta} S \mid \nabla_X S\rangle \\ -\int_M |\nabla S|^{2k-2}\,\mathrm{Tr}_{X,Y}\langle \nabla_Y R^W_{(Y,X)} S \mid \nabla_X S\rangle \\ -\int_M |\nabla S|^{2k-2}\,\mathrm{Tr}_{X,Y}\langle R^W_{(Y,X)} \nabla_Y S \mid \nabla_X S\rangle \\ -\int_M |\nabla S|^{2k-2}\,\mathrm{Tr}_X\langle \nabla_{\mathrm{ric}(X)} S \mid \nabla_X S\rangle\end{aligned} \tag{3.3.2}$$

We estimate the four terms (A), (B), (C) and (D) of the right-hand side of (3.3.2) in reverse order:

3.4. *Control of* (D)

$$-\int_M |\nabla S|^{2k-2}\,\mathrm{Tr}_X\langle \nabla_{\mathrm{ric}(X)} S \mid \nabla_X S\rangle$$

Using an orthonormal basis $(e_1, \dots, e_n)$ in which ric is diagonal, we have

$$-\langle \nabla_{\mathrm{ric}\, e_i} S \mid \nabla_{e_i} S\rangle = -\mathrm{ric}(e_i, e_i)\,|\nabla_{e_i} S|^2 \le -\underline{\mathrm{ric}}\,|\nabla_{e_i} S|^2$$

and then

$$-\int_M |\nabla S|^{2k-2}\,\mathrm{Tr}_X\langle \nabla_{\mathrm{ric}(X)} S \mid \nabla_X S\rangle \le -\int_M \underline{\mathrm{ric}}\,|\nabla S|^{2k} \le \int_M \underline{\mathrm{ric}}^-\,|\nabla S|^{2k} \tag{3.4.1}$$

3.5. *Control of* (C)

$$-\int_M |\nabla S|^{2k-2}\,\mathrm{Tr}_{X,Y}\langle R^W_{(Y,X)} \nabla_Y S \mid \nabla_X S\rangle$$

It is straightforward that

$$\left|\int_M |\nabla S|^{2k-2} \operatorname{Tr}_{X,Y}\langle R^W_{(Y,X)}\nabla_Y S \mid \nabla_X S\rangle\right| \leq n \int_M |R^W|\,|\nabla S|^{2k} \tag{3.5.1}$$

as soon as we see

$$|\operatorname{Tr}_X\langle R^W_{(Y,X)}\nabla_Y S \mid \nabla_X S\rangle| = |\langle R^W_{(Y,\cdot)}\nabla_Y S \mid \nabla S\rangle| \leq |R^W|\,|\nabla S|^2$$

The estimation of the two other terms, though not much more difficult, is slightly longer.

3.6. *Control of* (*B*)

$$-\int_M |\nabla S|^{2k-2} \operatorname{Tr}_{X,Y}\langle \nabla_Y R^W_{(Y,X)} S \mid \nabla_X S\rangle$$

We have to get rid of $\nabla_Y R^W_{(Y,X)} S$, for this we use the compatibility of ∇ with $\langle\cdot \mid \cdot\rangle$ to obtain

$$\begin{aligned} -\langle \nabla_Y R^W_{(Y,X)} S \mid |\nabla S|^{2k-2}\nabla_X S\rangle &= -Y.\langle R^W_{(Y,X)} S \mid |\nabla S|^{2k-2}\nabla_X S\rangle \\ &\quad + \langle R^W_{(Y,X)} S \mid \nabla_Y(|\nabla S|^{2k-2}\nabla_X S)\rangle \end{aligned} \tag{3.61}$$

As the integral of a divergence vanishes, only the second term remains:

$$\begin{aligned} &-\int_M \operatorname{Tr}_{X,Y}\langle \nabla_Y R^W_{(Y,X)} S \mid |\nabla S|^{2k-2}\nabla_X S\rangle \\ &= \int_M \operatorname{Tr}_{X,Y}\langle R^W_{(Y,X)} S \mid |\nabla S|^{2k-2}\nabla_Y\nabla_X S\rangle \\ &\quad + \langle R^W_{(Y,X)} S \mid (2k-2)|\nabla S|^{2k-3} d_Y(|\nabla S|)\nabla_X S\rangle \end{aligned}$$

As $R^W_{(Y,X)} S$ is skew-symmetric, we obtain:

$$\begin{aligned} 2\operatorname{Tr}_{X,Y}\langle R^W_{(Y,X)} S \mid \nabla_Y\nabla_X S\rangle &= \operatorname{Tr}_{X,Y}\langle R^W_{(Y,X)} S \mid \nabla_Y\nabla_X S - \nabla_X\nabla_Y S\rangle \\ &= |R^W S|^2 \leq |R^W|^2|S|^2 \end{aligned}$$

which gives the estimate:

$$\begin{aligned}
&-\int_M \mathrm{Tr}_{X,Y}\langle \nabla_Y R^W_{(Y,X)}S \mid |\nabla S|^{2k-2}\nabla_X S\rangle \\
&\le \int_M |\nabla S|^{2k-2}(|R^W|^2|S|^2+(2k-2)|R^W||S||d(|\nabla S|)|) \\
&\le (2k-1)\int_M |\nabla S|^{2k-2}|R^W|^2|S|^2 \\
&\quad +\frac{k-1}{2}\int_M |\nabla S|^{2k-2}|d(|\nabla S|)|^2
\end{aligned} \tag{3.6.3}$$

