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## Lie algebras and coverings

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*Meinem Lehrer Peter Gabriel gewidmet*

### 1. Introduction

1.1. Let  $A$  be an associative unitary finite dimensional  $\mathbb{C}$ -algebra which is representation finite. This means that the number of isomorphism classes of indecomposable finite dimensional  $A$ -left modules is finite. Let us fix a set  $\mathcal{J}$  of representatives for these isomorphism classes.

We showed in [Rie] that the free  $\mathbb{Z}$ -module

$$L(A) = \bigoplus_{A \in \mathcal{J}} \mathbb{Z}v_A$$

generated by the symbols

$$\{v_A : A \in \mathcal{J}\}$$

can be made into a  $\mathbb{Z}$ -Lie algebra in the following way: set

$$[v_A, v_B] = \sum_{X \in \mathcal{J}} (\gamma_{A,B}^X - \gamma_{B,A}^X)v_X,$$

where

$$\gamma_{A,B}^X = \chi(V_A(A, B; X))$$

is the Euler-Poincaré characteristic of the algebraic variety

$$V_A(A, B; X) = \{0 \subseteq Y \subseteq X : Y \text{ is a } A\text{-submodule of } X \text{ isomorphic to } A \text{ with quotient module } X/Y \text{ isomorphic to } B\}.$$

This is the complex version of Ringel's construction of Lie algebras via Hall algebras over finite fields [Rin].

The construction of  $L(A)$  carries over easily to the case where  $A$  is a locally representation finite  $\mathbb{C}$ -category. We will list the most important definitions and

facts about locally representation finite categories and their coverings in chapter 3; the references for these results are [BG] and [Ga].

1.2. If the representation finite algebra – or more generally the locally representation finite  $\mathbb{C}$ -category – is simply connected [BG], the Lie algebra  $L(\mathcal{A})$  has a particularly simple structure. Indeed, we proved in [Rie] that in this case one of the numbers

$$\gamma_{A,B}^X \quad \text{and} \quad \gamma_{B,A}^X$$

is zero for any choice of  $A, B, X$  in  $\mathcal{S}$  and that for fixed  $A$  and  $B$  there is at most one  $X$  for which

$$\gamma_{A,B}^X \neq 0.$$

To any locally representation finite  $\mathbb{C}$ -category  $\mathcal{A}$  one can associate a locally representation finite one which is simply connected: its universal cover  $\tilde{\mathcal{A}}$  ([BG], [Ga]). The reason why we consider  $\mathbb{C}$ -categories instead of  $\mathbb{C}$ -algebras in this paper is that  $\tilde{\mathcal{A}}$  is rarely a  $\mathbb{C}$ -algebra.

It is tempting to try and use the simple structure of  $L(\tilde{\mathcal{A}})$  in order to compute  $L(\mathcal{A})$ . The aim of this paper is to show that this is actually possible: if we choose a set  $\tilde{\mathcal{S}}$  of representatives for the indecomposable  $\tilde{\mathcal{A}}$ -modules which is stable under the fundamental group  $G$  (see chapter 3), the set of  $G$ -orbits  $\{\bar{A} = GA : A \in \tilde{\mathcal{S}}\}$  is a set  $\mathcal{S}$  of representatives for the indecomposable  $\mathcal{A}$ -modules. Our goal is to prove:

$$\chi(V_{\mathcal{A}}(\bar{A}, \bar{B}; \bar{X})) = \sum_{g,h \in G} \chi(V_{\tilde{\mathcal{A}}}(A, g(B); h(X)))$$

for any  $A, B, X \in \tilde{\mathcal{S}}$ . Thus the structure constants for  $L(\mathcal{A})$  are sums of – more easily accessible – structure constants of  $L(\tilde{\mathcal{A}})$ .

In fact, we will define an “orbit” Lie algebra  $L/G$  in chapter 2 for an appropriate action of a group  $G$  on a Lie algebra  $L$  and show in chapter 4 that the action of the fundamental group  $G$  of  $\mathcal{A}$  on  $L(\tilde{\mathcal{A}})$  is appropriate. Our aim is then:

**THEOREM.** *Let  $\mathcal{A}$  be a locally representation finite  $\mathbb{C}$ -category with universal cover  $\tilde{\mathcal{A}}$  and fundamental group  $G$ . Then  $L(\mathcal{A})$  is isomorphic to  $L(\tilde{\mathcal{A}})/G$ .*

1.3. As a first application, let us prove again that for  $\mathcal{A} = \mathbb{C}[T]/(T^n)$  the bracket on  $L(\mathcal{A})$  is trivial [Rie]. The universal cover  $\tilde{\mathcal{A}}$  of  $\mathcal{A}$  is given by the quiver

$$\cdots - 1 \xrightarrow{\alpha_{-1}} 0 \xrightarrow{\alpha_0} 1 \xrightarrow{\alpha_1} 2 \cdots,$$

having  $\mathbb{Z}$  as its vertex set and containing an arrow  $\alpha_i : i \rightarrow i + 1$  for  $i \in \mathbb{Z}$ , and the ideal of relations generated by all paths of length  $n$ . The fundamental group  $G = \mathbb{Z}$  is generated by the shift  $i \mapsto i + 1$ . For  $\tilde{\mathcal{F}}$  we choose the set

$$\{(i, r) : i \in \mathbb{Z}, 1 \leq r \leq n\},$$

where  $(i, r)$  is “the indecomposable with top  $i$  of length  $r$ ” defined by

$$(i, r)(j) = \begin{cases} \mathbb{C} & \text{for } i \leq j < i + r, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$(i, r)(\alpha_j) = \begin{cases} \text{id}_{\mathbb{C}} & \text{for } i \leq j < i + r - 1, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that

$$V_{\tilde{\mathcal{A}}}((i, r), (j, s); (k, t)) = \begin{cases} 1 \text{ point} & \text{for } i = j + s, k = j, t = r + s, \\ \emptyset & \text{otherwise,} \end{cases}$$

for any triple of indecomposables. Therefore

$$\chi(V_{\tilde{\mathcal{A}}}((j + s, r), (j, s); (j, r + s))) = 1 = \chi(V_{\tilde{\mathcal{A}}}((i + r, s), (i, r); (i, r + s)))$$

give the only non-trivial contributions to the bracket  $[v_A, v_B]$  with  $A = (i, r)$ ,  $B = (j, s)$ .

