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Splitting the spectral flow and the Alexander matrix

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1. Introduction

This paper is concerned with a procedure for computing the spectral flow of a path of self-adjoint operators of the form $D_t = *d_{A_t} - d_{A_t}*$, where the A_t are $SU(2)$ connections on a 3-manifold Z which is split along a torus, and A_0 and A_1 are flat. Recent theorems of Yoshida [Y1, Y2] show how to carry this out when Z is obtained by surgery on a knot, under certain nondegeneracy conditions. Under the assumption that there is a path A_t of flat connections on the knot complement and that the space of flat connections modulo gauge transformation is a smooth 1-dimensional variety near this path, Yoshida shows with an explicit formula that the spectral flow is determined by the restriction of the path to the boundary torus.

As a consequence of our main result we show that when the path A_t has singularities, the spectral flow is *not* determined by its restriction to the boundary torus. We give explicit computations in §6 comparing the spectral flow on a surgery of a Whitehead double of a knot to the spectral flow on the corresponding surgery of the Whitehead double of the unknot. These examples have paths of flat connections on the knot complements whose restrictions to the boundary are the same, while their spectral flows differ.

Suppose $Z = X \cup Y$, where X is the complement of a knot in S^3. Let A_0 and A_1 be flat connections on Z whose restrictions to X are reducible. Then there are corresponding flat connections A'_0 and A'_1 on $Z' = X' \cup Y$, where X' is the unknot complement (i.e., a solid torus). In §4 we show that the difference between the spectral flow from A'_0 to A'_1 on Z' and the spectral flow from A_0 to A_1 on Z is a classical knot signature, and in fact is equal to the spectral flow of the Alexander matrix of the knot. Applying this theorem to satellite knots yields examples in

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which Yoshida's pillowcase result fails for paths of representations having singularities. For this special case, Theorem 4.4 gives the precise correction term needed to make Yoshida's theorem apply. In §6, we demonstrate how to calculate this correction in a number of examples.

By combining this splitting device with the technique of [FS2], we can compute the spectral flows between arbitrary flat connections on a large class of 3-manifolds split along tori, including graph manifolds. In §5 we describe the moduli space of flat irreducible connections on a manifold obtained by gluing together two Seifert-fibered homology knot complements X and Y along their boundaries, and show how to compute the spectral flow between any two such connections using the techniques of Fintushel and Stern. In this case we obtain an explicit formula which shows the spectral flow is a sum of 3 terms. The first term involves only X and is an analogue for Seifert-fibered manifolds with boundary of the $R(e)$ invariant of Fintushel and Stern. The second term is analogous and involves only Y. The third term is an "interaction" term arising from the restrictions to the boundary.

These results are applied in §6 to compute the Floer chain complexes of certain surgeries on twisted Whitehead doubles of torus knots.

To put our results in proper perspective we will explain briefly the algorithm of Yoshida, in the cases where his work applies. Here and for the rest of the paper, we write $R(\pi)$ for the space of representations of a group π in SU(2), modulo conjugation. An elementary but vital fact is that $R(\mathbf{Z}^2)$ is a 2-sphere with 4 singular points, called the *pillowcase*. Suppose that Z is a 3-manifold and that $X \subset Z$ is the exterior of a knot in Z. Suppose further that ρ_0 and $\rho_1 \in R(\pi_1(Z))$ are representations which happen to lie on a smooth 1-dimensional component of irreducible representations of $\pi_1(X)$ which we parameterize as ρ_t. Restricting ρ_t to the boundary torus gives a smooth path in the pillowcase $R(\mathbf{Z}^2)$. There is a tangent line field on the complement of the singular points in the pillowcase and Yoshida shows that the spectral flow is just the degree of the tangent vector field to the restriction of ρ_t in this line field.

The intuitive reason for this is that each operator in the path whose spectral flow we are interested in has 1-dimensional kernel when restricted to X, and when restricted to the neighborhood, S, of the knot. These kernels are identified with the aforementioned tangent fields, and an eigenvalue (for the operator on Z) passes through 0 whenever these tangent fields coincide.

Yoshida assumes that the two representations lie on a smooth, 1-dimensional component of the representation space of the knot complement and proceeds by drilling out holes, thus splitting along a higher genus surface where non-degeneracy is easier to verify. T. Mrowka [M] has a more general approach – he shows that in any case there is an infinite dimensional Maslov index which equals the spectral flow. Under the nondegeneracy conditions, Mrowka identifies this Maslov index (via "symplectic reduction") with the degree of the vector fields in the pillowcase. However, the reduction process breaks down when the path ρ_t passes through a singular point of the representation variety $R(\pi_1(X))$. More recently Cappell, Lee, and Miller [CLM] have announced a formula which expresses the spectral flow as a sum of 3 terms given a splitting of a 3-manifold along any surface. Our results can be viewed as explicitly identifying the terms in their formula in the special cases outlined above.

Our methods do not involve the delicate analysis of [Y1] but instead use the machinery of the Atiyah–Patodi–Singer index theorem for manifolds with boundary, together with Wall nonadditivity. Along the way, we clear up some delicate points about orientations, and about Chern–Simons invariants for $SO(3)$ bundles which seem confused in the literature.

A few general remarks can be made here. The results in this paper are of a computational nature, and as such they provide explicit computations of spectral flow which can then be combined with the abstract splitting results of [Y1], [M], and [CLM]. For example, Yoshida shows how to compute the Floer homology of any surgery on the figure 8 knot starting with only two pieces of data: the image of the space of $SU(2)$ representations of the figure 8 knot in the pillowcase, and Fintushel and Stern's computation of the Floer homology of $\Sigma(2, 3, 7)$. The computation then follows from an algorithm, as explained in [Y2]. Similarly, the results of this paper show how to relate the spectral flow for representations of 3-manifolds to the spectral flows for simpler 3-manifolds. These computations depend on understanding the representation varieties of 3-manifold groups, and although this is a hard problem in general there are many partial results. Of course, computing the Floer homology will require understanding the boundary operators in the Floer chain complex, a difficult problem. Although this paper does not address this question in general we point out that the correction term in Theorem 4.4 is usually even. (This fact is used to compute the Floer *homology* of certain graph manifolds.)

2. The ρ_α invariants and spectral flow

We will explain some of the terms which appear in our formula for spectral flow. See also [T], [F], [FS3]. Our first remark is about orientations. There are two conventions for orienting the boundary for a 4-manifold N. We will use the convention "outward normal first". This is convenient when dealing with differential-geometric objects, for example, with this convention Stokes' theorem says $\int_N d\omega = \int_{\partial N} \omega$.

Let $P \to Z$ be a principal SU(n) bundle over a compact, closed, and oriented **Q**-homology 3-sphere Z. The bundle P is trivial and a fixed trivialization enables us to identify the space $\mathscr{A}$ of connections on P with the lie algebra valued 1-forms $\Omega^1 \otimes su(n)$.

Given a connection $a \in \mathscr{A}$ we form its covariant derivative

$$d_a : \Omega^p \otimes su(n) \to \Omega^{p+1} \otimes su(n).$$

So $d_a b = db + [a, b]$ in the trivialization. Let d_a^* denote the adjoint. We then define:

$$D_a : \Omega^0 \otimes su(n) \oplus \Omega^1 \otimes su(n) \to \Omega^0 \otimes su(n) \oplus \Omega^1 \otimes su(n)$$

by the formula:

$$D_a(\phi, \tau) = (d_a^* \tau, *d_a \tau + d_a \phi).$$

The operator D_a is self-adjoint and elliptic, and has a discrete real spectrum.

In general, if D_t is a one-parameter family of self-adjoint operators with discrete spectrum on a Hilbert space, the *spectral flow* of the family from D_0 and D_1 is the intersection number of the graph of the eigenvalues of D_t with a line segment from $(0, -\delta)$ to $(1, \delta)$ in $[0, 1] \times \mathbf{R}$ where δ is a number such that $0 < \delta < \inf |\lambda|$, the infimum taken over the set of non-zero eigenvalues of D_0 and D_1. (Note that all eigenvalues are real since the operators are self-adjoint.) See Figure 1.

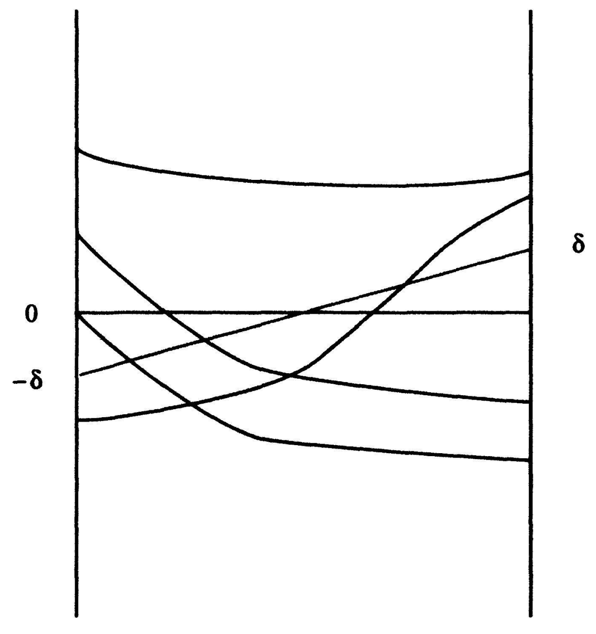


Figure 1

This is just the difference between the number of eigenvalues which change from negative to positive and the number of eigenvalues which change from non-negative to non-positive. (The δ is introduced in case D_0 or D_1 have kernel.)

Now if a_t, $t \in [0, 1]$ is a smooth 1-parameter family of connections we define the *Spectral flow from* a_0 *to* a_1 to be the spectral flow of the family of self-adjoint operators D_{a_t}. We denote this by $SF(a_0, a_1)$.

To resolve the dependency of $SF(\alpha_0, \alpha_1)$ on the choices made such as the choice of trivialization, the path a_t, and the basepoint of Z, we pass to the quotient $\mathscr{A}/\mathscr{G}$ of the space of connections modulo gauge transformations. Then the spectral flow becomes well defined in $\mathbf{Z}/2k_n\mathbf{Z}$, where k_n is an integer defined as follows. Let $\mathrm{ad} : SU(n) \to SU(n^2-1)$ be the (complexified) adjoint representation. Then:

$$\mathrm{ad}^* : H^4(BSU(n^2-1)) \to H^4(BSU(n))$$

takes c_2 to $k_n c_2$. For example $k_2 - 4$ so the spectral flow between two connections on an $SU(2)$ bundle is well defined mod 8. (See §7 for more details.)

The spectral flow has the following easily verified properties:

1. $SF(a, c) = SF(a, b) + SF(b, c) + \dim \operatorname{Ker} D_b$. In particular $SF(a, a) = -\dim \operatorname{Ker} D_a$.

2. If $-Z$ denotes Z with the opposite orientation, then

$$SF(a, b)(-Z) = -SF(a, b)(Z) - (\dim \operatorname{Ker} D_a + \dim \operatorname{Ker} D_b).$$

A more sophisticated invariant derived from the spectrum of D_a is its *eta*-invariant, $\eta_{D_a}(s)$, defined for $Re(s) \gg 0$ by:

$$\eta_{D_a}(s) = \sum_{\lambda \in \mathrm{Spec} D_a - 0} \mathrm{sign}(\lambda)|\lambda|^{-s}.$$

In [APS1] it is shown that $\eta_{D_a}(s)$ meromorphically continues to a function with a finite value at $s = 0$. Heuristically, $\eta_{D_a}(0)$ measures whether D_a has more positive or negative eigenvalues.

As a special case, suppose that $\alpha : \pi_1 Z \to SU(n)$ is a representation and let $\mathrm{ad}_\alpha : \pi_1 Z \to SU(n^2-1)$ denote the adjoint representation. Let a be a flat connection on P with holonomy α and let θ denote the trivial $SU(n)$ connection. Then the quantity $\eta_{D_a}(0) - \eta_{D_\theta}(0)$ is independent of the Riemannian metric and in fact equals the Atiyah–Patodi–Singer invariant $\rho_{\mathrm{ad}\,\alpha}(Z)$ introduced in [APS2]. In particular, if N is an oriented 4-manifold with oriented boundary Z and $\beta : \pi_1 N \to SU(n^2-1)$ extends $\mathrm{ad}\,\alpha$ then

$$\rho_{\mathrm{ad}\,\alpha}(Z) = (n^2-1)\,\mathrm{Sign}\, N - \mathrm{Sign}_\beta\, N.$$

In this formula $\mathrm{Sign}_\beta N$ denotes the signature of N with local coefficients in the flat bundle defined by β induced by the cup product and the invariant inner product on $su(n) \otimes \mathbf{C} \cong \mathbf{C}^{n^2-1}$.

The spectral flow may be expressed in terms of η-invariants using the main theorems of [APS1, 2]. One more quantity is needed, the *Chern–Simons* invariant of a connection. It is defined for a connection $a \in \mathscr{A}$ by

$$cs(a) = \frac{1}{8\pi^2} \int_Z Tr\left(da \wedge a + \frac{2}{3} a \wedge a \wedge a \right)$$

where we think of $a \in \Omega^1 \otimes su(n)$. In this formula wedging of $su(n)$ valued forms means to wedge the form parts and matrix-multiply the coefficients. Taken in $\mathbf{R}/\mathbf{Z}$, the Chern–Simons invariants are independent of the choice of trivialization or gauge transformation. Moreover, the Chern–Simons invariant of a flat connection is a flat cobordism invariant.

