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Manifolds of even dimension with amenable fundamental group

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0. Introduction

0.1. If the fundamental group G of a closed (orientable) 4-manifold X is *infinite and amenable* then the Euler characteristic $\chi(X)$ is ≥ 0. This has been proved in a previous paper [E] using the Følner criterion for amenability [F], in a geometrical version. If X is aspherical, i.e., an Eilenberg-MacLane space $K(G, 1)$ (whence G a Poincaré duality group of dimension 4, in short a PD^4-group) then $\chi(X) = \chi(G) = 0$ by [E], Corollary 2.3.

The main purpose of the present paper is to examine, conversely, 4-manifolds X as above *assuming* $\chi(X) = 0$. We recall (see [E], Section 0.3) that infinite amenable groups G have one or two ends, i.e., $H^1(G; \mathbb{Z}G) = 0$ or $\mathbb{Z}$. It is easily seen that the universal cover $\tilde{X}$ of X has integral homology $H_1(\tilde{X}) = H_4(\tilde{X}) = 0$ and $H_3(\tilde{X}) \cong H^1(G; \mathbb{Z}G)$. We will prove (Theorem 3.4):

(A) *If* $\chi(X) = 0$ *then* $H_2(\tilde{X}) \cong H^2(G; \mathbb{Z}G)$, *the "second end-group" of* G. From this we get the result

(B) *If* $H^1(G; \mathbb{Z}G) = H^2(G; \mathbb{Z}G) = 0$ *then* $\chi(X) = 0$ *implies that* $\tilde{X}$ *is contractible, whence* $X = K(G, 1)$ *and* G *is a* PD^4*-group.*

These statements can be expressed in terms of the Hausmann-Weinberger invariant $q(G)$, see [H-W], for finitely presented groups G (Corollaries 2.5 and 3.6):

(C) *If* G *is infinite amenable then* $q(G)$ *is* ≥ 0. *If* $H^1(G; \mathbb{Z}G) = H^2(G; \mathbb{Z}G) = 0$ *then* $q(G) = 0$ *implies that* G *is a* PD^4*-group.*

In the context of these results it is of interest to look at 2-knot groups G since for these $q(G)$ is always $=0$; see Section 4 below.

0.2. The proofs make use of (reduced and non-reduced) l_2-cohomology of the infinite cell-complex $\tilde{X}$ combined with the free cocompact action of G on $\tilde{X}$. The main tool then is a lemma of Cheeger-Gromov [Ch-G], see Section 2.2. We apply it not only to get the results for $\chi(X) = 0$ but also to give a new proof of the statement $\chi(X) \geq 0$ above. This is done in the more general context of a closed manifold of even dimension $n = 2k \geq 4$ which, if $k > 2$, is *aspherical* up to the middle dimension k; for $n = 4$ there is no asphericity assumption.

These $2k$-manifolds can be used to define a new invariant $\gamma_k(G)$ for groups G of type F_k, $k \geq 2$, generalizing the Hausmann-Weinberger invariant $q(G)$. For G of type F_2 (i.e., finitely presented) one has $\gamma_2(G) = q(G)$.

0.3. Section 1 contains various facts concerning l_2-cohomology of $\tilde{X}$, ordinary cohomology of $\tilde{X}$, and G-cohomology of $\tilde{X}$ for G-module coefficients such as l_2G and $\mathbb{Z}G$. They go a little beyond the minimum necessary for the following sections in view of later use.

0.4. Section 2 deals with $\chi(X) \geq 0$ for the $2k$-manifolds as above and with $\gamma_k(G)$, Section 3 with the vanishing of $\chi(X)$ and the main results. Section 5 is an appendix on the "partial Euler characteristic" of groups G fulfilling certain finiteness conditions; the results appear already in [E] but are given new proofs by the l_2-cohomology methods of the present paper.

0.5. Our results on 4-manifolds should be compared with some of those given by Hillman [H] for the case of "elementary amenable" groups, which constitute a special, but important class of amenable groups. The results of [H] are, however, more general in another sense, namely that G need only have a non-trivial normal subgroup which is elementary amenable.