As a second example, consider the quotient  $\mathcal{A}$  of the algebra of the quiver  $\bullet \xrightarrow{\alpha} \bullet \circlearrowleft \beta$  by the ideal generated by  $\beta^3$ . In this case the quiver of  $\tilde{\mathcal{A}}$  is:

$$\begin{array}{ccccccc} \cdots & -1 & \xrightarrow{\beta_{-1}} & 0 & \xrightarrow{\beta_0} & 1 & \xrightarrow{\beta_1} & 2 & \cdots \\ & \uparrow \alpha_{-1} & & \uparrow \alpha_0 & & \uparrow \alpha_1 & & \uparrow \alpha_2 & \\ \cdots & -1' & & 0' & & 1' & & 2' & \cdots \end{array}$$

and the ideal of relations is generated by  $\{\beta_{i+2}\beta_{i+1}\beta_i : i \in \mathbb{Z}\}$ . The fundamental group is generated by the shift again. Let us consider the indecomposables  $A, B, X$



given by

$$\dim A(x) = \begin{cases} 1 & \text{for } x = 1, 2, 3, \\ 0 & \text{otherwise,} \end{cases}$$

$$\dim B(x) = \begin{cases} 1 & \text{for } x = 0, 1, 2, 1', 2', \\ 0 & \text{otherwise,} \end{cases}$$

$$\dim X(x) = \begin{cases} 2 & \text{for } x = 1, 2, \\ 1 & \text{for } x = 0, 3, 1', 2', \\ 0 & \text{otherwise,} \end{cases}$$

and

$$X(\beta_0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad X(\beta_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X(\beta_2) = (0 \ 1),$$

$$X(\alpha_1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad X(\alpha_2) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The variety of embeddings of  $\bar{A}$  into  $\bar{X}$  is 6-dimensional, and possible quotients are quite hard to determine. Over  $\tilde{\Lambda}$ , however, it is easy to see that the only way to embed  $A$  into a translate of  $X$  with quotient a translate of  $B$  is to choose a map  $f: A \rightarrow X$  of the form

$$f(1) = \begin{pmatrix} \lambda \\ \mu \end{pmatrix}, \quad f(2) = \begin{pmatrix} \lambda \\ \mu \end{pmatrix}, \quad f(3) = \mu$$

with  $\mu \neq 0$ . The quotient is isomorphic to  $B$  if and only if  $\lambda \neq 0, \mu$ . Hence

$$\chi(V_A(\bar{A}, \bar{B}; \bar{X})) = -1.$$

## 2. The “orbit” Lie algebra

2.1. Let  $L$  be a  $\mathbb{Z}$ -Lie algebra which is generated (as a  $\mathbb{Z}$ -module) by some basis  $\mathcal{B}$ :

$$L = \bigoplus_{b \in \mathcal{B}} \mathbb{Z}b.$$

Suppose the group  $G$  acts on  $L$  by Lie algebra automorphisms in such a way that it permutes the elements of  $\mathcal{B}$  and that the following condition is satisfied:

$$\forall b, c \in \mathcal{B} : \#\{g \in G : [b, g(c)] \neq 0\} < \infty. \tag{*}$$

This condition is obviously empty in case  $G$  is finite. The main example to have in mind here, however, is the fundamental group  $G$  of a locally representation finite  $\mathbb{C}$ -category  $\mathcal{A}$  acting on  $\text{ind } \tilde{\mathcal{A}}$  for a  $G$ -stable set of representatives, and this is a free group [BG].

Set

$$\bar{\mathcal{B}} = \{\bar{b} = G \cdot b : b \in \mathcal{B}\}$$

and

$$L/G = \bigoplus_{\bar{b} \in \bar{\mathcal{B}}} \mathbb{Z}\bar{b}.$$

Let

$$\bar{\gamma} : L \rightarrow L/G$$

be the  $\mathbb{Z}$ -linear map which takes  $b$  to  $\bar{b}$  for  $b \in \mathcal{B}$ .

The following result is easy to prove:

**PROPOSITION.** *The bracket*

$$[\bar{b}, \bar{c}] = \overline{\sum_{g \in G} [b, g(c)]}, \quad \bar{b}, \bar{c} \in \bar{\mathcal{B}},$$

*defines a Lie algebra structure on  $L/G$ .*

Note that the map  $\bar{\gamma}$  is *not* a Lie algebra homomorphism in general.

2.2. Comparing the structure constants of  $L/G$  with those of  $L$ , we find: if

$$[b, c] = \sum_{d \in \mathcal{B}} \gamma_{b,c}^d d$$

and

$$[\bar{b}, \bar{c}] = \sum_{\bar{d} \in \bar{\mathcal{B}}} \bar{\gamma}_{\bar{b},\bar{c}}^{\bar{d}} \bar{d},$$

then

$$\gamma_{\bar{b}, \bar{c}}^{\bar{d}} = \sum_{d' \in \bar{d}} \sum_{g \in G} \gamma_{b, g(c)}^{d'}.$$

In case the action of  $G$  on  $\mathcal{B}$  is free as it is in the case we are interested in this formula becomes

$$\gamma_{\bar{b}, \bar{c}}^{\bar{d}} = \sum_{g, h \in G} \gamma_{b, g(c)}^{h(d)}.$$

2.3. Let  $G$  be a group acting on the Lie algebra

$$L = \bigoplus_{b \in \mathcal{B}} \mathbb{Z}b$$

in such a way that the hypotheses of 2.1 are satisfied. If  $H \triangleleft G$  is a normal subgroup, they are satisfied for the action of  $H$  on  $L$  as well, so that we can consider the Lie algebra

$$L/H = \bigoplus_{\bar{b} \in \bar{\mathcal{B}}} \mathbb{Z}\bar{b}$$

with

$$\bar{\mathcal{B}} = \{\bar{b} = Hb : b \in \mathcal{B}\}.$$

Extending the action given by

$$\bar{g}(\bar{b}) = \overline{g(b)}$$

of  $\bar{G} = G/H$  on  $\bar{\mathcal{B}}$  by  $\mathbb{Z}$ -linearity, we obtain an action of  $\bar{G}$  on  $L/H$ . It is easy to see that it satisfies again the hypotheses of 2.1 and that the following proposition holds:

**PROPOSITION.** *The  $\mathbb{Z}$ -linear map  $L/G \rightarrow (L/H)/(G/H)$  sending the basis element  $Gb$  to  $(G/H)\bar{b}$  is an isomorphism of Lie algebras.*

### 3. Locally representation finite categories and coverings

The references for this chapter are [BG], [Ga].