With these definitions in place, we can now write the formula for spectral flow. Although this formula is well-known we could not find it explicitly derived in the literature and so we give an argument in the last section of this paper. We also show how to relate the spectral flow to the index of the self-duality operator on $Z \times I$, suitably oriented. In this formula we assume the group of the bundle is $SU(2)$.

$$SF(a, b) = 8(cs(b) - cs(a)) + \frac{1}{2}(\eta_{D_b}(0) - \eta_{D_a}(0))$$
$$- \frac{1}{2}(\dim \operatorname{Ker} D_a + \dim \operatorname{Ker} D_b).$$

In the special case where a and b are *flat* connections with holonomy representations α and β respectively then the kernel of D_a is just $H^0(Z; \operatorname{ad} \alpha) \oplus H^1(Z; \operatorname{ad} \alpha)$ by the Hodge theorem. We then denote the dimension of the kernel by h_α. In this case the formula becomes:

$$SF(\alpha, \beta) = 8(cs(\beta) - cs(\alpha)) + \frac{1}{2}(\rho_{\operatorname{ad} \beta}(Z) - \rho_{\operatorname{ad} \alpha}(Z)) - \frac{1}{2}(h_\alpha + h_\beta). \tag{2.1}$$

In [T] it is proven that if Z is a homology sphere such that $H^1(Z; \operatorname{ad}_\alpha)$ vanishes for all irreducible $SU(2)$ representations α, then Casson's homology sphere invariant is equal to

$$-\frac{1}{2} \sum_{[\alpha]} (-1)^{SF([\theta],[\alpha])}$$

where the sum is taken over the finite set of conjugacy classes of irreducible $SU(2)$ representations. More generally one must perturb the flatness equations to obtain a finite sum.

REMARK. Taubes shows his invariant is equal to Casson's invariant, not its negative. However, we are using the sign convention of [AM] for Casson's invariant. Akbulut and McCarthy first define Casson's invariant up to an overall sign which they later nail down by requiring the surgery formula to hold with respect to a specific normalization of the Alexander polynomial. It turns out that the sign (S on pages 65 and 125 of [AM]) equals -1. We will see this later in our computations.

In [F], Floer makes use of the fact that spectral flow is well-defined mod 8 to construct a $\mathbf{Z}/8$-graded chain complex whose generators in dimension k are those $SU(2)$ representations α such that $SF([\theta],[\alpha]) \equiv k$ Mod 8. The homology of this complex is called the *Floer homology* or *Instanton homology* of Z.

3. The basic geometric construction

We introduce the geometric construction which will be our main tool. The idea is simple: if we can decompose the 3-manifold Z into simpler pieces, say X and Y and find 4-manifolds with boundary containing X and Y over which the representations extend, then we can glue the 4-manifolds together to get a flat cobordism from Z to a less complicated space for which we can compute the terms appearing in the formula for spectral flow directly.

So, let Z_0 be an oriented rational homology sphere and let $T \subset Z_0$ be a torus separating Z_0 into two pieces X_0 and Y_0. Let $\beta : \pi_1 Z_0 \to SU(n)$ be a representation. (In our applications $\beta = \operatorname{ad} \alpha$ for some $SU(2)$ representation α.) Denote by β_X and β_Y the restrictions of β to X_0 and Y_0.

Suppose there exist 4-manifolds M_X and M_Y such that:

1. $X_0 \subset \partial M_X$.
2. β_X extends over $\pi_1 M_X$.
3. $\partial M_X = X_0 \cup (T^2 \times I) \cup X_1 \amalg L_X$ where L_X is some closed manifold, and X_1 is a rational homology knot complement. See Figure 2.

We view M_X as a rel boundary cobordism of X_0 to $X_1 + L_X$, and view M_Y similarly.

Then we can glue M_X to M_Y along $T \times I$ to get a flat cobordism N with boundary $Z_0 \cup Z_1 \cup L_X \cup L_Y$. Here $Z_1 = X_1 \cup Y_1$. We orient N so that:

$$\partial N = -Z_0 + Z_1 + L_X + L_Y.$$

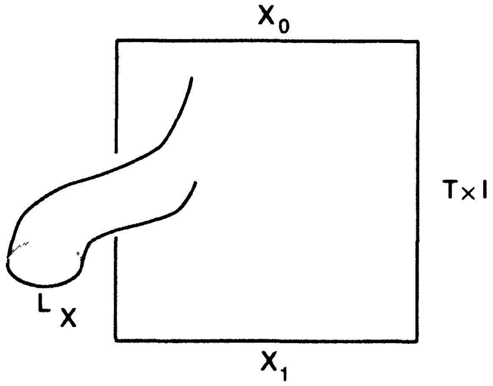


Figure 2

The main examples to keep in mind are the following:

1. X_0 is the complement of a knot in S^3, X_1 is the complement of the unknot, M_X is a flat cobordism from X_0 to X_1, and $M_Y = Y \times I$. Thus L_X and L_Y are empty.

2. X_0 is Seifert fibered, M_X is obtained from the mapping cylinder of the Seifert fibration by deleting neighborhoods of the singularities, and so L_X is a union of lens spaces and $X_1 = S^1 \times D^2$. We will take M_Y to be $Y_0 \times I$ or, if Y_0 is also Seifert fibered, we will construct M_Y from its mapping cylinder.

Consider now the terms appearing in the formula for the spectral flow between two flat $SU(2)$ connections. The Chern–Simons invariant can be computed for splittings along tori in various ways, for example using the results of [KK2]. The h_β terms are dimensions of cohomology groups which can usually be computed explicitly. This leaves the ρ_β invariants. These are not flat cobordism invariants but from the Atiyah–Patodi–Singer signature theorem we know:

$$\rho_\beta(Z_0) = \rho_\beta(Z_1) + \rho_\beta(L_X) + \rho_\beta(L_Y) - n \operatorname{Sign} N + \operatorname{Sign}_\beta N.$$

In our situation L_X and L_Y will either be empty or lens spaces, whose ρ_β invariants can be computed directly since they have finite fundamental groups. It remains then to compute the signature terms.

These are computed using Wall's non-additivity formula [Wa]. We are gluing M_X to M_Y along $T \times I$, so the signature of N differs from the sum of the signatures of M_X and M_Y by a correction term which we explain. Fix either trivial $\mathbf{R}$ or non-trivial flat coefficients. We have:

$$\operatorname{Sign} N = \operatorname{Sign} M_X + \operatorname{Sign} M_Y - \operatorname{Sign} \Psi$$

where Ψ is a non-degenerate bilinear form on the vector space defined as follows.

Consider:

$$A = \text{Image}\,(H^1(X_0 \cup X_1) \longrightarrow H^1(T \times \{0, 1\})),$$
$$B = \text{Image}\,(H^1(T \times I) \longrightarrow H^1(T \times \{0, 1\})),$$

and

$$C = \text{Image}\,(H^1(Y_0 \cup Y_1) \longrightarrow H^1(T \times \{0, 1\})).$$

Then Ψ is defined on:

$$\frac{B \cap (A + C)}{(B \cap A) + (B \cap C)}$$

Write $H^1(T \times \{0, 1\}) = H \oplus H$, so that B is just the diagonal subspace. We can write $A = A_0 \oplus A_1$ and $C = C_0 \oplus C_1$. There is an isomorphism:

$$\frac{B \cap (A + C)}{(B \cap A) + (B \cap C)} \cong \frac{(A_0 + C_0) \cap (A_1 + C_1)}{(A_0 \cap A_1) + (C_0 \cap C_1)}.$$

For the definition of the form Ψ we refer to Wall's paper. In the cases we consider we will use this isomorphism to show that Ψ is the zero form.

We end this section with a well-known lemma (see, for example, [H]).

LEMMA 3.1. *Let X be a 3-manifold with torus boundary, and let $\beta : \pi_1(X) \to G$ be a representation into some semi-simple Lie group G. Let E be a representation of G which has a non-degenerate, positive-definite, G-invariant inner product. Then the image of $H^1(X; E_\beta)$ in $H^1(\partial X; E_\beta)$ is half-dimensional.*

Proof. By Poincaré duality the composition of the cup product and the inner product $E \times E \to \mathbf{R}$

$$H^1(X; E_\beta) \times H^2(X, \partial X; E_\beta) \longrightarrow H^3(X, \partial X; \mathbf{R})$$

is non-degenerate. The orientation of X defines an isomorphism $H^3(X, \partial X; \mathbf{R}) \cong \mathbf{R}$ and one obtains an isomorphism:

$$H^1(X; E_\beta) \longrightarrow \text{Hom}\,(H^2(X, \partial X; E_\beta), \mathbf{R}).$$

Similarly we get $H^1(\partial X; E_\beta) \to \text{Hom}\,(H^1(\partial X; E_\beta), \mathbf{R})$.

Consider now the part of the long exact sequence:

$$H^1(X; E_\beta) \xrightarrow{f} H^1(\partial X; E_\beta) \xrightarrow{g} H^2(X, \partial X; E_\beta).$$

The maps f and g are dual maps with respect to Poincaré duality and so $\dim \operatorname{im}(f) = \dim \operatorname{Ker}(g) = \dim H^1(\partial X; E_\beta) - \dim \operatorname{im}(g) = \dim H^1(\partial X; E_\beta) - \dim \operatorname{im}(g^*) = \dim H^1(\partial X; E_\beta) - \dim \operatorname{im}(f)$ so that the image of f is half dimensional. □

4. Representations which are abelian on a knot complement

In this section we consider the set-up of the previous section where X_0 is the exterior of a knot in S^3 and $\alpha : \pi_1 Z_0 \to SU(2)$ restricts to an abelian representation on X_0. For example, suppose Z_0 is a homology sphere obtained from surgery on a satellite of a knot K. Then the companion torus splits Z_0 into the exterior of K and surgery on a knot in a solid torus. The representation space of $\pi_1 Z_0$ divides into two pieces, R_I and R_R depending on whether the restriction of a representation to the exterior of K is irreducible or reducible. The piece R_R is naturally homeomorphic to the space of representations of the corresponding satellite of the unknot. So if we write $Z_0 = X_0 \cup W \cup (S^1 \times D^2)$ where W is the exterior of a knot in a solid torus and we are given a path α_t of representations of $\pi_1(X_0 \cup W)$ which restrict to reducible representations on X_0 such that α_0 and α_1 extend over Z_0, there is a corresponding path of representations of $\pi_1(X_1 \cup W)$ where X_1 is the unknot complement. Restricting these two paths to the boundary $\partial(X_i \cup W)$ gives the identical path in the pillowcase, and so one might expect the spectral flow from α_0 to α_1 on Z_0 to agree with the spectral flow on $Z_1 = X_1 \cup W \cup S^1 \times D^2$ if the theorem of Yoshida continued to hold in this setting. We will show that this is not the case and that the difference is measured by equivariant signatures of K. In terms of splitting the spectral flow this should correspond to the spectral flow of the path α_t on X_0 being non-zero, since as we shall see the dimension of the cohomology of X_0 with coefficients in α_t jumps precisely when the Alexander matrix of the knot has kernel, i.e. when $\alpha(\mu)^2$ is a root of the Alexander polynomial of K, where μ is the meridian of K.

So let X_0 be the exterior of a knot K in S^3 and let Y_0 be a homology knot complement. Let $Z_0 = X_0 \cup Y_0$, glued in such a way that $H_1(Z_0; \mathbf{Z}) = 0$. (A good example to keep in mind is to let Z_0 be $1/n$ surgery on a satellite of K and we split Z_0 along the incompressible companion torus.) We suppose that $\alpha : \pi_1 Z_0 \to SU(2)$ is a representation whose restriction to X_0 is abelian.

Our first task is to find manifolds M_X, M_Y as in the previous section and compute their signatures. We thank Steve Boyer for suggesting the statements and proofs of the following lemma and theorem.

Consider the following 4-manifold. Let $U = D^4 \cup H$, where H is a 2-handle attached to D^4 along K with the zero framing. Choose a Seifert surface F for K and let $\bar{F}$ be the union of F pushed slightly into D^4 and the core of the 2-handle. Let $W = U - nbd(\bar{F})$. So $\partial W = (0\text{-surgery on } K) \cup (\bar{F} \times S^1)$. Let B be a handlebody of genus equal to the genus of $\bar{F}$. Then let

$$M_X = W \cup (B \times S^1).$$

The proof of the following lemma is then an application of Van Kampen's theorem.

LEMMA 4.1.

1. $\pi_1 M_X \cong \mathbf{Z}$.
2. *The map* $\pi_1(S^3 - K) \to \pi_1 M_X$ *induced by inclusion is* $\gamma \rightarrowtail [\gamma]$, *where* $[\gamma]$ *denotes the image of* γ *under the abelianization* $\pi_1(S^3 - K) \to H_1(S^3 - K) \cong \mathbf{Z}$.

Notice that M_X can be viewed as a rel boundary cobordism of X_0 to $X_1 \cong S^1 \times D^2$. Furthermore, from the previous lemma any abelian representation of $\pi_1 X_0$ extends over $\pi_1 M_X$. Extend α over M_X. We next wish to compute the signature and ad α-signature of M_X. Since $B \times S^1$ has a deformation retract $(\vee_i S^1) \times S^1$ in its boundary, Sign $(B \times S^1) = 0$ and $\text{Sign}_{\text{ad}\,\alpha}(B \times S^1) = 0$. Hence by Novikov additivity Sign M_X = Sign W and $\text{Sign}_{\text{ad}\,\alpha} M_X = \text{Sign}_{\text{ad}\,\alpha} W$.

To compute these signatures, let $\tilde{W}$ be the universal cover of W. Notice that $\pi_1 W \cong \mathbf{Z}$. Let

$$B : H_2\tilde{W} \times H_2\tilde{W} \to \mathbf{Z}[t, t^{-1}]$$

denote the equivariant intersection form of $\tilde{W}$.

THEOREM 4.2. $H_2(\tilde{W}; \mathbf{Z}) \cong (\mathbf{Z}[t, t^{-1}])^{2g} \oplus \mathbf{Z}$ *where* $\mathbf{Z}[t, t^{-1}]$ *acts trivially on the* $\mathbf{Z}$ *summand. The matrix for the equivariant intersection form, B, on the free summand is given by*

$$(1 - t)V + (1 - t^{-1})V^T$$

where V is the Seifert matrix for K, so $V_{ij} = lk(x_i, x_j^+)$. *Furthermore the* $\mathbf{Z}$ *summand is in* Ker B.

The proof of this theorem follows standard arguments and we only indicate the idea. Let C be obtained by cutting D^4 along $F \times I$ where $F \times 0 \subset \partial D^4$ and $F \times 1$ is the pushed in Seifert surface. Then the universal cover of $D^4 - F \times 1$ is obtained by gluing a $\mathbf{Z}$'s worth of copies of C in the usual way. This gives a manifold with second homology the free $\mathbf{Z}[\mathbf{Z}]$ module on the 2-cycles constructed from discs in C whose boundary in F are the generators of $H_1 F$. Thus the Seifert matrix determines the intersection form on this part of $\tilde{W}$ in the manner indicated.