0.6. Although this paper deals with amenable groups we want to emphasize that the results above on 4-manifolds and the invariant $q(G)$ are valid for other types of groups, in particular for all finitely presented groups with vanishing first l_2-Betti number; see Section 6 below (Addendum).

1. Infinite cell-complexes and l_2-cohomology

1.1. For a cell-complex X with $\pi_1 X = G$ and a G-module A we consider cohomology with local coefficients $H^i(X; A)$; i.e., G-cohomology $H^i_G(\tilde{X}; A)$ of the universal cover, relative to the G-module A (G operates on the cell complex $\tilde{X}$ and on A). A special situation occurs if X is a *finite* complex and G an *infinite* group, with regard to the coefficient modules $\mathbb{Z}G$ and l_2G (the Hilbert space of linear combinations $\Sigma_{x \in G} c_x x$, $c_x \in \mathbb{R}$, with $\Sigma_x c_x^2 < \infty$); G operates on $\mathbb{Z}G$ and on l_2G by left translations.

Namely, one has for the cochains $C^i(\tilde{X}; \mathbb{Z}G) = Hom_G(C_i(\tilde{X}), \mathbb{Z}G)$ and $C^i(\tilde{X}; l_2G) = Hom_G(C_i(\tilde{X}), l_2G)$ the isomorphisms

(1) $C^i(\tilde{X}, \mathbb{Z}G) \cong C^i_{\text{fin}}(\tilde{X}; \mathbb{Z})$,

(2) $C^i(\tilde{X}; l_2G) \cong C^i_{(2)}(\tilde{X}; \mathbb{R})$.

C^i_{fin} is the group of *finite cochains of* $\tilde{X}$, and $C^i_{(2)}$ the group of l_2*-cochains* (functions $f(\sigma_i)$ of the cells σ_i of $\tilde{X}$ with $\Sigma_{\sigma_i} f(\sigma_i)^2 < \infty$). The corresponding cohomology groups are respectively $H^i_{\text{comp}}(\tilde{X}; \mathbb{Z})$, cohomology with compact support; and $H^i_{(2)}(\tilde{X}; \mathbb{R})$, l_2-cohomology of $\tilde{X}$.

1.2. For the convenience of the reader we recall the proof of (1) an (2).

We choose a (finite) $\mathbb{Z}G$-basis $\{\tau_i\}$ of the chain group $C_i(\tilde{X})$ corresponding to the cells of X (one cell in each G-orbit). Given $f \in C^i(\tilde{X}; \mathbb{Z}G) = Hom_G(C_i(\tilde{X}), \mathbb{Z}G)$ we put $g(x\tau_i) = m_{x^{-1}} \in \mathbb{Z}$ where $f(\tau_i) = \Sigma_x m_x x$; clearly g is a finite cochain in $\tilde{X}$. Conversely, given $g \in C^i_{\text{fin}}\tilde{X}; \mathbb{Z})$ we put $f(\tau_i) = \Sigma_x g(x^{-1}\tau_i)x \in \mathbb{Z}G$. The correspondence $f \mapsto g$ yields the isomorphism (1). Note that it is independent of the choice of basis $\{\tau_i\}$: Indeed if we replace τ_i by $y\tau_i, y \in G$, then $g(x\tau_i) = g(xy^{-1}y\tau_i) = m'_{yx^{-1}}$ where $f(y\tau_i) = \Sigma_x m_x yx = \Sigma\, m'_x x$, i.e., $m'_x = m_{y^{-1}x}$; thus $g(x\tau_i) = m'_{yx^{-1}} = m_{x^{-1}}$ as before.

Similarly, given $f \in C^i(\tilde{X}; l_2G)$ we put $g(x\tau_i) = c_{x^{-1}}$ where $f(\tau_i) = \Sigma_x c_x x$ with $\Sigma_x c_x^2 < \infty$. Then

$$\sum_{\text{all } \sigma} g(\sigma)^2 = \sum_{\tau_i} \sum_{x} g(x\tau_i)^2 < \infty,$$

so g is an l_2-cochain. This yields the isomorphism (2). We summarize:

PROPOSITION 1.1. *For a finite cell complex* X (*with infinite fundamental group* G) *the cohomology groups with local coefficients* $H^i(X; \mathbb{Z}G)$ and $H^i(X; l_2G)$ *are isomorphic respectively to* $H^i_{\text{comp}}(\tilde{X}; \mathbb{Z})$ and $H^i_{(2)}(\tilde{X}; \mathbb{R})$ of the universal cover $\tilde{X}$ of X.