3.1. We begin by recalling some definitions:

A  $\mathbb{C}$ -category  $\Lambda$  is locally bounded if the following conditions are satisfied:

- (i)  $\Lambda(x, x)$  is a local  $\mathbb{C}$ -algebra for all objects  $x$  of  $\Lambda$ .
- (ii) Distinct objects of  $\Lambda$  are not isomorphic.
- (iii) For all objects  $x$  we have

$$\sum_y \dim_{\mathbb{C}} \Lambda(x, y) < \infty,$$

$$\sum_y \dim_{\mathbb{C}} \Lambda(y, x) < \infty,$$

where  $y$  ranges over the objects of  $\Lambda$ .

A finite dimensional  $\Lambda$ -left module is a covariant functor  $B : \Lambda \rightarrow \text{mod } \mathbb{C}$  with

$$\sum_x \dim_{\mathbb{C}} B(x) < \infty,$$

where  $x$  ranges over the objects of  $\Lambda$ .

We denote by  $\text{mod } \Lambda$  the category of finite dimensional  $\Lambda$ -modules and by  $\text{ind } \Lambda$  the full subcategory whose objects are a fixed set  $\mathcal{I}$  of representatives for the isomorphism classes of indecomposables in  $\text{mod } \Lambda$ .

A locally bounded  $\mathbb{C}$ -category  $\Lambda$  is locally representation finite if, for every object  $x$  of  $\Lambda$ , the number of indecomposables  $B$  in  $\mathcal{I}$  with  $B(x) \neq 0$  is finite.

$\mathbb{C}$ -algebras  $\Lambda$  which are sober, i.e. with  $\Lambda/\text{rad } \Lambda \xrightarrow{\sim} \mathbb{C} \times \cdots \times \mathbb{C}$ , correspond to locally bounded  $\mathbb{C}$ -categories with finitely many objects and representation finite  $\mathbb{C}$ -algebras to locally representation finite  $\mathbb{C}$ -categories with finitely many objects.

3.2. Let  $\tilde{\Lambda}$  be the universal cover of  $\Lambda$ , and choose a set  $\tilde{\mathcal{I}}$  of representatives for the indecomposable  $\tilde{\Lambda}$ -modules which is stable under the action of the fundamental group  $G$  of  $\Lambda$  on  $\text{mod } \tilde{\Lambda}$ . Then  $G$  acts on the full subcategory  $\text{ind } \tilde{\Lambda}$  of  $\text{mod } \tilde{\Lambda}$  whose objects are the elements of  $\tilde{\mathcal{I}}$  by  $\mathbb{C}$ -linear automorphisms. Moreover, we have

$$g(B) \neq B \quad \text{for every } B \text{ in } \tilde{\mathcal{I}} \text{ and every } g \neq 1 \text{ in } G$$

and

$$\#\{g \in G : \text{Hom}_{\tilde{\mathcal{A}}}(A, g(B)) \neq 0\} < \infty \quad \text{for every pair } A, B \text{ in } \tilde{\mathcal{F}}.$$

Under these circumstances  $\text{ind } \tilde{\mathcal{A}}$  has a quotient modulo  $G$ : its objects are the  $G$ -orbits of objects in  $\tilde{\mathcal{F}}$ , and the morphisms from the orbit of  $A$  to the orbit of  $B$  are families  $(f_{g,h})_{g,h \in G}$ ,

$$f_{g,h} : g(A) \rightarrow h(B)$$

with

$$l(f_{g,h}) = f_{lg, lh}$$

for all  $g, h, l$  in  $G$ .

Similarly, there exists a quotient  $\tilde{\mathcal{A}}/G$ , which is locally representation finite. The category  $(\text{ind } \tilde{\mathcal{A}})/G$  is isomorphic to  $\text{ind } (\tilde{\mathcal{A}}/G)$ , which is in turn isomorphic to the so called mesh category  $\mathbb{C}(\Gamma_{\mathcal{A}})$  associated with the Auslander-Reiten quiver of  $\mathcal{A}$ . So  $\tilde{\mathcal{A}}/G$  is the “standard form” of  $\mathcal{A}$ . But by [BGRS] non-standard algebras can exist only over ground fields of characteristic 2. Thus  $\tilde{\mathcal{A}}/G$  is isomorphic to  $\mathcal{A}$  and  $(\text{ind } \tilde{\mathcal{A}})/G$  to  $\text{ind } \mathcal{A}$ . We fix the set  $\mathcal{S}$  of  $G$ -orbits in  $\tilde{\mathcal{F}}$  as a set of representatives of the isomorphism classes of indecomposable  $\mathcal{A}$ -modules and identify  $\text{ind } \mathcal{A}$  with  $(\text{ind } \tilde{\mathcal{A}})/G$ .

### 3.3. The $\mathbb{C}$ -linear functor

$$F : \text{ind } \tilde{\mathcal{A}} \rightarrow (\text{ind } \tilde{\mathcal{A}})/G = \text{ind } \mathcal{A}$$

defined by

$$F(B) = G \cdot B$$

for  $B$  in  $\tilde{\mathcal{F}}$  and by associating to  $f : A \rightarrow B$ ,  $A, B \in \tilde{\mathcal{F}}$ , the family

$$Ff = (f'_{g,h}), \quad g, h \in G$$

with

$$f'_{g,h} = \begin{cases} g(f) & g = h \\ 0 & g \neq h \end{cases}$$

is a covering functor. This means that, for all  $A, B$  in  $\tilde{\mathcal{J}}$ ,  $F$  induces  $\mathbb{C}$ -linear isomorphisms

$$\bigoplus_{g \in G} \text{Hom}_{\tilde{\Lambda}}(A, g(B)) \xrightarrow{\sim} \text{Hom}_{\Lambda}(FA, FB),$$

$$\bigoplus_{g \in G} \text{Hom}_{\tilde{\Lambda}}(g(A), B) \rightarrow \text{Hom}_{\Lambda}(FA, FB).$$