To get $\tilde{W}$ one adds the universal cover of the 2-handle minus its core, which is homeomorphic to $D^2 \times I \times \mathbf{R}$. This last piece contributes the trivial $\mathbf{Z}$ in $H_2 \tilde{W}$, and its intersection with the other part of the homology is trivial since it is carried by a cycle which lives in the boundary $\partial \tilde{W}$.

For more details of this construction see [CG], or [Ka]. □

Let $B_K(t)$ denote the matrix $(1-t)V + (1-t^{-1})V^T$. (Of course $B_K(t)$ depends on the choice of Seifert surface F.) Recall that the symmetrized Alexander matrix for K is

$$A_K(t) = t^{1/2}V - t^{-1/2}V^T.$$

These are related by:

$$(t^{-1/2} - t^{1/2})A_K(t) = B_K(t).$$

Let $\beta : \pi_1 X_0 \to U(1)$ be a representation which sends the meridian μ_X to $e^{i\theta}$. Extend β over M_X. Notice that $B_K(e^{i\theta})$ is hermitian, and $A_K(e^{i\theta})$ is skew-hermitian.

It follows from the previous theorem that the signature of M_X with coefficients in the flat bundle determined by β is equal to the signature of $B_K(e^{i\theta})$. Furthermore, if θ is not a multiple of 2π, then $B_K(e^{i\theta})$ is singular if and only if the Alexander polynomial of K vanishes at $e^{i\theta}$.

Let us now return to the situation of the preceding section. Let $\alpha : \pi_1 Z_0 \to SU(2)$ be a representation whose restriction to X_0 is abelian. Think of M_X as a flat rel boundary cobordism of X_0 to $X_1 = S^1 \times D^2$. By gluing M_X to $M_Y = Y_0 \times I$ along $T \times I$, we obtain a cobordism N from $Z_0 = X_0 \cup Y_0$ to $Z_1 = S^1 \times D^2 \cup (Y_0 \times 1)$ over which α extends. Notice that Z_1 is in general a simpler manifold than Z_0 since it is just a Dehn filling of Y. (For example, if Z_0 is a surgery of a satellite of K, then Z_1 is a surgery on the corresponding satellite of the unknot.) By conjugating α we may assume $\alpha(\pi_1(T))$ lies in the circle of diagonal matrices. Thus if μ_X, λ_X denote the natural meridian and longitude of K, then $\alpha(\lambda_X) = 1$ and

$$\alpha(\mu_X) = \begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix}.$$

The corresponding adjoint representation $\operatorname{ad}\alpha$ takes λ_X to 1 and takes the meridian to the 3×3 matrix:

$$\operatorname{ad}\alpha(\mu_X) = \begin{pmatrix} e^{2i\theta} & & \\ & e^{-2i\theta} & \\ & & 1 \end{pmatrix}.$$

THEOREM 4.3. *Orient N so that $\partial N = -Z_0 + Z_1$. Then*:
1. $\operatorname{Sign} N = 0$.
2. $\operatorname{Sign}_{\operatorname{ad}\alpha} N = -2 \operatorname{Sign} B_K(e^{2i\theta})$.

REMARK. The idea of the proof is to use Wall's non-additivity theorem [W] to show that $\operatorname{Sign}_{\operatorname{ad}\alpha} N = \operatorname{Sign}_{\operatorname{ad}\alpha} M_X$.

Proof.

1. From the remarks immediately preceding Lemma 3.1 we know that $\operatorname{Sign} N = \operatorname{Sign} M_X - \operatorname{Sign}(\Psi)$, where Ψ is a form on $(A_0 + C_0) \cap (A_1 + C_1)/(A_0 \cap A_1) + (C_0 \cap C_1)$ (see §3 for the definitions). We claim that $A_0 = A_1$ and $C_0 = C_1$. This obviously implies that Ψ is the zero form.

Recall that $A_0 = \operatorname{Im}[H^1 X_0 \to H^1 T]$ and $A_1 = \operatorname{Im}[H^1(S^1 \times D^2) \to H^1 T]$ (say with real, untwisted coefficients). But since ∂M_X is just 0-surgery on K, the pair $(X_0 \cup (S^1 \times D^2), T)$ is homologically the same as $(S^1 \times S^2, S^1 \times S^1)$. Clearly then $A_0 = A_1$. Since $M_Y = Y_0 \times I$, $C_0 = C_1$.

Therefore $\operatorname{Sign}(\Psi) = 0$ and, since $\operatorname{Sign}(Y \times I) = 0$, Wall's formula gives $\operatorname{Sign} N = \operatorname{Sign} M_X$. One can see directly from the construction that $\operatorname{Sign} M_X = \operatorname{Sign} W = 0$; equivalently $\operatorname{Sign} M_X = -\operatorname{Sign} B_K(1) = 0$.

2. We will again show that $A_0 = A_1$ and $C_0 = C_1$. This time we must use local coefficients in $\operatorname{ad}\alpha$.

We can identify $H^1(A; \operatorname{ad}\alpha)$ with the group cohomology $H^1(\pi_1 A; \operatorname{ad}\alpha)$ for any path-connected space A and homomorphism $\pi_1 A \to SU(2)$. By taking the first cohomology with $\operatorname{ad}\alpha$ coefficients in the diagram of groups:

$$\begin{tikzcd} & \pi_1 X_0 \arrow[dr] & & \\ \pi_1 T \arrow[ur] \arrow[dr] & & \pi_1 W \arrow[r, "\alpha"] & SU(2) \\ & \pi_1(S^1 \times D^2) \arrow[ur, "\cong"'] & & \end{tikzcd}$$

one sees that the map $H^1(S^1 \times D^2; \operatorname{ad}\alpha) \to H^1(T; \operatorname{ad}\alpha)$ factors through $H^1(X_0; \operatorname{ad}\alpha)$ and so $A_1 \subset A_0$. But now Lemma 4.1 implies that $A_0 = A_1$, since they are both middle dimensional subspaces. Again $C_0 = C_1$ because $M_Y = Y_0 \times I$. So Ψ vanishes in this case also. Now $\operatorname{Sign}_{\operatorname{ad}\alpha}(Y \times I)$ vanishes since $Y \times I$ deforms to its boundary and so Wall's theorem implies that $\operatorname{Sign}_{\operatorname{ad}\alpha} N = \operatorname{Sign}_{\operatorname{ad}\alpha} M_X$.

The adjoint representation $\operatorname{ad}\alpha\colon \pi_1 M_X \to U(3)$ splits into three $U(1)$ representations sending the generator to $e^{2i\theta}$, $e^{-2i\theta}$, and 1. Thus the signature $\operatorname{Sign}_{\operatorname{ad}\alpha} M_X$ is the sum

$$\operatorname{Sign}_{\operatorname{ad}\alpha} M_X = -\operatorname{Sign} B_K(e^{2i\theta}) - \operatorname{Sign} B_K(e^{-2i\theta}) - \operatorname{Sign} B_K(1).$$

(the minus signs arise because of our choice of orientations. The orientation which N inherits from D^4 has Z_0 in its boundary. Since we want $\partial N = -Z_0 + Z_1$, we must give it the opposite orientation.) Now $\operatorname{Sign} B_K(1) = 0$, and since $B_K(t^{-1}) = B_K(t)^T$, the signatures $\operatorname{Sign} B_K(e^{2i\theta})$ and $\operatorname{Sign} B_K(e^{-2i\theta})$ are equal. Therefore

$$\operatorname{Sign}_{\operatorname{ad}\alpha} M_X = -2 \operatorname{Sign} B_K(e^{wi\theta})$$

as claimed. □

As a consequence of this theorem we can compare the $\rho_{\operatorname{ad}\alpha}$ invariants of Z_0 and Z_1. In fact,

$$\begin{aligned}\rho_{\operatorname{ad}\alpha}(Z_1) - \rho_{\operatorname{ad}\alpha}(Z_0) &= 3 \operatorname{Sign} N - \operatorname{Sign}_{\operatorname{ad}\alpha} N \\ &= 2 \operatorname{Sign} B_K(e^{2i\theta}).\end{aligned}$$

THEOREM 4.4. *Let α_0 and α_1 be $SU(2)$ representations of $\pi_1 Z_0$ which are abelian on the knot complement X_0. By conjugating we may assume that*

$$\alpha_j(\mu) = \begin{pmatrix} e^{i\theta_j} & \\ & e^{-i\theta_j} \end{pmatrix} \qquad \text{for } j = 0, 1,$$

with the $\theta_j \in [0, \pi]$. If α_j is non-central let $a_j = \operatorname{Dim} \operatorname{Ker} B_K(e^{2i\theta_j})$, so that $H^1(X_0; \operatorname{ad}\alpha_j) \cong \mathbf{R}^{1+2a_j}$. If α_j is central, let $a_j = 0$.

Then the difference of spectral flows

$$SF(\alpha_0, \alpha_1)(Z_0) - SF(\alpha_0, \alpha_1)(Z_1)$$

is equal to:

$$\operatorname{Sign} B_K(e^{2i\theta_0}) - \operatorname{Sign} B_K(e^{2i\theta_1}) - (a_0 + a_1).$$

If $e^{2i\theta_j}$ is not a root of the Alexander polynomial of K for $j = 0$ and 1 then this difference is even.

Proof. We use the formula of §2 which relates the spectral flow to the Chern–Simons invariants, the $\rho_{\mathrm{ad}\,\alpha}$ invariants, and the h_{α_j}. We first dispose of the h_{α_j}.

By computing with the Mayer–Vietoris sequences of (Z_0, X_0, Y) and $(Z_1, X_1, Y) = (Z_1, S^1 \times D^2, Y)$ with $\mathrm{ad}\,\alpha_j$ coefficients one can easily show that $H^0(Z_0, \mathrm{ad}\,\alpha_j) \cong H^0(Z_1; \mathrm{ad}\,\alpha_j)$. If the restriction of α_j to X_1 is trivial or if $e^{2i\theta_j}$ is not a root of the Alexander polynomial of K, then $H^1(X_0; \mathrm{ad}\,\alpha_j) \cong H^1(X_1; \mathrm{ad}\,\alpha_j)$. It follows that $H^1(Z_0; \mathrm{ad}\,\alpha_j) \cong H^1(Z_1; \mathrm{ad}\,\alpha_j)$. The only case in which the first cohomology groups do not agree is when $e^{2i\theta_j}$ is a root of the Alexander polynomial of K, in which case $H^1(X_0; \mathrm{ad}\,\alpha_j) \cong \mathbf{R}^{1+2a_j}$ and $H^1(X_1; \mathrm{ad}\,\alpha_j) \cong \mathbf{R}$. In this case $h_{\alpha_j}(Z_0) - h_{\alpha_j}(Z_1) = 2a_j$. So

$$-\frac{1}{2}(h_{\alpha_0}(Z_0) + h_{\alpha_1}(Z_0)) + \frac{1}{2}(h_{\alpha_0}(Z_1) + h_{\alpha_1}(Z_1)) = -(a_0 + a_1).$$

Next, we must compare the Chern–Simons invariants. Since the Chern–Simons invariants are flat cobordism invariants, $cs(\alpha_0)(Z_0) = cs(\alpha_0)(Z_1)$. Similarly for α_1.

Putting these facts together with the formula for spectral flow we obtain:

$$\begin{aligned} SF(\alpha_0, \alpha_1)(Z_0) - SF(\alpha_0, \alpha_1)(Z_1) &= \frac{1}{2}((\rho_{\mathrm{ad}\,\alpha_1}(Z_0) - \rho_{\mathrm{ad}\,\alpha_1}(Z_1)) - ((\rho_{\mathrm{ad}\,\alpha_0}(Z_0) \\ &\quad - \rho_{\mathrm{ad}\,\alpha_0}(Z_1)) - (a_0 + a_1) \\ &= \frac{1}{2}(-2\,\mathrm{Sign}\,B_K(e^{2i\theta_1}) + 2\,\mathrm{Sign}\,B_K(e^{2i\theta_0})) \\ &\quad - (a_0 + a_1) \\ &= \mathrm{Sign}\,B_K(e^{2i\theta_0}) - \mathrm{Sign}\,B_K(e^{2i\theta_1}) - (a_0 + a_1). \end{aligned}$$

If $e^{2i\theta_j}$ is not a root of the Alexander polynomial of K and is not equal to 1, then $B_K(e^{2i\theta_j})$ is a non-singular $2g$-dimensional matrix and hence has even signature; moreover $a_j = 0$. If $\theta_j = 0$ then $B_K(1)$ is the zero matrix so its signature is even. □

In the special case of a surgery on a satellite of a knot in S^3, there are two ways to interpret this formula as a splitting result depending on whether we think of the separating torus as the boundary of the satellite or the boundary of the companion. In this setting let W be the complement of the knot in the solid torus, so that Y_0

is a Dehn filling of one of the two boundary components of W. Then if we are given a path of representations of $\pi_1(X_0 \cup W)$ whose restrictions to X_0 are abelian, we can find a path of connections on Z_0 which are flat on $X_0 \cup W$. As above this gives rise to a path of connections on $X_1 \cup Y_0$ which is flat on $X_1 \cup W$, where X_1 is the unknot complement. If we restrict the paths to one of the tori in ∂W, the image $R(X_0 \cup W) \to R(T)$ coincides with the image $R(X_1 \cup W) \to R(T)$ *for either torus* T in ∂W. In particular, restricting to $\partial(X_0 \cup W)$ and $\partial(X_1 \cup W)$ gives examples of paths of representations of two knots in S^3 whose image in the pillowcase coincide, which are non-abelian in general (see the examples in §6), but for which the spectral flows (and Floer homology) are different. In particular, Yoshida's theorem fails to extend to this case.

The results of this section can be generalized to include surgeries of satellites of knots in arbitrary homology spheres Σ by replacing the pair (D^4, S^3) by (M, Σ) where M is a 4-manifold bounded by Σ. The correction term will then involve the signature of M as well as the Alexander matrix.

5. Splitting the spectral flow for graph manifolds

In this section we consider homology spheres $Z = X \cup Y$ where X and Y are Seifert fibered homology knot complements. For simplicity we will assume that X and Y are the complements of a tubular neighborhood of a regular fiber in a Seifert-fibered homology sphere. This restriction is not essential but makes some of the formulas less messy. Nor is it essential to take Z to be a homology sphere. Finally, one can do the computation for any graph manifold, that is, any 3 manifold obtained by gluing together Seifert-fibered spaces along tori in their boundary.