Remark. Everything above holds if instead of $\tilde{X}$ we take any free cocompact G-space (=cell complex) Y with $Y/G = X$; G is a factor group of $\pi_1 X$. The isomorphisms are of interest only if G is infinite.

1.3. We will also consider *reduced* l_2-cohomology of $\tilde{X}$, denoted by $\underline{\bar{H}^i(\tilde{X})}$. It differs from $H^i_{(2)}(X; \mathbb{R})$ by $\delta C^{i-1}_{(2)}(\tilde{X}; \mathbb{R})$ being replaced by its l_2-closure $\overline{\delta C^{i-1}_{(2)}}$. It imbeds equivariantly and isometrically in Z^i, the kernel of $\delta : C^i_{(2)} \to C^{i+1}_{(2)}$, and its von Neumann dimension relative to G is denoted by $\bar{\beta}_i(\tilde{X} \text{ rel. } G)$, cf. [Ch-G].

There is an obvious map Φ of $H^i_{(2)}(\tilde{X}; \mathbb{R})$, i.e. the G-cohomology group $H^i_G(\tilde{X}; l_2G)$ based on G-homomorphisms $C_i(\tilde{X}) \to l_2G$, into the *ordinary* cohomology group $H^i(\tilde{X}; l_2G)$ disregarding the G-action on $\tilde{X}$ and l_2G. Under that map Φ the closure of $\delta C^{i-1}(\tilde{X}; l_2G)$ goes to 0. Indeed, the l_2-limit f of a sequence of

i-coboundaries is $=0$ on the i-cycles; it thus defines $\varphi : \partial C_i(\tilde{X}) \to l_2G$ which can be extended to all of C_{i-1} (since l_2G is divisible, i.e. $\mathbb{Z}$-injective), and $\delta\varphi = f$.

PROPOSITION 1.2. *The natural map* $H^i_G(\tilde{X}; l_2G) \to H^i(\tilde{X}; l_2G)$ factors through the reduced l_2-cohomology group $\bar{H}^i(\tilde{X})$.

Of course $H^i(\tilde{X}; l_2G)$ can be regarded as a $\mathbb{Z}G$-module through the action of G on $\tilde{X}$ and on l_2G. The image of Φ lies in the invariant part $H^i(\tilde{X}; l_2G)^G$.

1.4. The map $\Phi : H^n_G(\tilde{X}; l_2G) \to H^n(\tilde{X}; l_2G)^G$ occurs in a well-known exact sequence, available if $\tilde{X}$ is $(n-1)$-connected, i.e., if $\pi_i(X) = 0$ for $1 < i < n$ (deduced from the spectral sequence of the covering $\tilde{X} \to X$):

$$0 \to H^n(G; l_2G) \to H^n_G(\tilde{X}; l_2G) \xrightarrow{\Phi} H^n(\tilde{X}; l_2G)^G \to H^{n+1}(G; l_2G) \to H^{n+1}_G(\tilde{X}; l_2G).$$

There is, of course, an analogous exact sequence for $\mathbb{Z}G$-coefficients. The coefficient map $\mathbb{Z}G \to l_2G$ by inclusion yields, in combination with Proposition 1.1, the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & H^n(G; \mathbb{Z}G) \arrow[r] \arrow[d] & H^n_{\mathrm{comp}}(\tilde{X}; \mathbb{Z}) \arrow[r, "\Phi'"] \arrow[d] & H^n(\tilde{X}; \mathbb{Z}G)^G \arrow[r] \arrow[d, "\Omega"] & H^{n+1}(G; \mathbb{Z}G) \arrow[d] \\ 0 \arrow[r] & H^n(G; l_2G) \arrow[r] & H^n_{(2)}(\tilde{X}; \mathbb{R}) \arrow[r, "\Phi"] & H^n(\tilde{X}; l_2G)^G \arrow[r] & H^{n+1}(G; l_2G) \end{tikzcd} \tag{4}$$

1.5. There is a further natural map $\Psi : H^i_{(2)}(\tilde{X}; \mathbb{R}) \to H^i(\tilde{X}; \mathbb{R})$; it clearly factors through $\bar{H}^i(\tilde{X})$ since the limit of a sequence of l_2-coboundaries is an ordinary coboundary.