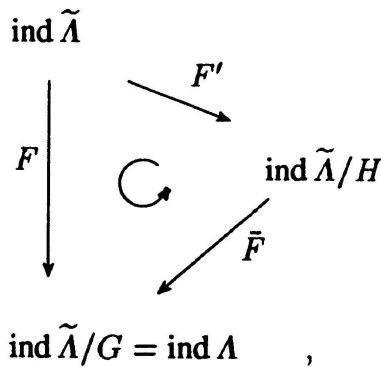
We will need also that  $F$  induces  $\mathbb{C}$ -linear isomorphisms:

$$\bigoplus_{g \in G} \text{Ext}_{\tilde{\Lambda}}^1(A, g(B)) \xrightarrow{\sim} \text{Ext}_{\Lambda}^1(FA, FB),$$

$$\bigoplus_{g \in G} \text{Ext}_{\tilde{\Lambda}}^1(g(A), B) \xrightarrow{\sim} \text{Ext}_{\Lambda}^1(FA, FB),$$

for all  $A, B$  in  $\tilde{\mathcal{J}}$ . This is an easy consequence of the isomorphisms for Hom-sets and the fact that  $F$  is an exact functor preserving projectivity.

3.4. If  $H \triangleleft G$  is a normal subgroup, it is the fundamental group of  $\tilde{\Lambda}/H$ , and again we identify  $(\text{ind } \tilde{\Lambda})/H$  with  $\text{ind } (\tilde{\Lambda}/H)$  and note this quotient simply  $\text{ind } \tilde{\Lambda}/H$ . There is a commutative triangle of covering functors:



where  $\bar{F}$  sends  $HB$  to  $GB$  for  $B$  in  $\tilde{\mathcal{J}}$  and a morphism

$$(f_{h_1, h_2} : h_1(A) \rightarrow h_1(B))_{h_1, h_2 \in H}$$

to

$$(f'_{g_1, g_2} : g_1(A) \rightarrow g_2(B))_{g_1, g_2 \in G}$$

with

$$f'_{g_1, g_2} = \begin{cases} g_2(f_{g_2^{-1}g_1, 1}) & \text{if } g_2^{-1}g_1 \in H, \\ 0 & \text{if not.} \end{cases}$$

As  $\bar{F}$  is a covering functor,  $\bar{F}$  induces  $\mathbb{C}$ -linear isomorphisms:

$$\bigoplus_{\bar{g} \in G/H} \text{Ext}_{\bar{\Lambda}/H}^1(F'A, \bar{g}(F'B)) \xrightarrow{\sim} \text{Ext}_{\Lambda}^1(FA, FB),$$

$$\bigoplus_{\bar{g} \in G/H} \text{Ext}_{\bar{\Lambda}/H}^1(\bar{g}(F'A), F'B) \xrightarrow{\sim} \text{Ext}_{\Lambda}^1(FA, FB),$$

for any two elements  $A, B$  in  $\tilde{\mathcal{J}}$ .

#### 4. The theorem – and its proof in some cases

4.1. Let  $\mathcal{A}$  be a locally representation finite  $\mathbb{C}$ -category with universal cover  $\tilde{\Lambda}$  and fundamental group  $G$ . Fix a  $G$ -stable set  $\tilde{\mathcal{J}}$  of representatives for the isomorphism classes of indecomposable  $\tilde{\Lambda}$ -modules and identify  $\text{ind } \mathcal{A}$  with  $\text{ind } \tilde{\Lambda}/G$ .

Extend the action of  $G$  on  $\tilde{\mathcal{J}}$  to a  $\mathbb{Z}$ -linear action of  $G$  on  $L(\tilde{\Lambda})$ . Note that, for  $A, B, X \in \tilde{\mathcal{J}}$  and  $g \in G$ , the varieties  $V_{\tilde{\Lambda}}(A, B; X)$  and  $V_{\tilde{\Lambda}}(g(A), g(B); g(X))$  are isomorphic and hence homeomorphic. Therefore  $G$  acts by Lie algebra automorphisms.

Moreover, the sets

$$\{h \in G : \text{Hom}_{\tilde{\Lambda}}(A, h(X)) \neq 0\}$$

and, for any  $h \in G$

$$\{g \in G : \text{Hom}_{\tilde{\Lambda}}(h(X), g(B)) \neq 0\}$$

are finite for any  $A, B, X$  by 3.2. This implies that the action of  $G$  on  $L(\tilde{\Lambda})$  satisfies the condition (\*) of 2.1 as well.

Now the statement of our theorem makes sense at least. In fact, both Lie algebras  $L(\mathcal{A})$  and  $L(\tilde{\Lambda})/G$  have as a basis the set of  $G$ -orbits in  $\tilde{\mathcal{J}}$ . The isomorphism is the identity on this basis.

4.2. We recall from [Rie] that there is another way to compute the structure constants of  $L(\mathcal{A})$ , which is more adapted to coverings: let  $A, B$  and  $X$  be indecomposable  $\mathcal{A}$ -modules. Then the following Euler-Poincaré characteristics coincide:

$$\chi(V_{\mathcal{A}}(B, A; X)) = \chi(\text{Ext}_{\mathcal{A}}^1(A, B)_X/\mathbb{C}^*).$$

The variety on the left hand side has been introduced in 1.1. As to the right hand side,  $\text{Ext}_{\mathcal{A}}^1(A, B)_X$  is the algebraic subset of equivalence classes of short exact

sequences in the  $\mathbb{C}$ -vector space  $\text{Ext}_{\Lambda}^1(A, B)$  whose middle term is isomorphic to  $X$ . It is stable under the action of  $\mathbb{C}^*$  by homotheties on  $\text{Ext}_{\Lambda}^1(A, B)$ .