We will give a "splitting theorem" for the spectral flow between two connections whose restrictions to X and Y are irreducible. This theorem expresses the spectral flow as the sum of 3 terms F_X, F_Y, and F_ϕ where F_X (resp. F_Y) depends only on the restriction of α to X (resp. Y) and F_ϕ is an interaction term involving the gluing map $\phi : \partial X \to \partial Y$.

This section closely parallels the computations of [FS2] for Seifert fibered homology spheres, in particular the starting point is the observation of Fintushel and Stern that Seifert fibered manifolds bound canonical 4-manifolds over which $SO(3)$ representations extend. Although our emphasis is different, the methods are similar. We refer the reader to their beautiful papers [FS2] and [FS1] for details.

Seifert fibered manifolds are characterized by the property that their fundamental groups have a cyclic center. Since the centralizer of any non-abelian subgroup of $SU(2)$ is just ± 1, it follows that any irreducible representation of the fundamental group of a Seifert fibered 3 manifold must send the generator of the center to ± 1, and hence the adjoint representation sends this element to 1.

Given a Seifert fibered manifold $X = S(F; (a_1, b_1), \ldots, (a_m, b_m))$, the mapping cylinder of the Seifert fibration $X \to F$ is a singular 4-manifold whose singularities are cones on $L(a_i, b_i)$. Removing neighborhoods of the singularities leaves a 4 manifold M_X whose boundary is the union $L_X = \bigcup_i L(a_i, b_i)$ together with the Dehn filling of X which caps off the generic fibers in ∂X.

The fundamental group of M is the quotient of $\pi_1 X$ by its center. In particular, if $\alpha : \pi_1 X \to SU(2)$ is an irreducible representation then ad α extends over M_X. This gives a canonical flat 4-manifold with X in its boundary.

Now let X be the complement of a regular fiber in a Seifert fibered homology sphere. So $X = S(D^2; (a_1, b_1), \ldots, (a_m, b_m))$ and

$$\pi_1 X = \langle x_1, \ldots, x_m, h \mid h \text{ central}, x_i^{a_i} h^{b_i} = 1 \rangle.$$

The center of $\pi_1 X$ is the cyclic subgroup generated by the generic fiber h.

We assume X has been given some fixed orientation.

Let $\alpha : \pi_1 X \to SU(2)$ be an irreducible representation and let θ denote the trivial $SU(2)$ representation. Both of these send the homotopy class of the regular fiber of X to $\pm 1 \in SU(2)$.

From the presentation of $\pi_1(X)$ we see that α must take each x_i to a $2a_i^{th}$ root of 1. Following [FS1], we can unambiguously define the *rotation numbers* of α to be the collection of integers p_i, $i = 1, \ldots, m$ so that $0 \le p_i \le a_i$ and $\alpha(x_i)$ is congugate to $\exp(2\pi i p_i / 2a_i)$ in $SU(2)$. To α we associate the integer $m_\alpha(X)$ which is the number of x_i which are not sent to ± 1 by α; equivalently the number of p_i strictly between 0 and a_i.

The manifold M_X has boundary $X \cup (T^2 \times I) \cup (S^1 \times D^2) \amalg L_X$. We orient M_X so that $-X \subset \partial M_X$. Thus we view M_X as a rel boundary cobordism from X to $S^1 \times D^2 + L_X$. The representation ad α extends over $\pi_1 M_X$.

LEMMA 5.1.
(1) $H^1(M_X; \mathbf{R}) = 0 = H^2(M_X; \mathbf{R})$.
(2) $H^1(X; \text{ad}\,\alpha) \cong H^1(M_X; \text{ad}\,\alpha) \cong \mathbf{R}^{2m_\alpha(X) - 3}$ *and* $H^2(M_X; \text{ad}\,\alpha) = 0$.

Proof.
(1) Let W denote the mapping cylinder of the Seifert fibration. Since X fibers over D^2, W is contractible. By excision, $H^n(W, M_X; \mathbf{R}) = H^n(cL_X, L_X; \mathbf{R})$, which is zero for $n = 2$ or 3. The exact sequence for the pair (W, M_X) shows that $H^1(M_X; \mathbf{R}) = 0 = H^2(M_X; \mathbf{R})$ since W is contractible.

(2) Let P be the complement of m small open discs in D^2 centered at the images (under the Seifert fibration) of the singular fibers in X. We decompose M_X as:

$$M_X = (P \times D^2) \cup \coprod_{i=1}^{m} \left(D^2 \times S^1 \times \left[0, \frac{1}{2}\right]\right)_i. \qquad (*)$$

In this decomposition, each $D^2 \times S^1 \times \{0\}$ corresponds to a neighborhood of a singular fiber in X. For each point $p \in D^2 \times S^1$, the arc $p \times [0, \frac{1}{2}]$ corresponds to half of the mapping cylinder arc emanating from p. Define

$$\left(T^2 \times \left[0, \frac{1}{2}\right]\right)_i = (P \times D^2) \cap \left(D^2 \times S^1 \times \left[0, \frac{1}{2}\right]\right)_i.$$

In what follows, we will repeatedly use that fact that if Q is a CW-complex and $\rho : \pi_1 Q \to \mathrm{Aut}\,(V)$ is a homomorphism, then for $i = 0, 1$,

$$H^i(Q; V_\rho) \cong H^i(\pi_1 Q; V_\rho)$$

where the latter denotes group cohomology.

We have the presentation

$$\pi_1 M_X = \langle x_1, \ldots, x_m \mid x_i^{a_i} = 1 \text{ for } i = 1, \ldots, m\rangle.$$

We compute $H^1(\pi_1 M_X; \mathrm{ad}\,\alpha)$ using the usual bar resolution as follows. A 1-cocycle $\sigma : \pi_1 M_X \to su(2)$ is determined by its values on the generators $\{x_1, \ldots, x_m\}$. Using the cocycle condition on the relations implies that these values must satisfy the equations

$$0 = \sigma(x_i^{a_i}) = (1 + \mathrm{ad}\,\alpha(x_i) + \cdots + \mathrm{ad}\,\alpha(x_i^{a_i - 1})) \cdot \sigma(x_i)$$

If $\alpha(x_i) = \pm 1$, this implies $\sigma(x_i) = 0$. If $\alpha(x_i) \neq \pm 1$, this implies $\sigma(x_i) \in \mathbf{R}^2 =$ the orthogonal complement in $su(2)$ of the one-dimensional subspace fixed pointwise by $\mathrm{ad}\,\alpha(x_i)$.

It follows that $Z^1(\pi_1 M_X; \mathrm{ad}\,\alpha) \cong \mathbf{R}^{2m_\alpha(X)}$. Since α and hence $\mathrm{ad}\,\alpha$ are irreducible, $B^1(\pi_1 M_X; \mathrm{ad}\,\alpha) \cong \mathbf{R}^3$, so $H^1(M_X; \mathrm{ad}\,\alpha) \cong \mathbf{R}^{2m_\alpha(X) - 3}$.

Similar (but easier) computations give:

$$H^1\left(\left(S^1 \times D^2 \times \left[0, \frac{1}{2}\right]\right)_i; \mathrm{ad}\,\alpha\right) \cong \begin{cases} \mathbf{R}^3 & \text{if } \alpha(x_i) = \pm 1 \\ \mathbf{R} & \text{if } \alpha(x_i) \neq \pm 1 \end{cases}$$

$$H^1(T_i^2; \mathrm{ad}\,\alpha) \cong \begin{cases} \mathbf{R}^6 & \text{if } \alpha(x_i) = \pm 1 \\ \mathbf{R}^2 & \text{if } \alpha(x_i) \neq \pm 1 \end{cases}$$

Also, $H^0((S^1 \times D^2)_i; \mathrm{ad}\,\alpha) \to H^0(T_i^2; \mathrm{ad}\,\alpha)$ is surjective since $\mathrm{im}\,(\alpha \mid \pi_1(S^1 \times D^2)_i) = \mathrm{im}\,(\alpha \mid \pi_1 T_i^2)$. Clearly,

$$H^2(P^2 \times D^2; \mathrm{ad}\,\alpha) = H^2((S^1 \times D^2)_i; \mathrm{ad}\,\alpha) = 0$$

since these spaces are homotopy equivalent to 1-complexes. Plugging this data into the Mayer–Vietoris sequence for the decomposition (*) above implies that $H^2(M_X; \mathrm{ad}\,\alpha) = 0$.

Finally, to see that $H^1(\pi_1 X; \mathrm{ad}\,\alpha) \cong H^1(\pi_1 M_X; \mathrm{ad}\,\alpha)$, it suffices to check that if σ is a 1-cocycle on $\pi_1 X$, then the relations imply that $\sigma(h) = 0$. □

Now let $Y = S(D^2; (c_1, d_1), \ldots, (c_n, d_n))$ be the complement of the regular fiber in the Seifert fibered homology sphere. Assume Y has been given a fixed orientation and let M_Y, L_Y be defined as for X. As before,

$$\pi_1 Y = \langle y_1, \ldots, y_n, k \mid k \text{ central}, y_i^{c_i} k^{d_i} = 1 \rangle.$$

We glue X to Y using an orientation reversing map $\phi : \partial X \to \partial Y$ to form the oriented homology sphere Z_ϕ. We can then glue M_X to M_Y along $T \times I$ using $\phi \times I$ to obtain the 4-manifold $N_\phi = M_X \bigcup_\phi M_Y$. The oriented boundary of N_ϕ is $-Z_\phi + L_X + L_Y + L_\phi$, where L_ϕ is the lens space obtained by gluing two solid tori together along T so that h bounds in one and k bounds in the other. (So for example, if Z_ϕ is itself Seifert fibered, then $\phi(h) = k^{\pm 1}$ so that $L_\phi \cong S^1 \times S^2$. Another easy case is when $\phi(h)$ and k intersect in one point in which case $L_\phi \cong S^3$.)

Let $\alpha : \pi_1 Z_\phi \to SU(2)$ be a representation whose restrictions α_X and α_Y to X and Y are irreducible. So $\mathrm{ad}\,\alpha$ extends to $\pi_1 N_\phi$. We denote by p_i the rotation numbers of α_X and by q_i the rotation numbers of α_Y. We also define $m_\alpha(X)$, $m_\alpha(Y)$ as before.

LEMMA 5.2.

(1) $H^2(N_\phi; \mathbf{R}) \cong \mathbf{R}^2$.

(2) *If Z_ϕ is not Seifert-fibered, then* $H^2(N_\phi; \mathrm{ad}\,\alpha) = 0$.

(3) *If Z_ϕ is Seifert fibered, then the* 4*-manifold obtained by gluing $S^1 \times D^3$ to N_ϕ along $L_\phi = S^1 \times S^2$ is just M_Z (i.e. the mapping cyclinder with the cone points removed).*

Proof.

1. Using the Mayer–Vietoris sequence for $N_\phi = M_X \cup M_Y$ and the previous lemma we see that $H^1(T; \mathbf{R}) \to H^2(N_\phi; \mathbf{R})$ is an isomorphism.

2. With $\mathrm{ad}\,\alpha$ coefficients, the Mayer–Vietoris sequence is:

$$\longrightarrow H^1(M_X; \mathrm{ad}\,\alpha) \oplus H^1(M_Y; \mathrm{ad}\,\alpha) \longrightarrow H^1(T; \mathrm{ad}\,\alpha) \longrightarrow H^2(N_\phi; \mathrm{ad}\,\alpha) \longrightarrow 0.$$

Suppose first that the restriction of α to $\pi_1 T$ is not central. Then $H^1(T; \mathrm{ad}\,\alpha) \cong \mathbf{R}^2$. By conjugating α we may assume that $\alpha(\pi_1 T)$ lies in the circle subgroup of diagonal matrices. The image $H^1(M_X; \mathrm{ad}\,\alpha) \to H^1(T; \mathrm{ad}\,\alpha)$ is the same as $H^1(X; \mathrm{ad}\,\alpha) \to H^1(T; \mathrm{ad}\,\alpha)$, which is half-dimensional. Thus it suffices to show that the images $H^1(X; \mathrm{ad}\,\alpha) \to H^1(T; \mathrm{ad}\,\alpha)$ and $H^1(Y; \mathrm{ad}\,\alpha) \to H^1(T; \mathrm{ad}\,\alpha)$ are transverse.

Let $z \in Z^1(\pi_1 T; \mathrm{ad}\,\alpha)$ be a cocycle which extends to z_X and z_Y on $\pi_1 X$ and $\pi_1 Y$. By subtracting a coboundary we may assume $z(x) \in i\mathbf{R}$ for $x \in \pi_1 T$, where we identify $su(2)$ with the pure quaternions. Since α is diagonal on the fundamental group of T, the adjoint action on z is trivial and so z is just an ordinary homomorphism from $\mathbf{Z}^2 \to i\mathbf{R}$. We have $z(h) = z_X(h) = 0$ and $z(k) = z_Y(k) = 0$ since z extends over X and Y. But since Z_ϕ is not Seifert fibered $h \neq k^{\pm 1}$ and so z vanishes on a (rational) basis which implies $z = 0$.

This argument modifies to handle the case when α restricts to a central representation of $\pi_1 T$ since in that case the adjoint action is trivial. Thus a cocycle $z \in Z^1(\pi_1(T); \mathrm{ad}\,\alpha)$ is just a homomorphism $\pi_1(T) \to \mathbf{R}^3$. Again this is trivial if it vanishes on h and k.

3. We leave this to the reader. Notice that if Z_ϕ is Seifert fibered, then $H^2(N_\phi; \mathrm{ad}\,\alpha)$ is 1-dimensional. However, by gluing $S^1 \times D^3$ to the boundary we obtain M_Z with $H^2(M_Z; \mathrm{ad}\,\alpha) = 0$. □

We will temporarily assume that Z_ϕ is not Seifert fibered. Since $H^2(N_\phi; \mathrm{ad}\,\alpha) = 0$, $\mathrm{Sign}_{\mathrm{ad}\,\alpha} N_\phi = 0$. Computing the ordinary signature of N_ϕ is a bit trickier, and depends on the choice of orientations. We digress momentarily to discuss the orientation point.