1.6. There is an l_2-*homology* analogue of the above statements for l_2-cohomology; we leave it to the reader. We just remark that it is based on the boundary operator $\partial : C^i_{(2)} \to C^{i-1}_{(2)}$ instead of the coboundary $\delta : C^i_{(2)} \to C^{i+1}_{(2)}$; and that the reduced homology groups $\bar{H}_i(\tilde{X})$ are isometrically isomorphic to the $\bar{H}^i(\tilde{X})$ – indeed, they are both isomorphic to the intersection $Z^i(\tilde{X}) \cap Z_i(\tilde{X})$ in $C^i_{(2)}$, where Z^i denotes the cocycle subspace, Z_i the cycle subspace of $C^i_{(2)}$, and $Z^i(\tilde{X}) \cap Z_i(\tilde{X})$ is (a) the orthogonal complement of $\overline{\delta C^{i-1}_{(2)}}$ in Z^i, (b) the orthogonal complement of $\overline{\partial C^{i+1}_{(2)}}$ in Z_i (Hodge-de Rham decomposition of $C^i_{(2)}$). We further remark that this yields a simple proof of l_2-*Poincaré duality* for a closed n-manifold X by using (2) and ordinary Poincaré duality of X; one gets $\bar{H}^i(\tilde{X}) \cong \bar{H}_{n-i}(\tilde{X}) \cong \bar{H}^{n-i}(\tilde{X})$ as Hilbert G-modules.

2. Closed manifolds of dimension $n = 2k$ and an invariant for groups of type F_k

2.1. We take for X a closed orientable (differentiable) n-manifold, $n = 2k \geq 4$ which if $k > 2$ is $(k-1)$-aspherical; i.e., with $\pi_i(X) = 0$ for $1 < i < k$. We assume again $G = \pi_1(X)$ infinite.

We note that $H_i(\tilde{X}) = 0$ for $1 \leq i < k$, and that $H_{2k}(\tilde{X}) = 0$ since G is infinite (if in ordinary homology coefficients are not indicated they are meant to be $\mathbb{Z}$).

PROPOSITION 2.1. *For $k < i \leq 2k$ one has $H_i(\tilde{X}) \cong H^{2k-i}(G; \mathbb{Z}G)$.*

Proof. $H_i(\tilde{X}) \cong H^{2k-i}_{\text{comp}}(\tilde{X}) \cong H^{2k-i}(X; \mathbb{Z}G)$ by Poincaré duality. But since X is $(k-1)$-aspherical $H^i(X; \mathbb{Z}G) \cong H^i(G; \mathbb{Z}G)$ for $0 \leq i < k$. If $n = 2k = 4$, there are no asphericity assumptions, and one simply has $H_3(\tilde{X}) \cong H^1(X; \mathbb{Z}G) \cong H^1(G; \mathbb{Z}G)$.

If the "end-groups" $H^i(G; \mathbb{Z}G)$ are 0 for $0 \leq i < k$ then $H_k(\tilde{X})$ is the only homology group of $\tilde{X}$ which is possibly non-zero. If moreover $H_k(\tilde{X}) = 0$ then $\tilde{X}$ is contractible, X is a $K(G, 1)$, and G is a PD^{2k}-group.