**4.3. PROPOSITION.** *Let  $\Lambda$  be a locally representation finite  $\mathbb{C}$ -category with universal cover  $\tilde{\Lambda}$  and fundamental group  $G$ , and suppose that the set*

$$\{g \in G : \text{Ext}_{\Lambda}^1(A, g(B)) \neq 0\}$$

*has at most one element for any pair  $A, B$  in  $\tilde{\mathcal{F}}$ . Then  $L(\Lambda)$  is isomorphic to  $L(\tilde{\Lambda})/G$ .*

*Proof.* Let

$$F : \text{ind } \tilde{\Lambda} \rightarrow \text{ind } \tilde{\Lambda}/G = \text{ind } \Lambda$$

be the orbit covering functor. Choose  $A$  and  $B$  in  $\tilde{\mathcal{F}}$  in such a way that

$$\text{Ext}_{\Lambda}^1(FA, FB) \neq 0.$$

According to our hypothesis and 3.3 there is a unique element  $g \in G$  such that  $\text{Ext}_{\tilde{\Lambda}}^1(A, g(B)) \neq 0$ , and  $F$  induces a  $\mathbb{C}$ -linear isomorphism

$$\text{Ext}_{\tilde{\Lambda}}^1(A, g(B)) \xrightarrow{\sim} \text{Ext}_{\Lambda}^1(FA, FB).$$

Clearly the inverse image of  $\text{Ext}_{\Lambda}^1(FA, FB)_{FX}$  under this isomorphism is the disjoint union

$$\dot{\bigcup}_{h \in G} \text{Ext}_{\tilde{\Lambda}}^1(A, g(B))_{h(X)}$$

for any  $X$  in  $\tilde{\mathcal{F}}$ .

As the characteristic  $\chi(\mathcal{C})$  of a finite disjoint union  $\mathcal{C} = \dot{\bigcup} \mathcal{C}_i$  of constructible subsets of a variety  $\mathcal{C}$  is the sum  $\sum \chi(\mathcal{C}_i)$ , we conclude that

$$\begin{aligned} \chi(\text{Ext}_{\Lambda}^1(FA, FB)_{FX}/\mathbb{C}^*) &= \sum_{h \in G} \chi(\text{Ext}_{\tilde{\Lambda}}^1(A, g(B))_{h(X)}/\mathbb{C}^*) \\ &= \sum_{g, h \in G} \chi(\text{Ext}_{\tilde{\Lambda}}^1(A, g(B))_{h(X)}/\mathbb{C}^*). \end{aligned}$$



*Remarks.*

(i) As  $\tilde{\Lambda}$  is simply connected, there is in fact at most one  $h \in G$  with  $\text{Ext}_{\tilde{\Lambda}}^1(A, g(B))_{h(X)} \neq 0$ , as indecomposables are determined by their composition factors [Ha].

(ii) Note also that in case

$$\{g \in G : \text{Ext}_{\tilde{\Lambda}}^1(A, g(B)) \neq 0\}$$

contains at most one element for just a pair  $A, B$  in  $\tilde{\mathcal{F}}$ , it is still true that

$$\chi(\text{Ext}_{\tilde{\Lambda}}^1(FA, FB)_{FX}/\mathbb{C}^*) = \sum_{g, h \in G} \chi(\text{Ext}_{\tilde{\Lambda}}^1(A, g(B))_{h(X)}/\mathbb{C}^*).$$

## 5. $\mathbb{C}^*$ -actions

We fix a locally representation finite  $\mathbb{C}$ -category  $\Lambda$  with universal cover  $\tilde{\Lambda}$  and fundamental group  $G$  as well as a  $G$ -stable set  $\tilde{\mathcal{F}}$  of representatives for the isomorphism classes of indecomposable  $\tilde{\Lambda}$ -modules. We denote by  $F$  the orbit covering functor

$$F : \text{ind } \tilde{\Lambda} \rightarrow \text{ind } \tilde{\Lambda}/G = \text{ind } \Lambda.$$

5.1. Any map  $\lambda : G \rightarrow \mathbb{Z}$  gives rise to a  $\mathbb{C}$ -linear  $\mathbb{C}^*$ -action on the  $\mathbb{C}$ -vector space

$$\bigoplus_{g \in G} \text{Ext}_{\tilde{\Lambda}}^1(A, g(B))$$

by

$$t \cdot (\varepsilon_g)_{g \in G} = (t^{\lambda(g)} \varepsilon_g)_{g \in G}$$

for  $t \in \mathbb{C}^*$ ,  $\varepsilon_g \in \text{Ext}_{\tilde{\Lambda}}^1(A, g(B))$  and for any  $A, B \in \tilde{\mathcal{F}}$ . A line through the origin in this vector space is stable under  $\mathbb{C}^*$  if and only if there exists an integer  $n$  such that the line lies in

$$\bigoplus_{\substack{g \in G \\ \lambda(g) = n}} \text{Ext}_{\tilde{\Lambda}}^1(A, g(B)).$$

Using the  $\mathbb{C}$ -isomorphism

$$\bigoplus_{g \in G} \text{Ext}_{\tilde{\Lambda}}^1(A, g(B)) \rightarrow \text{Ext}_{\tilde{\Lambda}}^1(FA, FB),$$

induced by  $F$  (see 3.3) we obtain a  $\mathbb{C}^*$ -action on

$$\text{Ext}_A^1 (FA, FB) / \mathbb{C}^*,$$

whose fixed points are the disjoint union

$$\dot{\bigcup}_{n \in \mathbb{Z}} \left[ F \left[ \bigoplus_{\substack{g \in G \\ \lambda(g) = n}} \text{Ext}_A^1 (A, g(B)) / \mathbb{C}^* \right] \right]$$

Recall that the Euler-Poincaré characteristic of a variety  $Z$  admitting an algebraic action of  $\mathbb{C}^*$  equals the characteristic of the fixed point set  $Z^{\mathbb{C}^*}$ .

Therefore our theorem would be proved if we could exhibit a map  $\lambda : G \rightarrow \mathbb{Z}$  satisfying:

- (i) the middle term of a short exact sequence in  $\text{Ext}_A^1 (FA, FB)$  changes only up to isomorphism under the  $\mathbb{C}^*$ -action defined by  $\lambda$ .
- (ii) for each integer  $n$  there is at most one  $g \in G$  with  $\lambda(g) = n$ .