First note that a Seifert fibered homology sphere $\Sigma(a_1, \ldots, a_m)$ has a canonical orientation as the link of a complex singularity [JN]. If M_Σ denotes the deleted mapping cyclinder of $\Sigma \to S^2$ then $H^2(M_\Sigma; \mathbf{R}) \cong \mathbf{R}$ and the orientation which makes M_Σ negative definite gives Σ the correct boundary orientation. We are assuming $X = \Sigma(a_1, \ldots, a_m)$-regular fiber, and $Y = \Sigma(c_1, \ldots, c_n)$-regular fiber, but we are not assuming the orientations are compatible. So define ε_X to be ± 1 according to whether or not the fixed orientation of X agrees with the induced orientation as a subspace of $\Sigma(a_1, \ldots, a_m)$ and similarly define ε_Y. We have oriented M_X so that $\partial M_X = -X + L_X$ (rel boundary) where L_X is the union of $L(a_i, b_i)$. These lens spaces have a canonical orientation, namely the one induced by the covering $S^3 \to L(a, b)$ where S^3 is oriented as the boundary of the unit ball in $\mathbf{C}^2$. With this orientation, $L_X = \varepsilon_X(\amalg_i L(a_i, b_i))$.

At this point we can express the gluing map ϕ in coordinates as follows. The pair of curves $m_X = x_1, \ldots, x_m$, $l_X = h^{\varepsilon_X}$ form an oriented symplectic basis for

$\pi_1(\partial X)$. Similarly $m_Y = y_1, \ldots, y_n$ and $l_Y = k^{\varepsilon_Y}$. (The l_X are not longitudes in the usual sense; they are not nullhomologous in X and Y.) Then ϕ can be written as a 2×2 integer matrix with determinant -1 using these bases. Write:

$$\phi = \begin{pmatrix} u & v \\ w & z \end{pmatrix},$$

then $L_\phi = L(v, z)$, and Z_ϕ is a homology sphere if and only if

$$au + \varepsilon_X v - c\varepsilon_Y(aw + \varepsilon_X z) = \pm 1$$

where $a = \Pi_i a_i$ and $c = \Pi_i c_i$.

LEMMA 5.3. *The signature of* N_ϕ *is equal to* $\varepsilon_X + \varepsilon_Y$.

Proof. Recall that we have oriented N_ϕ so that $-Z_\phi \subset \partial N_\phi$. We have seen that $H^2(N_\phi; \mathbf{R}) \cong \mathbf{R}^2$. It is convenient to split N_ϕ into $N_1 = M_X \cup (Y \times I)$ and $N_2 = (S^1 \times D^2) \cup M_y$ as in Figure 3.

Then $H^2(N_1; \mathbf{R}) \cong \mathbf{R} \cong H^2(N_2; \mathbf{R})$ and since N_ϕ is the union of N_1 and N_2 along a closed submanifold of their boundary, Sign N_ϕ = Sign N_1 + Sign N_2. That Sign $N_1 = \varepsilon_X$ is a consequence of the fact proven in [FS1] that the mapping cylinder M_Σ for $\Sigma(a_1, \ldots, a_m)$ oriented with $\partial M_\Sigma = -\Sigma(a_1, \ldots, a_m)$ is positive definite. □

The following lemma is easily proven using the Atiyah–Singer G-signature theorem and the formula from [APS2] which expresses the ρ_β invariants as the "Fourier transforms" of the G-signature defects.

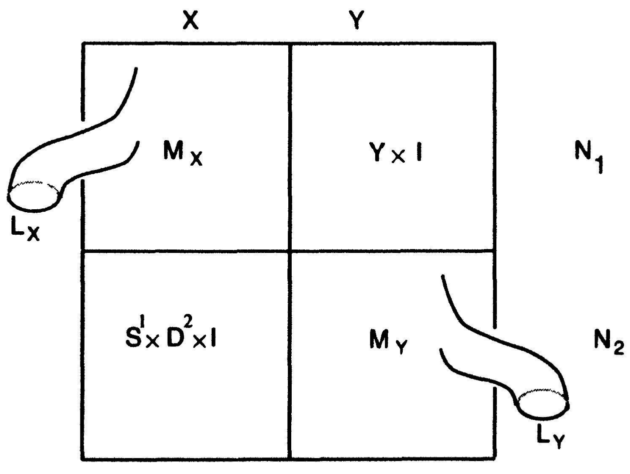


Figure 3

LEMMA 5.4. *Let $L(a, b)$ be a lens space, oriented as the quotient of $S^3 = \partial B^4 \subset B^4 \subset \mathbf{C}^2$. Let $\beta : \pi_1 L(a, b) \to U(1)$ be a representation which takes the generator g to $e^{(2\pi i p/a)}$. Then*:

$$\rho_\beta(L(a, b)) = -\frac{2}{a}\sum_{k=1}^{a-1} \cot\left(\frac{\pi bk}{a}\right)\cos\left(\frac{\pi k}{a}\right)\sin^2\left(\frac{\pi pk}{a}\right).$$

Let $\beta : \pi_1 L(a, b) \to SO(3)$ be a representation. Orient $L(a, b)$ as the quotient of $S^3 = \partial B^4 \subset B^4 \subset \mathbf{C}^2$ where the action is $\xi_a \cdot (z, w) = (\xi_a^b z, \xi_a w)$. Suppose $\beta(\xi_a)$ is a rotation of angle $2p\pi/a \neq 0$. Then the complexification splits into three $U(1)$ representations with rotation numbers $2p$, $-2p$ and 0. Thus

$$\rho_\beta = -\frac{4}{a}\sum_{k=1}^{a-1} \cot\left(\frac{\pi bk}{a}\right)\cot\left(\frac{\pi k}{a}\right)\sin^2\left(\frac{\pi pk}{a}\right).$$

LEMMA 5.5. *Let $\beta : \pi_1 L(a, b) \to SO(3)$ be a representation with rotation number p as above. Let r be an inverse for b* Mod *a. Then*

$$-\frac{2rp^2}{a} + \frac{1}{2}\rho_\beta(L(a, b))$$

is an integer.

Proof. Let $E \to L(a, b)$ denote the flat bundle defined by β. Since $H_3(BSO(3); \mathbf{Z}) = 0$, this bundle extends over some 4-manifold W. The Pontriagin form of a connection on this bundle extending β then defines the $SO(3)$ Chern–Simons invariant much as in the $SU(2)$ case. Using the Atiyah–Patodi–Singer theorem (as in the final section) we see that the index of the self-duality operator on W is congruent mod $\mathbf{Z}$ to -2 times this $SO(3)$ Chern–Simons invariant plus $1/2\beta_\beta(L(a, b))$. The computation of the Chern–Simons invariant may be done in several ways: by directly writing down the integral as in §13 in [MMR], or via the method of [KK3]. For the reader's convenience we sketch the latter. Notice that the bundle over either of the two solid tori S_1, S_2 which make up $L(a, b)$ is trivial. So given a flat connection on $L(a, b)$, there are paths of flat connections on the S_i to the trivial $SO(3)$ connections. Use these paths to construct a connection on

$$L(a, b) \times I = (L(a, b) \times I) \cup_{L(a,b) \times 0} (S_1 \times I) \cup_{L(a,b) \times 0} (S_2 \times I)$$

extending the given flat connection. As in [KK3], Chern–Simons invariant may be computed as a sum of two integrals which reduce via Stokes' theorem to integrals on $T^2 \times I$. The result of the calculation is rp^2/a, yielding the formula in the lemma. □

REMARK. In the case when the representation β lifts to an $SU(2)$ representation, then we can take p even and $-2rp^2/a$ is just 8 times the Chern–Simons invariant of the $SU(2)$ connection. The only subtlety occurs when the bundle over $L(a, b)$ is non-trivial. This can happen only when a is even.

Let X, Y, α, ϕ be as above. Define

$$e_X = \sum_i \frac{ap_i}{a_i}$$

and

$$e_Y = \sum_i \frac{cq_i}{c_i}$$

where $a = \Pi_i a_i$ and $c = \Pi_i c_i$.

The next lemma appears as Theorem 4.5 of [KK3].

LEMMA 5.6. *The Chern–Simons invariant of* (Z_ϕ, α) *is equal to*

$$-\varepsilon_X \frac{e_X^2}{4a} - \varepsilon_Y \frac{e_Y^2}{4c} - \frac{p_\phi^2 u}{4v} - \frac{w\kappa^2}{4}(2p_\phi + z)$$

where κ *is defined by* $\alpha(k) = (-1)^\kappa$. □

The basic idea is to compute the Chern–Simons integral separately on X and Y by using an explicit path of flat connections from the given connection to the trivial connection for each piece. One then applies Stokes theorem and the definition of Chern–Simons invariants as the integral of a 4-form over $X \times I$. See [KK3] for the detailed proof.

A few brief remarks are in order. The first two terms in this formula are the analogues of the Chern–Simons invariants of Seifert-fibered homology spheres. In particular, a representation of the homology sphere $\Sigma(a_1, \ldots, a_m)$, with rotation numbers p_i has Chern–Simons invariant $e_X^2/4a$. The third term is a Chern–Simons invariant of the extra lens space L_ϕ, at least when the $SO(3)$ representation lifts to an $SU(2)$ representation.

The last term can be considered a correction term in the following way. The flat $SO(3)$ cobordism N_ϕ shows that the $SO(3)$ Chern–Simons invariants of Z_ϕ equal those of the union of lens spaces $L_X \cup L_Y \cup L_\phi$. However, one cannot conclude that the $SU(2)$ Chern–Simons invariants coincide, firstly because some of the lens spaces may have even order and so the $SO(3)$ representations do not lift to $SU(2)$, but also because even if they do, the flat cobordism need not lift to an $SU(2)$ flat cobordism. The best one can say at this point is that these Chern–Simons invariants are equal mod $\frac{1}{4}\mathbf{Z}$. However, we need the result mod $\mathbf{Z}$. Notice that this last term has denominator equal to 4. This brings to light the technical point that one loses too much information using the $SO(3)$ cobordism to reach conclusions about the mod 8 $SU(2)$ spectral flow. This $SO(3)$ cobordism could at best give the spectral flow mod 2.

We can now prove the main result of this section.

THEOREM 5.7. *Let $\alpha : \pi_1(Z_\phi) \to SU(2)$ be a representation of the graph manifold $Z_\phi = X \cup_\phi Y$ whose restriction to each piece is irreducible. Assume Z_ϕ is not Seifert fibered. Choose $p_\phi \in \{0, \dots, v\}$ so that $\alpha(y_1 \cdots y_n)$ is conjugate to* $\exp(2\pi i(p_\phi/2v))$. *Choose κ so that $\alpha(k) = (-1)^\kappa$. Finally let $C = 2$ if the restriction of α to $\pi_1 T$ is non-central and 3 if it is central. Then*:

$$\begin{aligned} SF(\theta, \alpha; Z_\phi) \equiv &-\varepsilon_X\left(\frac{2e_X^2}{a} + \sum_{i=1}^{m} \frac{2}{a_i} \sum_{k=1}^{a_i-1} \cot\left(\frac{\pi b_i k}{a_i}\right) \cot\left(\frac{\pi k}{a_i}\right) \sin^2\left(\frac{pp_i k}{a_i}\right)\right) - m_X(\alpha) \\ &-\varepsilon_Y\left(\frac{2e_Y^2}{c} + \sum_{i=1}^{n} \frac{2}{c_i} \sum_{k=1}^{c_i-1} \cot\left(\frac{\pi d_i k}{c_i}\right) \cot\left(\frac{\pi k}{c_i}\right) \sin^2\left(\frac{\pi q_i k}{c_i}\right)\right) - m_Y(\alpha) \\ &-(2wk(2p_\phi + z) + \frac{2p_\phi^2 u}{v} - \frac{2}{v} \sum_{k=1}^{v-1} \cot\left(\frac{\pi z k}{v}\right) \cot\left(\frac{\pi k}{v}\right) \sin^2\left(\frac{\pi p_\phi k}{v}\right)) \\ &-\frac{3}{2}(\varepsilon_X + \varepsilon_Y) + C \pmod 8. \end{aligned}$$

Proof. Suppose first that α is non-central. Then $H^0(T; \operatorname{ad}\alpha) \cong \mathbf{R}$ and $H^1(T; \operatorname{ad}\alpha) \cong \mathbf{R}^2$. Since Z_ϕ is not Seifert fibered, it follows that $H^1(X; \operatorname{ad}\alpha) \oplus H^1(Y; \operatorname{ad}\alpha) \to H^1(T; \operatorname{ad}\alpha)$ is onto. From the Mayer–Vietoris sequence one then has

$$\frac{1}{2}(h_\alpha(Z_\phi) + h_\theta(Z_\phi)) = m_X(\alpha) + m_Y(\alpha) - 2.$$

If α restricts to a central representation of $\pi_1(T)$, then similarly:

$$\frac{1}{2}(h_\alpha(Z_\phi) + h_\theta(Z_\phi)) = m_X(\alpha) + m_Y(\alpha) - 3.$$

The $\rho_{\mathrm{ad}\,\alpha}$ invariants are computed using the fact that

$$\rho_{\mathrm{ad}\,\alpha}(Z_\phi) = \rho_{\mathrm{ad}\,\alpha}(L_X) + \rho_{\mathrm{ad}\,\alpha}(L_Y) + \rho_{\mathrm{ad}\,\alpha}(L_\phi) - 3\,\mathrm{Sign}(N_\phi) + \mathrm{Sign}_{\mathrm{ad}\,\alpha}(N_\phi).$$

The terms are computed in Lemmas 5.2 and 5.4.

The Chern–Simons invariant is given by Lemma 5.6. The result follows using equation 2.1. □

REMARKS.