2.2. We now consider the Euler characteristic $\chi(X) = \Sigma_{i=0}^n (-1)^i \alpha_i = \Sigma_{i=0}^n (-1)^i \beta_i(X)$; α_i is the number of i-cells of a cell-decomposition of X, and $\beta_i(X) = \dim_{\mathbb{Q}} H_i(X; \mathbb{Q})$ the i-th Betti number. We recall ([Ch-G] and [E]) that $\chi(X)$ can also be expressed by the reduced Betti numbers $\bar{\beta}_i(\tilde{X} \text{ rel. } G)$ as

$$\chi(X) = \sum_{i=0}^{n} (-1)^i \bar{\beta}_i(\tilde{X} \text{ rel. } G).$$

$\bar{\beta}_i(\tilde{X} \text{ rel. } G)$ is the von Neumann dimension of $\bar{H}^i(\tilde{X})$ considered as a Hilbert G-module.

A lemma of Cheeger-Gromov [Ch-G] tells that if G is amenable then the natural map $\bar{H}^i(\tilde{X}) \to H^i(\tilde{X}; \mathbb{R})$ is *injective*. From our assumptions it follows that $H^i(\tilde{X}; \mathbb{R}) = 0$ for $0 < i < k$ whence $\bar{H}^i(\tilde{X}) = 0$ and $\bar{\beta}_i(\tilde{X} \text{ rel. } G) = 0$ for $0 \leq i < k$ ($\bar{\beta}_0 = 0$ since G is infinite). By Poincaré duality for the $\bar{\beta}_i$ (cf. 1.6, or [L-L], Proposition 4.2) it follows that $\bar{\beta}_i(\tilde{X} \text{ rel. } G) = 0$ for $k < i \leq 2k$. The Euler characteristic can thus be expressed by $\bar{\beta}_k$ alone:

THEOREM 2.2. *Let X be a closed orientable n-manifold, $n = 2k$, which for $k > 2$ is $(k-1)$-aspherical, and with infinite amenable fundamental group G. Then*

$$\chi(X) = (-1)^k \bar{\beta}_k(\tilde{X} \text{ rel. } G).$$

COROLLARY 2.3. *For X as in Theorem 2.2 one has*

$$(-1)^k \chi(X) \geq 0.$$

This is due to the fact that $\bar{\beta}_k$ is a non-negative real number.

In the case $n=4$ there are no asphericity assumptions and we get the result proved by a different method ("Følner sequence") in [E]:

THEOREM 2.4. *Let X be a closed orientable 4-manifold with infinite amenable fundamental group G. Then $\chi(X)$ is ≥ 0.*

Or in terms of the Hausmann-Weinberger invariant $q(G)$:

COROLLARY 2.5. *For a finitely presented infinite amenable group G the invariant $q(G)$ is ≥ 0.*

2.3. For manifolds X as considered in 2.1 the fundamental group $G=\pi_1(X)$ is *of type F_k* (finitely presented and of type FP_k). Indeed, the (finite) k-skeleton of a cell-decomposition of X can be extended to a $K(G, 1)$ by attaching cells of dimensions $>k$.

Conversely there exists for any group G of type F_k, $k \geq 2$, a closed orientable $2k$-manifold with $\pi_1(X)=G$ and $\pi_i(X)=0$ for $1<i<k$. To find X one starts with any closed orientable differentiable $2k$-manifold M with $\pi_1(M)=G$. For $k>2$, type FP_k of G guarantees that $\pi_2(M)=H_2(\tilde{M})$ is finitely generated as a $\mathbb{Z}G$-module. Thus $\pi_2(M)$ can be annihilated by a finite number of surgeries in M (see [M]), and there results a closed manifold M' with $\pi_1(M')=G$, $\pi_2(M')=0$. If $k>3$ then $\pi_3(M')$ is finitely generated over $\mathbb{Z}G$, and the procedure can be repeated until one has a manifold X as required.

Now we define for a group G of type F_k, $k \geq 2$, the invariant $\gamma_k(G)$ to be the minimum of $(-1)^k\chi(X)$ for all $2k$-manifolds as above with $\pi_1(X)=G$, $\pi_i(X)=0$ for $1<i<k$. The minimum exists since

$$\begin{aligned}(-1)^k\chi(X) &= \beta_k(X)+2\sum_0^{k-1}(-1)^{i+k}\beta_i(X)\\ &= \beta_k(X)+2\sum_0^{k-1}(-1)^{i+k}\beta_i(G)\end{aligned}$$

and $\beta_k(X) \geq \beta_k(G)$.