Indeed, such a  $\lambda$  would give rise to a  $\mathbb{C}^*$ -action stabilizing  $\text{Ext}_A^1 (FA, FB)_{FX}$  for any  $X \in \tilde{\mathcal{F}}$  by (i), and we could write

$$\begin{aligned} \chi(\text{Ext}_A^1 (FA, FB)_{FX} / \mathbb{C}^*) &= \chi((\text{Ext}_A^1 (FA, FB)_{FX} / \mathbb{C}^*)^{\mathbb{C}^*}) \\ &= \chi \left[ \dot{\bigcup}_{\substack{n \in \mathbb{Z} \\ \exists g_n : \lambda(g_n) = n}} \dot{\bigcup}_{h \in G} \text{Ext}_A^1 (A, g_n(B))_{h(X)} / \mathbb{C}^* \right] \\ &= \sum_{g, h \in G} \chi(\text{Ext}_A^1 (A, g(B))_{h(X)} / \mathbb{C}^*). \end{aligned}$$

Here we used again that the inverse image of  $\text{Ext}_A^1 (FA, FB)_{FX}$  in  $\text{Ext}_A^1 (A, g_n(B))$  is the disjoint union

$$\dot{\bigcup}_{h \in G} \text{Ext}_A^1 (A, g_n(B))_{h(X)}.$$

Unfortunately, such  $\lambda$ 's need not exist. We will concentrate first on the condition (i), which is indispensable.

5.2. For  $A, B, U \in \tilde{\mathcal{F}}$  we consider the pull-back map

$$\pi : \text{Ext}_A^1 (FA, FB) \times \text{Hom}_A (FU, FA) \rightarrow \text{Ext}_A^1 (FU, FB)$$

which associates to an exact sequence

$$\varepsilon : 0 \rightarrow FB \rightarrow Z \rightarrow FA \rightarrow 0$$

with  $Z$  in  $\text{mod } \Lambda$  and a homomorphism  $f : FB \rightarrow FU$  the pull-back sequence  $\pi(\varepsilon, f)$  in  $\text{Ext}_\Lambda^1(FU, FB)$ :

$$\begin{array}{ccccccccc} \varepsilon & : & 0 & \rightarrow & FB & \rightarrow & Z & \rightarrow & FA & \rightarrow & 0 \\ & & & & & & \parallel & & \uparrow & & \uparrow f \\ \pi(\varepsilon, f) & : & 0 & \rightarrow & FB & \rightarrow & Z' & \rightarrow & FU & \rightarrow & 0. \end{array}$$

By Auslander's criterion [AR] two  $\Lambda$ -modules  $Z_1$  and  $Z_2$  are isomorphic if and only if

$$\dim_{\mathbb{C}} \text{Hom}_\Lambda(FU, Z_1) = \dim_{\mathbb{C}} \text{Hom}_\Lambda(FU, Z_2)$$

for all indecomposables  $FU$  over  $\Lambda$ . Thus two exact sequences

$$\begin{array}{ccccccccc} \varepsilon_1 & : & 0 & \rightarrow & FB & \rightarrow & Z_1 & \rightarrow & FA & \rightarrow & 0 \\ \varepsilon_2 & : & 0 & \rightarrow & FB & \rightarrow & Z_2 & \rightarrow & FA & \rightarrow & 0 \end{array}$$

have isomorphic middle terms if and only if

$$\dim_{\mathbb{C}} \ker \pi(\varepsilon_1, ?) = \dim_{\mathbb{C}} \ker \pi(\varepsilon_2, ?)$$

for the two maps  $\pi(\varepsilon_1, ?)$  and  $\pi(\varepsilon_2, ?)$  from  $\text{Hom}_\Lambda(FU, FA)$  to  $\text{Ext}_\Lambda^1(FU, FB)$  and all indecomposables  $FU$ .

Let  $\lambda : G \rightarrow \mathbb{Z}$  be a map and consider the  $\mathbb{C}^*$ -action on

$$\bigoplus_{l \in G} \text{Hom}_\Lambda(l(U), A)$$

given by

$$t \cdot (f_l)_{l \in G} = (t^{-\lambda(l)} f_l)_{l \in G},$$

for  $B, U$  in  $\mathcal{F}$ . The  $\mathbb{C}$ -isomorphism

$$\bigoplus_{l \in G} \text{Hom}_\Lambda(l(U), A) \xrightarrow{\sim} \text{Hom}_\Lambda(FU, FA)$$

allows us to transfer this action of  $\mathbb{C}^*$  to  $\text{Hom}_\Lambda(FU, FA)$ .

LEMMA. Let  $\lambda : G \rightarrow \mathbb{Z}$  be a group homomorphism. Then the map

$$\pi : \text{Ext}_\lambda^1 (FA, FB) \times \text{Hom}_\lambda (FU, FA) \rightarrow \text{Ext}_\lambda^1 (FU, FB)$$

is  $\mathbb{C}^*$ -equivariant, where on the left  $\mathbb{C}^*$  acts diagonally.

*Proof.* It suffices to check that, for  $g, l$  in  $G$ , the pull-back map

$$\pi : \text{Ext}_\lambda^1 (A, g(B)) \times \text{Hom}_\lambda (l(U), A) \rightarrow \text{Ext}_\lambda^1 (l(U), g(B))$$

has the property

$$\pi(t^{\lambda(g)}\varepsilon_g, t^{-\lambda(l)}f_l) = t^{\lambda(l^{-1}g)}\pi(\varepsilon_g, f_l),$$

for  $t \in \mathbb{C}^*$ . This is clear, as  $\lambda(l^{-1}g) = \lambda(g) - \lambda(l)$ .

COROLLARY. If  $\lambda : G \rightarrow \mathbb{Z}$  is a group homomorphism, the  $\mathbb{C}^*$ -action on  $\text{Ext}_\lambda^1 (FA, FB)$  associated with  $\lambda$  stabilizes  $\text{Ext}_\lambda^1 (FA, FB)_{FX}$ , for all  $A, B, X$  in  $\tilde{\mathcal{F}}$ .

Thus our first condition is satisfied. But it is clear that a group homomorphism will rarely satisfy the second one.

5.3. PROPOSITION. Let  $\lambda : G \rightarrow \mathbb{Z}$  be a group homomorphism. Then  $L(\Lambda)$  is isomorphic to  $L(\tilde{\Lambda}/\ker \lambda)/(G/\ker \lambda)$ .

COROLLARY. If the fundamental group  $G$  is  $\mathbb{Z}$ ,  $L(\Lambda)$  is isomorphic to  $L(\tilde{\Lambda})/G$ .