1. This theorem expresses the spectral flow as a sum of three parts. The first part involves only X, the second only Y and the third depends only on the map ϕ, the restriction of α to $\pi_1 T$, and the orientations.
2. This formula generalizes the result of [FS2] for Seifert fibered homology spheres. Their result is:

$$\mathrm{SF}(\theta, \alpha; \Sigma(a_1, \ldots, a_n)) = -3 - R(\alpha)$$

where

$$R(\alpha) = \frac{2e^2}{a} - 3 + m + \sum_{i=1}^{m} \frac{2}{a_i} \sum_{k=1}^{a_i - 1} \cot\left(\frac{\pi b_i k}{a_i}\right) \cot\left(\frac{\pi k}{a_i}\right) \sin^2\left(\frac{\pi p_i k}{a_i}\right).$$

3. We point out the following discrepancy in the literature. From a Kirby calculus argument it is easy to see that $\Sigma(2, 3, 5) = -(+1$ surgery on the right handed trefoil). Thus Casson's invariant, as defined in [AM], of $\Sigma(2, 3, 5)$ must equal -1. On the other hand, Taubes in [T] defines the spectral flow $SF(\theta, \alpha)$ with the same convention that we do here and takes his invariant to be the sum over the irreducible representations of the spectral flow mod 2. Since $R(\alpha)$ is always odd by [FS2], it follows that Taubes' invariant is actually equal to -2 times Casson's invariant. This arises because in order to make the surgery formula for Casson's invariant work out with respect to the correct normalization of the Alexander polynomial, there is an extra factor of -1 introduced in the definition of Casson's invariant (the S on pp. 125 of [AM] is equal to -1).

6. Examples

The results of the previous two sections can be used to compute the spectral flows of representations of certain homology spheres by reducing the computation to previously known cases. This enables us to compute the Floer chain complexes and in certain cases the Floer homology of these spaces. We will consider -1 surgeries on k-twisted Whitehead doubles of torus knots. This class of examples lends itself well to the methods developed above because they are graph manifolds, namely the exterior of the p, q torus knot union the exterior of the left-handed trefoil. We remark that one can also consider ± 1 surgery on the positive or negative clasp Whitehead doubles of torus knots. In this case one also needs the computations of spectral flow carried out in [Y2] for the surgeries on the Figure 8. For background on representation spaces of knot groups see [K1] and [K2].

We will carry out three computations explicitly. The first is -1 surgery on the 10-twisted Whitehead double of the 5, 2 torus knot. This homology sphere has the property that every non-trivial representation restricts to an abelian representation of the companion knot. Thus all spectral flows can be computed using Theorem 4.4. By comparison with the representations of the 10-twisted Whitehead double of the unknot, this example shows that the spectral flow is not determined by the image in the pillowcase. For this homology sphere we can compute the Floer *homology* since all boundary operators turn out to be trivial.

We will also compute the spectral flows and Floer chain groups for -1 surgery on the untwisted and 5-twisted Whitehead doubles of the trefoil. For the untwisted example, every non-trivial representation restricts to an irreducible representation of the companion, so that the spectral flows are all computed using Theorem 5.7. The example involving the 5-twisted double has non-trivial representations of both types, so the full computation uses both Theorems 4.4 and 5.7.

One technical point will arise when trying to compute the Floer chain complex, namely one must perturb the Chern–Simons function if it is not a Morse function. In general, the representation space $R(\pi_1 Z)$ forms the critical points of the Chern–Simons function. If $R(Z)$ is a smooth submanifold of $\mathcal{A}/\mathcal{G}$ with nondegenerate normal bundle then the Floer chain complex can be constructed by taking as a basis the critical points of a Morse function f on $R(Z)$ and assigning to a critical point p the index $SF(\theta, \alpha(p)) + \mu_p(f)$ where $\alpha(p) \in R(Z)$ lies in the component containing p and $\mu_p(f)$ is the index of f at p. See [FS2] for the proof that this gives a complex whose homology is Floer homology.

We begin with some general observations. Let $p, q > 0$ and choose $k \in \mathbf{Z}$. The three sphere S^3 can be Seifert fibered so that the (p, q) torus knot is a regular fiber. Let $\Sigma_{p,q}(k)$ denote the homology 3-sphere obtained by performing -1 surgery on the k-twisted positive clasp Whitehead double of the (p, q) torus knot. See Figure 4 for $(p, q) = (2, 3), k = 3$.

So $\Sigma_{p,q}(k)$ is the union of the exterior X_0 of the (p, q) torus knot and a Dehn filling Y_0 of the Whitehead link complement.

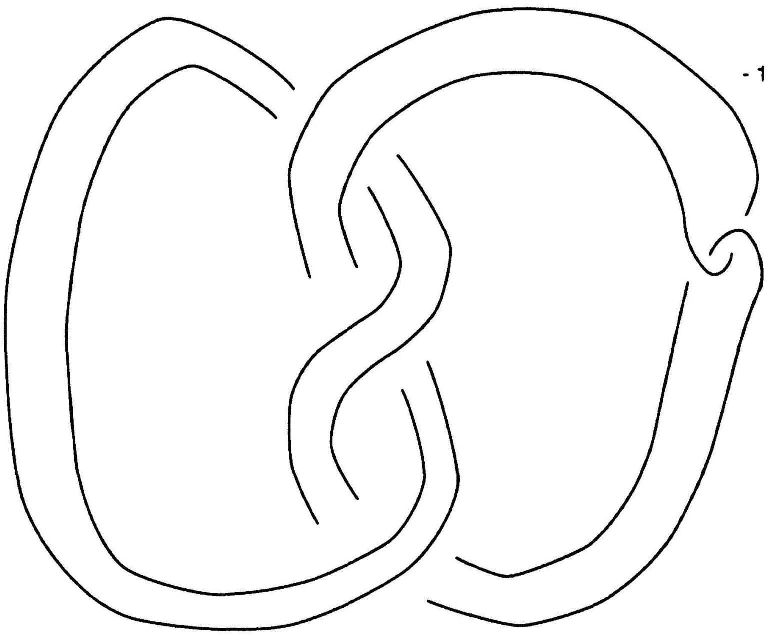


Figure 4

PROPOSITION 6.1. *The homology sphere $\Sigma_{p,q}(k)$ is a graph manifold. In fact X_0 in the exterior of the p, q torus knot and Y_0 is the exterior of the left-handed trefoil knot. Furthermore, the gluing map ϕ is given in the natural meridian-longitude coordinates by the matrix*:

$$\phi = \begin{pmatrix} 0 & 1 \\ 1 & -k \end{pmatrix}.$$

(*This means that $\phi(\mu_X) = \lambda_Y$ and $\phi(\lambda_X) = \mu_Y - k\lambda_Y$.*)

Proof. Figure 5 shows a Kirby calculus computation which shows that Y_0 is the exterior of the left-handed trefoil.

Taking a k-twisted double means that if μ_W and λ_W are a meridian and longitude pair for the Whitehead link exterior which is being glued to X_0 along T,

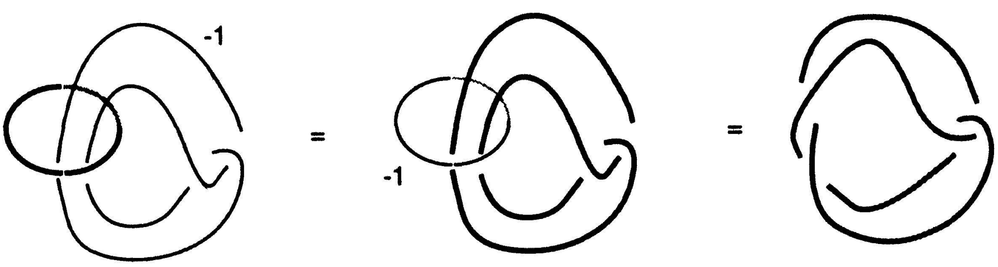


Figure 5

then μ_X is identified with λ_W and λ_X is glued to $\mu_W - k\lambda_W$. But if we do -1 surgery on the other component of the Whitehead link, then μ_W and λ_W become a meridian-longitude pair for the left-handed trefoil. □

Now the regular fiber for the Seifert fibration of X_0 (restricted to T) is

$$k_X = pq\mu_X + \lambda_X + \lambda_X = \mu_Y + (pq - k)\lambda_Y.$$

The regular fiber for the Seifert fibration of Y_0 (restricted to T) is

$$h_Y = -6\mu_Y + \lambda_Y.$$

Thus if L_ϕ denotes the lens space obtained by capping off the regular fibers in T,

$$L_\phi = -L(-6(pq-k) - 1, pq - k).$$

Write $Z_\phi = \Sigma_{p,q}(k) = X_0 \cup_\phi Y_0$. Notice that if the Seifert fibrations were compatible then $\phi(h_X) = h_Y^{\pm 1}$. But this obviously cannot happen. One might wonder if Z_0 is Seifert-fibered in some other way, but this is not possible since if it was the incompressible torus T separating X_0 from Y_0 would be horizontal. But then $Z_\phi - T$ is a union of I-bundles. However, we know that Y_0 is not an I-bundle. (For most Z_ϕ we could instead use Lemma 6.3 below which shows that the $SU(2)$ representation space of $\pi_1 Z_\phi$ contains circles, and thus Z_ϕ cannot be Seifert fibered by the main theorem of [KK1].)

We first give a convenient set of coordinates for the pillowcase. If $\gamma_1, \gamma_2 \in \pi_1 T$ is a pair of curves which generate $\pi_1 T$, then the map

$$\mathbf{R}^2 \to R(T)$$

which takes the pair (x, y) to the conjugacy class of the representation

$$\gamma_1 \mapsto \begin{pmatrix} e^{\pi i x} & \\ & e^{-\pi i x} \end{pmatrix}, \quad \gamma_2 \mapsto \begin{pmatrix} e^{\pi i y} & \\ & e^{-\pi i y} \end{pmatrix}$$

is a branched cover. A fundamental domain for the action is the strip $[0, 2] \times [0, 1]$. The pillowcase is then seen as the identification space of this strip by folding it in two along the segment $1 \times [0, 1]$ and identifying the edges.

LEMMA 6.2. *If $\alpha : \pi_1 Z_\phi \to SU(2)$ is a non-trivial representation, then the restriction of α to $\pi_1 Y$ is always irreducible.*

Proof. Suppose the restriction of α to Y_0 is abelian. Then in particular the restriction of α to the Whitehead link complement is abelian. But then α must take the longitude $\lambda_W = \lambda_Y$ to 1 since λ_Y lies in the commutator subgroup. Since $\phi(\mu_X) = \lambda_W$, α sends the meridian of X to 1. But the meridian normally generates $\pi_1 X_0$ and so α restricts to the trivial representation of $\pi_1 X_0$. Thus α is in fact abelian and since Z_0 is a homology sphere α must be the trivial representation. □

For a space A, let $R^*(A)$ denote the space of conjugacy classes of non-trivial representations of $\pi_1 A$ into $SU(2)$. Write:

$$R^*(Z_\phi) = R_R \cup R_I$$

according to whether $\alpha \in R^*(Z_\phi)$ restricts to a reducible or irreducible representation of $\pi_1(X_0)$.

If $\alpha \in R_R$ then the construction of §4 provides us with a flat cobordism N from Z_ϕ to Z_1, where Z_1 is obtained by replacing the exterior of K by the exterior of the unknot. In fact, Z_1 is just the manifold obtained from $-1/k$ surgery on the left-handed trefoil knot. Thus

$$Z_1 = \begin{cases} -\Sigma(2,3,6k-1) & \text{if } k > 0: \\ \Sigma(2,3,1-6k) & \text{if } k < 0;. \\ S^3 & \text{if } k = 0 \end{cases}$$

The representation spaces and spectral flows were computed for these manifolds by [FS2]. (It is also easy to compute the spectral flow between two connections using Yoshida's theorem [Y2] since the representation space of the trefoil is connected.) Using Theorem 4.4 we can compute the spectral flow for any $\alpha \in R_R$.

On the other hand, if $\alpha \in R_I$ then by the previous lemma the restriction of α to both X_0 and Y_0 is irreducible. Thus the construction of §5 applies and we obtain a flat cobordism N from Z_ϕ to $L_X + L_Y + L_\phi$. Theorem 5.7 then gives the spectral flow in this case.

LEMMA 6.3. *The space R_R is discrete. If $\alpha(\mu_X) = e^{i\theta}$ and $e^{2i\theta}$ is not a root of the Alexander polynomial of the p, q torus knot then α is a non-degenerate representation, i.e. $H^1(Z_\phi; \mathrm{ad}\,\alpha) = 0$.*

The space R_I is a union of smooth, non-degenerate circles; in particular if $\alpha \in R_I$ then $H^1(Z_\phi; \mathrm{ad}\,\alpha) \cong \mathbf{R}$.

Proof. Suppose $\alpha \in R_R$. Such representations are in $1-1$ correspondence with representations of $\pi_1 Z_1$. Since Z_1 is a Seifert-fibered homology sphere with 3 exceptional fibers, its representation space is discrete. It follows from the previous lemma that the restriction of α to $\pi_1 T$ is non-central. Thus $H^0(T; \mathrm{ad}\,\alpha) \cong \mathbf{R}$ and $H^1(T; \mathrm{ad}\,\alpha) \cong \mathbf{R}^2$. The images of $H^1(X_0; \mathrm{ad}\,\alpha) \to H^1(T; \mathrm{ad}\,\alpha)$ and $H^1(Y_0; \mathrm{ad}\,\alpha) \to H^1(T; \mathrm{ad}\,\alpha)$ are lines, which must be transverse: If they were tangent, then there would be an arc of representations in R_R passing through α. (This is not true for general knots; it holds for torus knots because (cf. [KK3]) the image of their representation spaces in the pillowcase are straight lines.)

Now $H^0(X_0; \mathrm{ad}\,\alpha) \to H^0(T; \mathrm{ad}\,\alpha) \cong \mathbf{R}$ is an isomorphism, $H^1(X_0; \mathrm{ad}\,\alpha) \cong \mathbf{R}$ (this is where we use the hypothesis on the Alexander polynomial), and $H^1(Y_0; \mathrm{ad}\,\alpha) \cong \mathbf{R}$ (This can be computed directly using group cohomology. Alternatively the space of conjugacy classes of irreducible representations of a torus knot is a smooth 1-manifold, see [K1].) The result now follows from the Mayer–Vietoris sequence.