Clearly $\gamma_2(G)=q(G)$.

COROLLARY 2.6. *For an infinite amenable group G of type F_k, $k \geq 2$, the invariant $\gamma_k(G)$ is ≥ 0.*

3. The vanishing of $\chi(X)$

3.1. We return to a closed orientable manifold X of even dimension $n = 2k$ as in Section 2, aspherical up to the middle dimension k (if $k > 2$) and with infinite amenable fundamental group.

If $\chi(X) = 0$ then by Theorem 2.2 $\bar{\beta}_k(\tilde{X}$ rel. $G) = 0$, whence $\bar{H}^k(\tilde{X}) = 0$. We will show that this implies, in addition to Proposition 2.1, $H_k(\tilde{X}) \cong H^k(G; \mathbb{Z}G)$.

Since $\tilde{X}$ is $(k-1)$-connected we can use (part of) diagram (4) with exact rows

$$\begin{tikzcd} 0 \arrow[r] & H^k(G; \mathbb{Z}G) \arrow[r] \arrow[d] & H^k_{\text{comp}}(\tilde{X}; \mathbb{Z}) \arrow[r, "\Phi'"] \arrow[d] & H^k(\tilde{X}; \mathbb{Z}G)^G \arrow[d, "\Omega"] \\ 0 \arrow[r] & H^k(G; l_2G) \arrow[r] & H^k_{(2)}(\tilde{X}; \mathbb{R}) \arrow[r, "\Phi"] & H^k(\tilde{X}; l_2G)^G \end{tikzcd}$$

Since Φ factors through $\bar{H}^k(\tilde{X})$ (see Proposition 1.2) which is 0 if $\chi(X) = 0$ the map

$$H^k_{\text{comp}}(\tilde{X}; \mathbb{Z}) \xrightarrow{\Phi'} H^k(\tilde{X}; \mathbb{Z}G)^G \xrightarrow{\Omega} H^k(\tilde{X}; l_2G)$$

is $=0$. The coefficient map Ω is injective since $H^{k-1}(\tilde{X}; -) = 0$. Thus $\Phi' = 0$ and $H^k(G; \mathbb{Z}G) \cong H^k_{\text{comp}}(\tilde{X}; \mathbb{Z}) \cong H_k(\tilde{X})$.

THEOREM 3.1. *Let X be a compact orientable n-manifold, $n = 2k$, which for $k > 2$ is $(k-1)$-aspherical, and with infinite amenable fundamental group G. If $\chi(X) = 0$ then*

$$H_k(\tilde{X}) \cong H^k(G; \mathbb{Z}G).$$

We recall that $H_i(\tilde{X}) = 0$ for $0 < i < k$, and that $H_i(\tilde{X}) \cong H^{2k-i}(G; \mathbb{Z}G)$ for $k < i < 2k$ (by Proposition 2.1); whence

COROLLARY 3.2. *Let X be as in Theorem 3.1. If $\chi(X) = 0$ and $H^i(G; \mathbb{Z}G) = 0$ for $0 \le i \le k$ then $\tilde{X}$ is contractible, X a $K(G, 1)$, and G is a PD^{2k}-group.*

In terms of the invariant $\gamma_k(G)$ defined in 2.3:

COROLLARY 3.3. *If G is an infinite amenable group of type F_k, $k \ge 2$, with $H^i(G; \mathbb{Z}G) = 0$ for $0 \le i \le k$, then $\gamma_k(G) = 0$ implies that G is a PD^{2k}-group.*

3.2. Again $n = 2k = 4$ does not require any asphericity assumptions:

THEOREM 3.4. *Let X be a closed orientable 4-manifold with infinite amenable fundamental group G. If $\chi(X) = 0$ then $H_2(\tilde{X}) \cong H^2(G; \mathbb{Z}G)$.*

COROLLARY 3.5. *If for X as in Theorem 3.3, $H^1(G; \mathbb{Z}G) = H^2(G; \mathbb{Z}G) = 0$ and $\chi(X) = 0$ then X is a PD^4-group.*