*Proof of the proposition.* We have to show that, for any  $A, B, X$  in  $\tilde{\mathcal{F}}$ ,

$$\chi(\text{Ext}_\lambda^1 (FA, FB)_{FX}/\mathbb{C}^*) = \sum_{\bar{g}, \bar{h} \in G/\ker \lambda} \chi(\text{Ext}_{\tilde{\Lambda}/\ker \lambda}^1 (F'A, \bar{g}(F'B))_{\bar{h}(F'X)}/\mathbb{C}^*)$$

where  $F' : \text{ind } \tilde{\Lambda} \rightarrow \text{ind } \tilde{\Lambda}/\ker \lambda$  is the orbit functor. This follows easily from the formula for fixed points in 5.1, as the inverse image of  $\text{Ext}_\lambda^1 (FA, FB)_{FX}$  in  $\text{Ext}_{\tilde{\Lambda}/\ker \lambda}^1 (F'A, \bar{g}(F'B))$  is the disjoint union

$$\bigcup_{\bar{h} \in G/\ker \lambda} \text{Ext}_{\tilde{\Lambda}/\ker \lambda}^1 (F'A, \bar{g}(F'B))_{\bar{h}(F'B)}.$$

**6. The proof**

6.1. The last ingredient for our proof is the following:

**PROPOSITION.**  *$L(A)$  is isomorphic to  $L(\tilde{\Lambda}/G')/(G/G')$ , where  $G'$  is the commutator subgroup of  $G$ .*

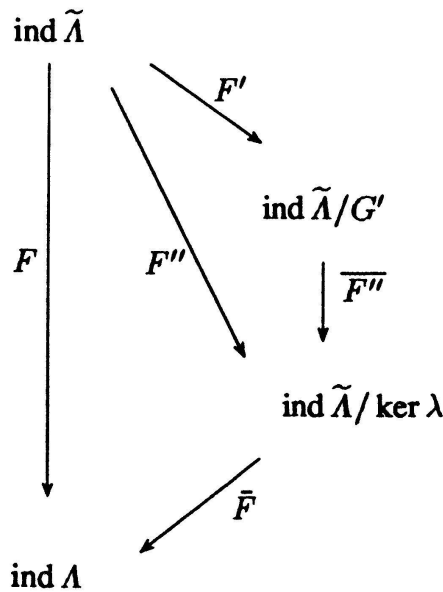
*Proof.* As  $G$  is free [BG], the quotient  $G/G'$  is free abelian. Let  $\rho : G \rightarrow G/G'$  be the projection.

Fix  $A$  and  $B$  in  $\mathcal{F}$ , and let  $S$  be the finite subset

$$S = \{g \in G : \text{Ext}_{\tilde{\Lambda}}^1(A, g(B)) \neq 0\}$$

of  $G$ . As  $G/G'$  is free abelian, there exists a group homomorphism  $\bar{\lambda} : G/G' \rightarrow \mathbb{Z}$  whose restriction to  $\rho(S)$  is injective. Choose for  $\lambda : G \rightarrow \mathbb{Z}$  the composition  $\bar{\lambda} \circ \rho$ .

The following picture explains the notations we choose for orbit functors related to the groups  $G' \subseteq \ker \lambda \subseteq G$ :



We denote the residue class of an element  $g$  in  $G$  modulo  $G'$  by  $\bar{g}$  and modulo  $\ker \lambda$  by  $\bar{\bar{g}}$ .

We know from 5.3 that, for any  $X$  in  $\mathcal{F}$ ,

$$\chi(\text{Ext}_{\Lambda}^1(FA, FB)_{FX}/\mathbb{C}^*) = \sum_{\bar{\bar{g}}, \bar{\bar{h}} \in G/\ker \lambda} \chi(\text{Ext}_{\tilde{\Lambda}/\ker \lambda}^1(F''A, \bar{\bar{g}}(F''B))_{\bar{\bar{h}}(F''X)}/\mathbb{C}^*).$$

$\overline{F''}$  Applying 3.4 to  $\tilde{\Lambda}/\ker \lambda$ ,  $G' \subseteq \ker \lambda$  and the elements  $A, g(B) \in \tilde{\mathcal{F}}$ , we find that  $\overline{F''}$  induces an isomorphism

$$\bigoplus_{\bar{l} \in \ker \lambda/G'} \text{Ext}_{\tilde{\Lambda}/G'}^1(F'A, \bar{l}(F'(gB))) \rightarrow \text{Ext}_{\tilde{\Lambda}/\ker \lambda}^1(F''A, F''(gB)).$$

We claim that, by our choice of  $\lambda$ , there is a unique  $\bar{l} \in \ker \lambda/G'$  for which

$$\text{Ext}_{\tilde{\Lambda}/G'}^1(F'A, \bar{l}(F'(gB))) \neq 0$$

provided that

$$\text{Ext}_{\tilde{\Lambda}/\ker \lambda}^1(F''A, F''(gB)) \neq 0.$$

Indeed, for  $l \in \ker \lambda$ , the space

$$\text{Ext}_{\tilde{\Lambda}/G'}^1(F'A, F'(lgB))$$

is isomorphic to

$$\bigoplus_{g' \in G'} \text{Ext}_{\tilde{\Lambda}}^1(A, g'lgB).$$

If now for  $l_1, l_2 \in \ker \lambda$  there exists  $g'_1, g'_2 \in G'$  such that

$$\text{Ext}_{\tilde{\Lambda}}^1(A, g'_i l_i gB) \neq 0, \quad i = 1, 2,$$

the elements  $g'_i l_i g$  both belong to  $S$ , and their residue class modulo  $\ker \lambda$  is  $\bar{g}$ . Thus their classes modulo  $G'$  coincide, and therefore  $\bar{l}_1 = \bar{l}_2$ .

Suppose now that

$$\text{Ext}_{\tilde{\Lambda}/\ker \lambda}^1(F''A, F''(gB)) \neq 0,$$

and fix  $l$  in  $\ker \lambda$  with

$$\text{Ext}_{\tilde{\Lambda}/G'}^1(F'A, F'(lgB)) \neq 0.$$

Then  $\overline{F''}$  induces an isomorphism

$$\text{Ext}_{\tilde{\Lambda}/G'}^1(F'A, F'(lgB)) \rightarrow \text{Ext}_{\tilde{\Lambda}/\ker \lambda}^1(F''A, F''(gB)).$$

The inverse image of

$$\text{Ext}_{\tilde{\Lambda}/\ker \lambda}^1(F''A, F''(gB))_{F''(hX)}$$

under this isomorphism is the disjoint union

$$\dot{\bigcup}_{\bar{k} \in \ker \lambda/G'} \text{Ext}_{\tilde{\Lambda}/G'}^1(F'A, F'(lgB))_{\bar{k}F'(hX)}.$$

Summing up we find

$$\chi(\text{Ext}_{\tilde{\Lambda}}^1(FA, FB)_{FX}/\mathbb{C}^*) = \sum_{\bar{g}, \bar{h} \in G/G'} \chi(\text{Ext}_{\tilde{\Lambda}/G'}^1(F'A, \bar{g}(F'B))_{\bar{h}(F'X)}/\mathbb{C}^*).$$

6.2. The higher commutator subgroups  $G^{(i)}$  of  $G$ ,  $i \in \mathbb{N}$ , are defined inductively by

$$G^{(i+1)} = [G^{(i)}, G^{(i)}].$$

They are normal subgroups of  $G$ . As a consequence of Magnus' theorem on the lower central series, they intersect in the neutral element of  $G$ , since  $G$  is free.