Now suppose $\alpha \in R_I$. Again the restriction of α to $\pi_1 T$ is non-central. So $H^0(T; \mathrm{ad}\,\alpha) \cong \mathbf{R}$ and $H^1(T; \mathrm{ad}\,\alpha) \cong \mathbf{R}^2$. As shown in the proof of Lemma 5.2 the images of $H^1(X_0; \mathrm{ad}\,\alpha) \to H^1(T; \mathrm{ad}\,\alpha)$ and $H^1(T; \mathrm{ad}\,\alpha) \to H^1(T; \mathrm{ad}\,\alpha)$ are transverse lines. As above $H^1(X_0; \mathrm{ad}\,\alpha) \cong \mathbf{R} \cong H^1(Y_0; \mathrm{ad}\,\alpha)$. The Mayer–Vietoris sequence now implies that the boundary map $H^0(T; \mathrm{ad}\,\alpha) \to H^1(Z_\phi; \mathrm{ad}\,\alpha)$ is an isomorphism. Thus the Zariski tangent space to R_I at α is 1 dimensional. By "bending" the representation along T we see that α can be deformed into a 1-parameter family. Thus R_I is smooth and since it is compact it must consist entirely of circles. □

REMARK. One interesting (and well-known) consequence of this fact and the results of the previous sections, together with the perturbation argument is that Casson's invariant of $\Sigma_{p,q}(k)$ is the same as Casson's invariant of Z_1. This can of course be checked directly using Casson's surgery formula. Alternatively, the circles in the previous theorem give rise to pairs of generators of the Floer chain complex which lie in adjacent dimensions using the perturbation of [FS2]. Therefore, only the points of R_R contribute to the euler characteristic of the Floer chain complex, which equals -2 times Casson's invariant. Each point contributes with the same sign as the corresponding point in $R(Z_1)$ using Theorem 4.4. However, we will see in our examples that although the parity of the spectral flow of these points does not change, its mod 8 refinement does.

In the following example we use Theorem 4.4 to compute the Floer homology of $\Sigma_{5,2}(10)$, the homology sphere obtained by performing -1 surgery on the 10-twisted double of the $(5, 2)$ torus knot.

THEOREM 6.4. *The Floer homology of* $\Sigma_{5,2}(10)$ *is* $\mathbf{Z}^5$ *in each even dimension,* 0 *in each odd dimension.*

We begin by proving the following lemma.

LEMMA 6.5. *Each representation* $\alpha : \pi(\Sigma_{5,2}(10)) \to SU(2)$ *is abelian when restricted to* $\pi_1(X_0)$.

Proof. Suppose that α is non-abelian on $\pi_1(X_0)$. Recall that X_0 is Seifert fibered with regular fiber $h = \lambda_X \mu_X^{10}$. It follows that $\alpha(\mu_Y) = \alpha(\lambda_X \mu_X^{10}) = \pm 1$ in $SU(2)$; hence α is abelian on $\pi_1(Y_0)$. Then $\alpha(\mu_X) = \alpha(\lambda_Y) = 1$, which contradicts the hypothesis since μ_X normally generates $\pi_1(X_0)$. □

To be consistent with the notation of the previous sections, we write $\Sigma_{5,2}(10) = Z_0$. Lemma 6.5 implies that there is a natural one-to-one correspondence between $R(Z_0)$ and $R(Z_1)$, where we define $Z_1 = (S^1 \times D^2) \cup_\partial Y_0$, with a meridian of $S^1 \times D^2$ being glued to $\mu_Y \lambda_Y^{-10}$; in other words Z_1 is the result of $(-1/10)$-surgery on the left-handed trefoil. We will now calculate the elements of $R(Z_1)$ and their spectral flow invariants, and then use Theorem 4.4 to calculate the spectral flow invariants of the corresponding elements of $R(Z_0)$.

Recall (see, for example, [K1]) that the set of irreducible representations of $\pi_1(Y_0)$ mod conjugacy (which we will denote by $R_i(Y_0)$) is a single arc of the form $\{\alpha_\beta\}_{1/6 < \beta < 5/6}$, where

$$\alpha_\beta(\mu_Y) = \begin{pmatrix} e^{i\pi\beta} & \\ & e^{-i\pi\beta} \end{pmatrix}.$$

Since Y_0 is Seifert fibered with regular fiber $\lambda_Y \mu_Y^{-6}$, it follows that $\alpha_\beta(\lambda_Y) = -\alpha_\beta(\mu_Y)^6$. (Here we are using the fact that these representations take the regular fiber to -1.) Hence the image of $R_i(Y_0)$ in the pillowcase is the arc of slope 6 shown in Figure 6.

Clearly $\alpha : \pi_1(Y_0) \to SU(2)$ factors through to give a representation of Z_1 if and only if $\alpha(\mu^Y \lambda_Y^{-10}) = 1$. This condition may be visualized in the pillowcase by intersecting the image of $R_i(Y_0)$ with the line $\mathscr{L}$ of slope 1/10, also shown in Figure 6. It follows that $R(Z_1) = \{\alpha_\beta\}$ for $\beta = 2j/59$ where $5 \le j \le 24$. From now on we write α_j to denote $\alpha_{2j/59}$. Using the formula of [FS2] or the technique of Yoshida, we compute the spectral flow $SF(\theta, \alpha_j)(Z_1)$ for each of these representations of $Z_1 = \Sigma(2, 3, 59)$, and assemble the following table:

n	**generators of $\mathbf{FH_n(Z_1)}$**
0	$\alpha_5, \alpha_7, \alpha_9, \alpha_{16}, \alpha_{18}$
2	$\alpha_{10}, \alpha_{12}, \alpha_{14}, \alpha_{21}, \alpha_{23}$
4	$\alpha_6, \alpha_8, \alpha_{15}, \alpha_{17}, \alpha_{19}$
6	$\alpha_{11}, \alpha_{13}, \alpha_{22}, \alpha_{24}$

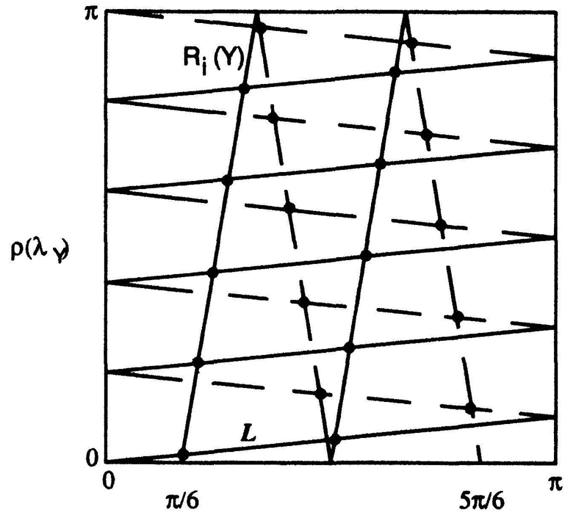


Figure 6

Next, we compute the adjustments to these spectral flows required to compute $FH_*(\Sigma_{5,2}(10))$ using Theorem 4.4. The Seifert matrix of the (5, 2)-torus knot with respect to an obvious genus two Seifert surface is easily computed to be

$$V = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

The four eigenvalues of the matrix $B_K(t) = (1-t)V + (1-t^{-1})V^T$ are given by

$$\sqrt{2(1-\cos\theta)}\,\{-\sqrt{2(1-\cos\theta)} \pm \sqrt{(3\pm\sqrt{5})/2}\},$$

where $t = e^{i\theta}$ and the two $\pm$ signs are taken in all four possible combinations. In Figure 7 we graph these eigenvalues as functions of θ.

For any value of θ, we can read off Sign $B_K(e^{i\theta})$ directly from this graph. Note that the values of θ at which this signature changes are $\pi/5$ and $3\pi/5$. Recalling that $\alpha_\beta(\mu_X) = \alpha_\beta(\lambda_Y) = -\alpha_\beta(\mu_Y)^6$

$$= \begin{pmatrix} e^{(6\beta-1)\pi i} & \\ & e^{-(6\beta-1)\pi i} \end{pmatrix},$$

we find (by Theorem 4.4) that $SF(\theta, \alpha_j)(Z_0) - SF(\theta, \alpha_j)(Z_1) = -\text{Sign}\, B_K(e^{2(6\cdot(2j/59)-1)\pi i})$

$$= \begin{cases} 0 & \text{for } j = 5, 10, 15, 20 \\ 2 & \text{for } j = 6, 9, 11, 14, 16, 19, 21, 24 \\ 4 & \text{for } j = 7, 8, 12, 13, 17, 18, 22, 23 \end{cases}$$

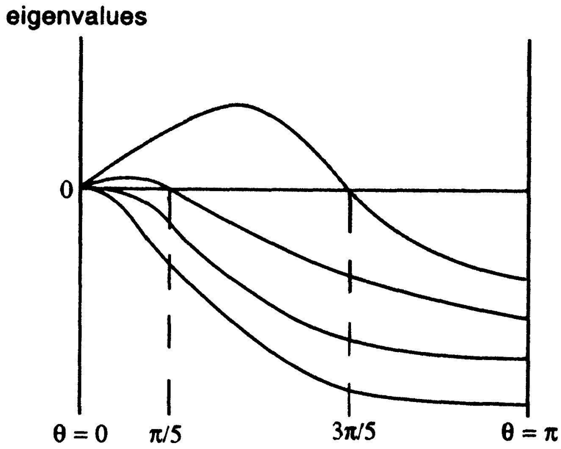


Figure 7

In summary, for each n, of the five generators of $FH_n(Z_1)$ one remains a generator of $FH_n(\Sigma_{5,2}(10))$, two become generators of $FH_{n+2}(\Sigma_{5,2}(10))$, and two become generators of $FH_{n+4}(\Sigma_{5,2}(10))$. The theorem follows. □

We now compute the spectral flow for -1 surgery on the untwisted double of the right handed trefoil. The representation space has 8 components; each is a smooth circle. This manifold is a graph manifold obtained by gluing a right handed trefoil complement to a left handed trefoil complement, identifying meridians with longitudes and vice versa. In Figure 8 we have drawn the image in the pillowcase of the representation spaces of X and Y. The coordinates used are μ_Y and λ_Y, the

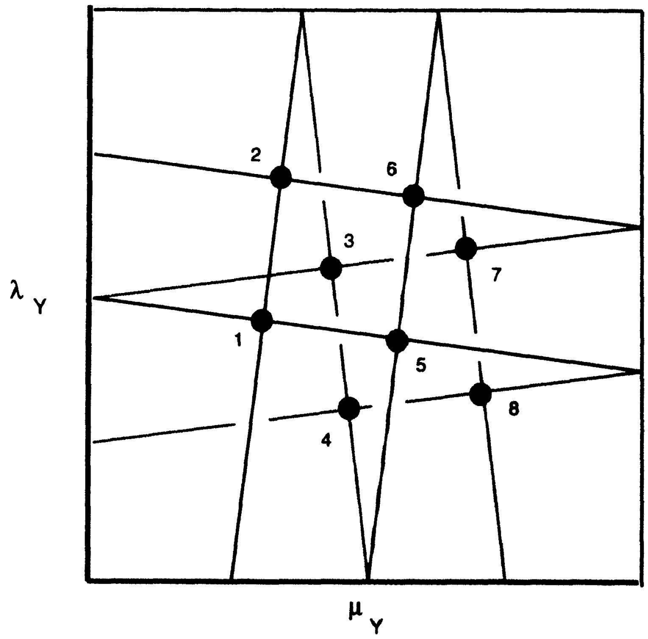


Figure 8

meridian and longitude of Y. The representation space of a trefoil complement consists of an arc of abelian representations (sending the longitude to 1) and an arc of non-abelian representations limiting at the endpoints on an abelian representation; see [K1]. From Figure 8 we see that there are exactly 8 components of the representation space of $\Sigma_{2,3}(0)$, and each representation restricts to a non-abelian representation of both X and Y. By Lemma 6.3, each component is a circle. We label the components α_i, $i = 1, \ldots, 8$.

We will apply Theorem 5.7 and so we need the various quantities appearing in the formula. First we need the gluing map ϕ in terms of the curves $m_X = \mu_X$, $l_X = h$, $m_Y = \mu_Y$, and $l_Y = k$. Using the relationship between the meridians, longitudes, and the regular fibers we obtain:

$$\phi = \begin{pmatrix} u & v \\ w & z \end{pmatrix} = \begin{pmatrix} 6 & 37 \\ 1 & 6 \end{pmatrix}.$$

Since $X = -Y$ we have $\varepsilon_X = 1$, $\varepsilon_Y = 1$, $\varepsilon_Y = -1$. The rotation numbers for the restrictions of each representation to X or Y are $(1, 1)$. (These numbers are independent of the representation since the space of irreducible representations of the trefoil is connected. They are easy to compute for any particular representation. See for example [KK1].) For each representation the regular fiber is sent to -1, so that $\kappa = 1$. Each representation is non-central when restricted to the torus since none of the eight points lie on a corner of the pillowcase, so that the terms C are equal to 2 for each representation. The terms $m_X(\alpha)$ and $m_Y(\alpha)$ are equal to 2. It remains to compute the rotation numbers for each of the eight representations and then apply the formula of Theorem 5.7. These are easily computed from Figure 8. Notice that most of the terms in the formula of Theorem 5.7 cancel since $X = -Y$.

n	$\mathbf{p}_\phi(\alpha_n)$	$SF(\theta, \alpha_n)$
1	9	1
2	11	5
3	15	7
4	17	3
5	21	5
6	23	1
7	27	3
8	29	7

The Floer chain complex can be obtained from this by perturbing the circles. We obtain:

$$FC_* = (\mathbf{Z}^2, \mathbf{Z}^2, \mathbf{Z}^2, \mathbf{Z}^2, \mathbf{Z}^2, \mathbf{Z}^2, \mathbf{Z}^2, \mathbf{Z}^2).$$

Computing the Floer homology is of course harder, since one must compute the boundary operators.

We next show how to compute the spectral flows and the Floer chain complex of -1 surgery on the 5-twisted positive clasped Whitehead double of the right-handed trefoil.

In Figure 9 we have drawn the image of the representaiton spaces of $\pi_1 X_0$ and $\pi_1 Y_0$ in the pillowcase $R(T)$ using the coordinates μ_Y, λ_Y.

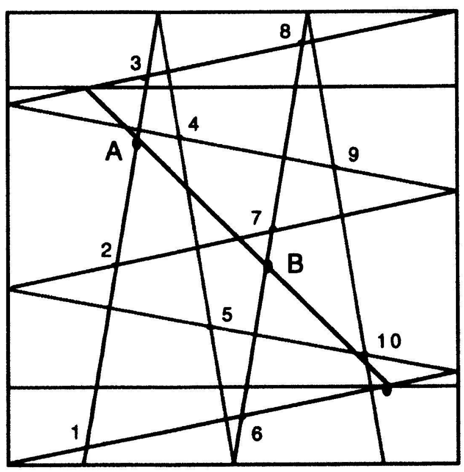
The lines drawn on $R(X)$ are the images of the restriction maps $R(X) \to R(T)$ and $R(Y) \to R(T)$. As in the previous example $R(X)$ and $R(Y)$ are constructed from two intervals: one interval consists entirely of abelian representations and the other interval consists of irreducible representations except for its endpoints which are abelian.