We recall that $H^1(G; \mathbb{Z}G)$ must be 0 or $\mathbb{Z}$; it is $= \mathbb{Z}$ if and only if G is virtually infinite cyclic; whence

COROLLARY 3.6. *If G is a finitely presented infinite amenable group, not virtually infinite cyclic, with $H^2(G; \mathbb{Z}G) = 0$, then $q(G) = 0$ implies that G is PD^4-group.*

4. Amenable 2-knot groups

4.1. A 2-knot, or a knot in dimension 4, is a differentiable embedding $f: S^2 \to S^4$ of the 2-sphere into the 4-sphere. The group G is called a 2-knot group if there is a 2-knot such that the fundamental group $\pi_1(S^4 - f(S^2))$ of the complement is $\cong G$. For such a group one has $H_1(G) = \mathbb{Z}$ and $H_2(G) = 0$ (cf. Kervaire [K]).

Let C be the closed complement of $f(S^2)$ in S^4, obtained by removing an open tubular neighborhood of $f(S^2)$. Clearly $\pi_1 C = G$, and ∂C is homeomorphic to $f(S^2) \times S^1$. Attaching a handle $V^3 \times S^1$ to ∂C ("surgery along $f(S^2)$") yields a closed 4-manifolds X, with $\pi_1 X = G$, $H_1 X = H_1 G = \mathbb{Z}$, and $H_2 X = 0$. The invariant $q(G)$ is $\geq 2 - 2\beta_1(G) + \beta_2(G) = 0$, and $q(G) \leq \chi(X) = 0$.

Thus one has quite generally $q(G) = 0$ for *all* 2-knot groups.

4.2. If the 2-knot group G is amenable then Theorem 3.3 can be applied, whence

THEOREM 4.1. *Let G be an amenable 2-knot group, not virtually $\mathbb{Z}$, and X the closed 4-manifold obtained by surgery from a 2-knot with fundamental group G. Then $H^2(G; \mathbb{Z}G) = H_2(\tilde{X})$.*

COROLLARY 4.2. *If G is an amenable 2-knot group with $H^1(G; \mathbb{Z}G) = H^2(G; \mathbb{Z}G) = 0$ then $\tilde{X}$ is contractible, and G is a PD^4-group.*

4.3. *Remark.* Since $H_1(G) = \mathbb{Z}$ for a 2-knot group (actually for any knot group) one can write G as an HNN extension over a finitely generated group; if G is amenable the HNN extension must be ascending, i.e. $G = H_{*H,p}$ (cf. [E], p. 389). Here H also being amenable is either finite or has one or two ends.

If H is finite then G is virtually infinite cyclic, i.e. G has two ends. If H has one end, and if we assume that H is almost finitely presented, then $H^1(G; \mathbb{Z}G) = H^2(G; \mathbb{Z}G) = 0$ by [B-G], thus G is a PD^4-group. If H has 2 ends it must be infinite cyclic $= \langle a \rangle$; this yields $G = \langle a, p \mid pap^{-1} = a^k \rangle$ where $H_1(G) = \mathbb{Z}$ forces k to be $= 2$.

4.4. *Remark.* All 2-knot groups with 2 ends are determined by Hillman in [H2], Chapter 4. All *elementary amenable* 2-knot groups which are PD^4-groups are virtually solvable (cf. [H-L]) and thus torsion-free virtually polycyclic; all such 2-knot groups have been determined in [H2], Chapter 6.

5. Partial Euler characteristic of groups

5.1. In this appendix we use the method of l_2-cohomology to prove results concerning the "partial Euler characteristic" of an amenable group G which were already established earlier [E], partly by an entirely different method.

We assume that G is of type F_m; i.e., G admits a $K(G, 1)$ which has a finite m-skeleton (G is of type FP_m and finitely presented if $m \geq 2$). We denote by X the m-skeleton of $K(G, 1)$ and consider its Euler characteristic $\chi(X)$. The minimum value of $(-1)^m\chi(X)$ for all such $K(G, 1)$ is written $q_m(G)$. The minimum exists since $\beta_i(X) = \beta_i(G)$ for $i < m$ and $\beta_m(X) \geq \beta_m(G)$.