**COROLLARY.** *For any  $i \in \mathbb{N}$ ,  $L(\Lambda)$  is isomorphic to  $L(\tilde{\Lambda}/G^{(i)})/(G/G^{(i)})$ .*

*Proof.* Indeed, proposition 6.1 applied to  $\tilde{\Lambda}/G^{(i)}$  tells us that

$$L(\tilde{\Lambda}/G^{(i)}) \xrightarrow{\sim} L(\tilde{\Lambda}/G^{(i+1)})/(G^{(i)}/G^{(i+1)})$$

for all  $i$ . We conclude by induction applying 2.3.

6.3. **PROPOSITION.** *If  $\Lambda$  has finitely many objects there exists a natural number such that  $L(\tilde{\Lambda}/G^{(t)})$  is isomorphic to  $L(\tilde{\Lambda})/G^{(t)}$ .*

*Proof.* In view of proposition 4.3 (applied to  $\tilde{\Lambda}/G^{(t)}$ ) we only need to find  $t \in \mathbb{N}$  such that

$$\{g \in G^{(t)} : \text{Ext}_{\tilde{\Lambda}}^1(A, g(B)) \neq 0\}$$

has at most one element for all  $A, B \in \tilde{\mathcal{J}}$ . For  $A, B \in \tilde{\mathcal{J}}$  we set

$$S(A, B) = \{g \in G : \text{Ext}_{\tilde{\Lambda}}^1(A, g(B)) \neq 0\}$$

and

$$T(A, B) = \{g_1 g_2^{-1} : g_1, g_2 \in S(A, B)\}.$$

Clearly we have

$$S(h(A), l(B)) = hS(A, B)l^{-1}$$

and

$$T(h(A), l(B)) = hT(A, B)h^{-1}.$$

Since  $\mathcal{A}$  has finitely many objects,  $\tilde{\mathcal{J}}$  contains only finitely many  $G$ -orbits, and therefore the set

$$T = \bigcup_{A, B \in \tilde{\mathcal{J}}} T(A, B)$$

is a finite union of conjugacy classes in  $G$ .

Use now that the intersection  $\bigcap_{i \in \mathbb{N}} G^{(i)}$  is reduced to  $\{1\}$ . Fix an integer  $t$  with

$$T \cap G^{(t)} = \{1\}.$$

Then the set

$$S(A, B) \cap G^{(t)} = \{g \in G^{(t)} : \text{Ext}_{\tilde{\mathcal{A}}}^1(A, g(B)) \neq 0\}$$

contains at most one element for all  $A, B \in \tilde{\mathcal{J}}$ , and our proof is complete.

6.4. In case  $\mathcal{A}$  is finite the preceding proposition proves our theorem. Indeed, we have a chain of isomorphisms

$$L(\mathcal{A}) \xrightarrow{\sim} L(\tilde{\mathcal{A}}/G^{(t)})/(G/G^{(t)}) \xrightarrow{\sim} (L(\tilde{\mathcal{A}})/G^{(t)})/(G/G^{(t)}) \xrightarrow{\sim} L(\tilde{\mathcal{A}})/G.$$

In general, there is no reason why proposition 6.3 should hold. But “ $t$  exists locally”, and this suffices to prove our theorem: for  $A, B \in \tilde{\mathcal{J}}$  there exists  $t = t(A, B)$  such that

$$T(A, B) \cap G^{(t)} = \{1\},$$

as  $T(A, B)$  is finite.

Again this implies that

$$\{g \in G^{(t)} : \text{Ext}_{\tilde{\mathcal{A}}}^1(A, g(B)) \neq 0\}$$



contains at most one element. By the second remark in 4.2 the Lie bracket of  $v_{G^{(t)} \cdot A}$  with  $v_{G^{(t)} \cdot B}$  is “the same” in  $L(\tilde{\Lambda}/G^{(t)})$  as in  $L(\tilde{\Lambda})/G^{(t)}$ . We finish the proof as in case  $\Lambda$  is finite, comparing the brackets of  $v_{G \cdot A}$  and  $v_{G \cdot B}$  in  $L(\Lambda)$ ,  $L(\tilde{\Lambda}/G^{(t)})/(G/G^{(t)})$ ,  $(L(\tilde{\Lambda})/G^{(t)})/(G/G^{(t)})$  and  $L(\tilde{\Lambda})/G$ .

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### REFERENCES

- [AR] AUSLANDER, M. and REITEN, I., *Modules determined by their composition factors*, Illinois J. Math. 29 (1985), 289–301.
- [BGRS] BAUTISTA, R., GABRIEL, P., ROITER, A. V. and SALMERON, L., *Representation finite algebras and multiplicative bases*, Invent. Math. 81 (1985), 217–285.
- [BG] BONGARTZ, K. and GABRIEL, P., *Covering spaces in representation theory*, Invent. Math. 65 (1982), 331–378.
- [GA] GABRIEL, P., *The universal cover of a representation-finite algebra*, in “Representations of algebras”, Springer Lecture Notes 903, 1981.
- [Ha] HAPPEL, D., *Composition factors for indecomposable modules*, Proc. AMS 86 (1982), 29–31.
- [Rie] RIEDTMANN, CH., *Lie algebras generated by indecomposables*, Prépublication de l’Institut Fourier n° 170, Grenoble, 1991 to appear in J. of Algebra.
- [Rin] RINGEL, C. M., *Hall algebras*, in “Topics in Algebra”, Banach Center Publications 26 (1990), 433–447.

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