We can glue together representations of X to Y provided they agree on T. From the figure we see that there are 10 isolated, nondegenerate representations whose restriction to X are abelian; these are labeled 1–10. So R_R consists of 10 points. To compute their spectral flow we may use the method of §4.

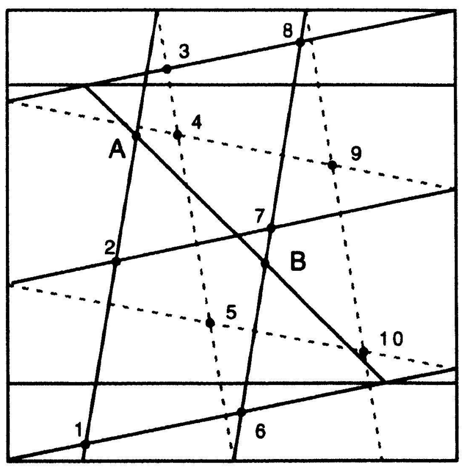
There are two circles in R_I corresponding to where the image of the arc of irreducible representation of X intersects the arc of irreducible representations of Y. Their image in $R(T)$ are the two points labeled A and B. The two horizontal lines delineate where the matrix $B_K(e^{2i\theta})$ has kernel; they correspond to the roots of the Alexander polynomial of the trefoil $e^{\pi i/3}$ and $e^{-\pi i/3}$.

We first deal with the representations 1–10. We see that the representations labeled 2, 4, 5, 7, 9, and 10 lie in the region where the signature of $B_K(e^{2i\theta})$ is -2. The others lie in the region where this signature is 0.

Let S_i be the spectral flow from the trivial representation to the ith representation on $Z_1 = \Sigma(2, 3, 29)$. Then $S_1 = 4$, $S_2 = 0$, $S_3 = 4$, $S_4 = 2$, $S_5 = 6$, $S_6 = 0$, $S_7 = 4$,



(a)



(b)

Figure 9

$S_8 = 0$, $S_9 = 6$, $S_{10} = 2$. One way to see this is to compute using the formula of Fintushel and Stern (see the second remark following Theorem 5.7). However, there is a much easier way to compute this using the main result of Yoshida's paper [Y2]. The following algorithm enables us to compute the spectral flow between any two representations of any homology sphere surgery of the trefoil. Let a and b be representations and let γ be the loop in the pillowcase made up of the path from a to b in the space of irreducible representations of the trefoil followed by the path from b to a of (abelian) representations of the surgery solid torus. Then the spectral flow from a to b is equal (Mod 8) to 2 times the number of corners contained in the region in $R(T) = S^2$ to the right of the curve. In Figure 10 we show $SF(1, 2) = 4$, $SF(3, 4) = 6$, $SF(5, 6) = 2$ for the manifold obtained by $-1/5$ surgery on the left handed trefoil.

We now use Theorem 4.4 to relate the spectral flow $SF(\theta, i)(\Sigma(2, 3, 29))$ to $SF(\theta, i)(Z_\phi)$. If T_i denotes the spectral flow from θ to the ith representation on $Z_\phi = \Sigma_{2,3}(10)$, then by Theorem 4.4 $T_i = S_i - 2$ if $i = 2, 3, 5, 7, 9$, or 10, otherwise $T_i = S_i$. So we get the table:

n	$SF(\theta, \alpha_n)$
1	4
2	6
3	2
4	2
5	4
6	0
7	2
8	0
9	4
10	0

Thus the part of the Floer complex corresponding to the representations in R_R is

$$(\mathbf{Z}^3, 0, \mathbf{Z}^2, 0, \mathbf{Z}^3, 0, \mathbf{Z}^2, 0).$$

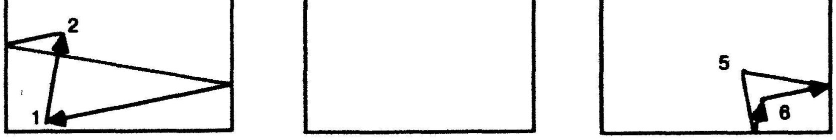


Figure 10

We next consider the two circles A and B. Since $X_0 = -Y_0$, it follows that $\varepsilon_X = -\varepsilon_Y$. Moreover, the rotation numbers for X_0 equal those for Y_0, since the space of irreducible representations of a trefoil (right or left handed) is connected. (In fact the rotation numbers are (1,1).)

The numbers $m_X(\alpha)$ and $m_Y(\alpha)$ are equal to 2, for $\alpha = A$ or B. Applying Theorem 5.7 we see that most of the terms cancel since rotation numbers for each side coincide by $\varepsilon_X = -\varepsilon_Y$. The gluing matrix is given as

$$\phi = \begin{pmatrix} u & v \\ w & z \end{pmatrix} = \begin{pmatrix} 6 & 7 \\ 1 & 1 \end{pmatrix}$$

in the coordinates x_1x_2, h and y_1y_2, k.

Thus $L_\phi = -L(7, 1)$. An examination of Figure 10 above shows that the rotation number for A is 2 and the rotation number for B is 4. Using Proposition 5.5 one computes:

$$\begin{aligned} SF(\theta, \alpha_A) &= -2 - 2 - 2(2 \cdot 2 + 1) - \frac{2 \cdot 4 \cdot 6}{7} + \frac{2}{7} \sum_{k=1}^{6} \cot^2\left(\frac{\pi k}{7}\right) \sin^2\left(\frac{2\pi k}{7}\right) + 2 \\ &= -15 \equiv 1 \pmod 8. \end{aligned}$$

Similarly

$$SF(\theta, \alpha_B) = 5 \pmod 8.$$

Thus we can perturb this non-degenerate critical level to conclude that the Floer chain complex of -1 surgery on the 5 twisted positive clasp Whitehead double of the right handed trefoil is:

$$(\mathbf{Z}^3, \mathbf{Z}, \mathbf{Z}^3, 0, \mathbf{Z}^3, \mathbf{Z}, \mathbf{Z}^3, 0).$$

Again, the boundary operators must be understood in order to compute the Floer homology.

7. The spectral flow formula

In this section we derive formula 2.1 for spectral flow in terms of the ρ_α and Chern–Simons invariants.

Let X be an oriented 4-manifold and let $Y = \partial X$ oriented using the convention "outward normal first". Give X the product metric near Y and let u be the inward coordinate, normalized so that $\|du\| = 1$. So near its boundary, $d\,\mathrm{vol}_X = d\,\mathrm{vol}_Y \wedge du$.

Write d for the exterior derivative on X and use δ on Y to distinguish it from d. We will suppress the notation indicating a connection in an auxiliary bundle but you should think of d and δ as covariant derivatives with respect to a connection.

For any connection on a bundle E, we have the self-adjoint operator

$$D : \Omega^0_Y \oplus \Omega^1_Y \to \Omega^0_Y \oplus \Omega^1_Y$$

defined by $(a, b) \mapsto (\delta^* b, *\delta b + \delta a)$, acting on $\operatorname{ad}(E)$-valued forms. We take $\eta_D(s)$ to be the η-invariant of D.

Consider first the signature operator $d + d^* : \Omega^+_X \to \Omega^-_X$. In [APS2] it is shown that (with respect to a $U(n)$ representation α)

$$\operatorname{Sign}_\alpha X = n \int_X L - \eta_{B^e_\alpha}(0)$$

where

$$B^e : \Omega^0_Y \oplus \Omega^2_Y \to \Omega^0_Y \oplus \Omega^2_Y$$

is the twisted signature operator defined on bundle-valued forms by

$$B^e(a, b) = (- *\delta b, \delta * b + *\delta a).$$

(This operator is sometimes written $(-1)^{p/2}(*\delta - \delta *)$.)

Notice that the spectrum of D associated to a representation β is equal to the spectrum of the operator B^e associated to $\operatorname{ad}(\beta)$, and so in the signature formula we can replace the η invariant of B^e by the η invariant of D. In particular, for a flat $SU(2)$ connection b with holonomy β we see that

$$\eta_{D_b}(0) - \eta_{D_\theta}(0) = 3 \operatorname{Sign} X - \operatorname{Sign}_\beta X = \rho_{\operatorname{ad}\beta}(Y).$$

We next want to relate $\eta_D(0)$ to the self-duality operator. So let

$$S : \Omega^1 \to \Omega^0 \oplus \Omega^2_-$$

be the self-duality operator on X defined by $\omega \mapsto (d^*\omega, P_-(d\omega))$ where P_- denotes projection onto the anti-self-dual 2-forms.

Consider the bundle isomorphisms:

$$\Phi : \Lambda^0_Y \oplus \Lambda^1_Y \to \Lambda^1_{X|Y}$$

given by $\Phi(a, b) = a\,du + b$ and

$$\Psi : \Lambda^0_Y \oplus \Lambda^1_Y \to \Lambda^0_{X|Y} \oplus \Lambda^2_{X|Y}$$

defined by $\Psi(a, b) = (-a, P_-(b\,du))$.

Then a careful computation reveals that near the boundary

$$S \circ \Phi = \Psi\left(C + \frac{\partial}{\partial u}\right)$$

where $C : \Omega^0_Y \oplus \Omega^1_Y \to \Omega^0_Y \oplus \Omega^1_Y$ is defined by $C(a, b) = (-\delta^* b, *\delta b - \delta a)$. The spectrum of C equals that of D and so the Atiyah–Patodi–Singer theorem says that

$$\text{Index } S = \int_X \hat{A}(X)ch(V_-)ch(\text{ad } E) - \frac{1}{2}(\eta_D(0) + \dim \text{Ker } D)$$

where S is given the global boundary conditions

$$\xi_{|Y} \in \text{Span}\,\{\phi_\lambda \mid C\phi_\lambda = \lambda\phi_\lambda, \lambda < 0\}.$$

We now apply this formula to $X = Z \times I$ where Z is an oriented rational homology sphere and $Z \times I$ is oriented as $(\mathcal{O}_Z, dt)$. (Notice that with respect to this orientation $\partial(Z \times I) = Z \times 0 - Z \times 1$.)

THEOREM 7.1. *Let a and b be connections on Z. Choose a path a_t, $t \in I$ joining a to b and let A be the corresponding connection on $Z \times I$. Let $SF(a, b)$ denote the spectral flow of the family of operators D_{a_t}. Then*:

$$SF(a, b) = \text{Index } S_A$$

where the index is taken with respect to the global boundary conditions of [APS1].

We sketch the proof:

Divide the spectrum of D_t into a finite part F_t and its complement G_t continuously with respect to t so that all eigenvalues which pass through 0 lie in F_t. Let $f_t(s)$ and $g_t(s)$ be the corresponding eta-invariants so that $\eta_{D_t}(s) = f_t(s) + g_t(s)$.

Let $h_i = \dim \ker D_i$, $i = 0, 1$. Then it is easy to see that

$$SF(a, b) = \frac{1}{2}(f_1(0) - f_0(0) - h_1 - h_0)$$

with respect to the convention described in section 2. Therefore, the spectral flow equals

$$\frac{1}{2}(\eta_{D_1}(0) - \eta_{D_0}(0)) - \frac{1}{2}(h_1 + h_0) - \frac{1}{2}(g_1(0) - g_0(0)).$$

Our orientation convention implies that $\partial(Z \times I) = Z \times 0 - Z \times 1$. Therefore,

$$\text{Index } S_A = \int_{Z \times I} \alpha_0(x) - \frac{1}{2}(\eta_{D_0}(0) - \eta_{D_1}(0) + h_0 + h_1).$$

where $\alpha_0(x) = \hat{A}(Z \times I)ch(V_-)ch(\text{ad } A)$. We can then write:

$$\text{Index } S_A - SF(a, b) = \int_{Z \times I} \alpha_0(x) + \frac{1}{2}(g_1(0) - g_0(0)).$$

The left side is an integer and the right side is a real number which varies continuously along the path a_t. Since $g_t(0) = g_0(0)$ for t small it follows that the right side is zero and so the spectral flow equals the index. □

Restrict now to an $SU(2)$ connection. The integrand appearing in the index formula can be split up:

$$\hat{A}(Z \times I)ch(V_-)ch(\text{ad } A) = 3\hat{A}(Z \times I)ch(V_-) - 2c_2(\text{ad } A).$$

in this formula $c_2(\text{ad } A)$ means

$$\frac{1}{8\pi^2} Tr(F^{\text{ad } a} \wedge F^{\text{ad } a}).$$

where $F^{\text{ad } a}$ means the curvature of the corresponding connection in the adjoint bundle. In particular, if a and b are $SU(2)$ connections,

$$\int_{Z \times I} c_2(\text{ad } A) = 4(cs(a) - cs(b))$$

by Stokes' theorem.

The other term appearing in the inetgrand, $3\hat{A}(Z \times I)ch(V_-)$ has zero integral. This is most easily seen by considering the spectral flow $SF(\theta, \theta)$ of the trivial connection to itself. By our conventions this is equal to -3, by the previous theorem it equal $3 \int \hat{A}(Z \times I)ch(V_-) - h_\theta$. But $h_\theta = 3$ by the Hodge theorem.

Therefore we obtain the formula:

$$SF(a, b) = -8(cs(a) - cs(b)) - \frac{1}{2}(\eta_{D_a} - \eta_{D_b}) - \frac{1}{2}(\dim \operatorname{Ker} D_a + \dim \operatorname{Ker} D_b).$$

Finally, if a and b are flat connections with holonomy α, β, let $h_\alpha = \dim (H^0(Z; \operatorname{ad} \alpha) + H^1(Z; \operatorname{ad} \alpha))$ and similarly h_β. We then get:

$$SF(\alpha, \beta) = 8(cs(\beta) - cs(\alpha)) + \frac{1}{2}(\rho_{\operatorname{ad} \beta} - \rho_{\operatorname{ad} \alpha}) - \frac{1}{2}(h_\alpha + h_\beta).$$

This is the desired formula.

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