3.7. *Control of* (A)

$$\int_M |\nabla S|^{2k-2}\,\mathrm{Tr}_X\langle \nabla_X \bar{\Delta} S \mid \nabla_X S\rangle$$

As in (3.6.1), in order to get rid of $\nabla_X \bar{\Delta} S$, we use the compatibility of ∇ with $\langle . \mid . \rangle$ to obtain

$$\begin{aligned}
\langle \nabla_X \bar{\Delta} S \mid |\nabla S|^{2k-2}\nabla_X S\rangle &= X.\langle \bar{\Delta} S \mid |\nabla S|^{2k-2}\nabla_X S\rangle \\
&\quad -\langle \bar{\Delta} S \mid \nabla_X(|\nabla S|^{2k-2}\nabla_X S)\rangle
\end{aligned} \tag{3.7.1}$$

Using again the fact that the integral of a divergence vanishes, we have:

$$\begin{aligned}
&\int_M \mathrm{Tr}_X\langle \nabla_X \bar{\Delta} S \mid |\nabla S|^{2k-2}\nabla_X S\rangle \\
&= -\int_M \mathrm{Tr}_X\langle \bar{\Delta} S \mid |\nabla S|^{2k-2}\nabla_X\nabla_X S\rangle \\
&\quad +\langle \bar{\Delta} S \mid (2k-2)|\nabla S|^{2k-3}d_X(|\nabla S|)\nabla_X S\rangle \\
&= \int_M |\nabla S|^{2k-2}|\bar{\Delta} S|^2-(2k-2)|\nabla S|^{2k-3}\langle \bar{\Delta} S \mid \nabla_{\nabla(|\nabla S|)}S\rangle
\end{aligned}$$

We thus obtain the estimate:

$$\begin{aligned}
&\int_M \mathrm{Tr}_X \langle \nabla_X \bar{\Delta} S \mid |\nabla S|^{2k-2} \nabla_X S \rangle \\
&\quad \le \int_M |\nabla S|^{2k-2} (|\bar{\Delta} S|^2 + (2k-2)|\bar{\Delta} S|\, |d(|\nabla S|)|) \\
&\quad \le (2k-1) \int_M |\nabla S|^{2k-2} |\bar{\Delta} S|^2 \\
&\qquad + \frac{k-1}{2} \int_M |\nabla S|^{2k-2} |d(|\nabla S|)|^2
\end{aligned} \tag{3.7.2}$$

3.8. *End of the proof of lemma* 3.15

We sum up the four estimates of (A), (B), (C) and (D) respectively given in (3.7.2), (3.6.2), (3.5.1) and (3.4.1). Replacing them in (3.3.2) we obtain:

$$\begin{aligned}
\int_M |\nabla S|^{2k-2} \langle \bar{\Delta} \nabla S \mid \nabla S \rangle \le & \int_M (\underline{\mathrm{ric}}^- + n|R^W|) |\nabla S|^{2k} \\
& + (2k-1) \int_M |\nabla S|^{2k-2} (|R^W|^2 |S|^2 + |\bar{\Delta} S|^2) \\
& + (k-1) \int_M |\nabla S|^{2k-2} |d(|\nabla S|)|^2
\end{aligned} \tag{3.8.1}$$

The inequality (3.2.1)

$$(2k-1) \int_M |\nabla S|^{2k-2} |d(|\nabla S|)|^2 \le \int_M |\nabla S|^{2k-2} \langle \nabla S \mid \bar{\Delta} \nabla S \rangle$$

gives

$$\begin{aligned}
k \int_M |\nabla S|^{2k-2} |d(|\nabla S|)|^2 \le & \int_M (\underline{\mathrm{ric}}^- + n|R^W|) |\nabla S|^{2k} \\
& + (2k-1) \int_M |\nabla S|^{2k-2} (|R^W|^2 |S|^2 + |\bar{\Delta} S|^2)
\end{aligned}$$

Now, if we use the two Hölder inequalities

$$\int_M f|\nabla S|^{2k} \leq \|f^+\|_{\frac{s}{s-1}} \|\nabla S\|_{2ks}^{2k}$$

$$\int_M g^2|S|^2|\nabla S|^{2k-2} \leq \|g\|_{\frac{2s'}{s'-1}}^2 \|S\|_\infty^2 \|\nabla S\|_{(2k-2)s'}^{2k-2}$$

where we set s' such that $(2k-2)s' = 2ks$, (hence $\frac{2s'}{s'-1} = \frac{2ks}{1+k(s-1)}$), we obtain:

$$\begin{aligned}\frac{1}{k}\|d(|\nabla S|^k)\|_2^2 &= k\int_M |\nabla S|^{2k-2}|d|\nabla S||^2 \\ &\leq (\|\underline{\mathrm{ric}}^-\|_{\frac{s}{s-1}} + n\|R^W\|_{\frac{s}{s-1}})\|\nabla S\|_{2ks}^{2k} \\ &\quad + (2k-1)(\|R^W\|_{\frac{2ks}{1+k(s-1)}}^2 \|S\|_\infty^2 + \|\bar{\Delta} S\|_{\frac{2ks}{1+k(s-1)}}^2)\|\nabla S\|_{2ks}^{2k-2} \qquad \square\end{aligned}$$

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