5.2. Since $H_i(\tilde{X}) = 0$ for $0 < i < m$ the Cheeger-Gromov lemma yields, for amenable G, $\bar{H}^i(\tilde{X}) = 0$ for $0 \leq i < m$, whence $\bar{\beta}_i(\tilde{X} \text{ rel. } G) = 0$ for these i. Thus

$$\chi(X) = (-1)^m \bar{\beta}_m(\tilde{X} \text{ rel. } G).$$

THEOREM 5.1. *For an infinite amenable group G of type F_m the group invariant $q_m(G)$ is* ≥ 0.

We recall that this yields explicit results of the following type: If G is a finitely presented infinite amenable group then the defect $d(G)$ is ≤ 1, cf. [E].

5.3. The vanishing of $q_m(G)$ is of special interest. It means that there is a certain $K(G, 1)$ – with finite m-skeleton X – such that $\chi(X) = 0$.

From 5.2 it follows that this implies $\bar{\beta}_m(\tilde{X} \text{ rel. } G) = 0$, whence $\bar{H}^m(\tilde{X}) = 0$. The map $\Psi : H^m_{(2)}(\tilde{X}; \mathbb{R}) \to H^m(\tilde{X}; \mathbb{R})$, see (5) in 1.5, factors through $\bar{H}^m(\tilde{X})$ and is therefore $= 0$.

We now consider an arbitrary *finite* subcomplex S of $\tilde{X}$. The restriction of $\tilde{X}$ to S yields the commutative diagram

$$\begin{array}{ccc} H^m_{(2)}(\tilde{X}; \mathbb{R}) & \xrightarrow{\Psi = 0} & H^m(\tilde{X}; \mathbb{R}) \\ \downarrow & & \downarrow \\ H^m_{(2)}(S; \mathbb{R}) & \xrightarrow{=} & H^m(S; \mathbb{R}) \end{array}$$

The vertical maps are surjective due to exactness of the relative sequence of $\tilde{X}$ modulo S, and to the fact that there are no $(m+1)$-cells.

Thus $H^m(S; \mathbb{R}) = Hom(H_m(S), \mathbb{R}) = 0$. As $H_m(S)$ is $\mathbb{Z}$-free, it must be 0. Thus $H_m(\tilde{X}) = 0$, and $\tilde{X}$ is contractible; i.e., we can take $X = K(G, 1)$.

THEOREM 5.2. *If for an infinite amenable group G of type F_m the group invariant $q_m(G) = 0$ then G admits a finite $K(G, 1)$-complex of dimension $\leq m$; in particular the cohomology dimension cdG is $\leq m$.*

5.4. We finally remark that results such as Theorems 2.2 and 5.1 hold in the more general setting of [E], Section 5: namely for a group G of the appropriate type which need not be amenable, but is an *extension* $G/N = A$ of an infinite amenable group A by a normal subgroup N with $\beta_i(N)$ *finite* for the respective i. These results can be established by the l_2-cohomology methods of the present paper. One takes, instead of $\tilde{X}$, the covering space Y corresponding to the subgroup N of G, which is a free cocompact A-space. Since $H^i(Y; \mathbb{R}) = H^i(N; \mathbb{R})$ has finite $\mathbb{R}$-dimension and $\bar{H}^i(Y) \to H^i(Y; \mathbb{R})$ is injective, $\bar{H}^i(Y)$ must be 0 ($\bar{H}^i(Y)$ is an invariant subspace of $C^i_{(2)}(Y; \mathbb{R})$ and cannot be of finite $\mathbb{R}$-dimension unless it is 0). Thus $\bar{\beta}_i(Y \text{ rel. } A) = 0$ and the arguments are as before. – These remarks, of course, do not apply to the "converse" statements concerning the vanishing of the Euler characteristic.

6. Addendum*) on groups with vanishing first l_2-Betti number

6.1. For any finite complex X with fundamental group G, i.e., for any finitely presented group, $\bar{\beta}_1(\tilde{X} \text{ rel. } G)$ depends on G only; it can be written $\bar{\beta}_1(G)$. If X is a closed orientable 4-manifold with $\pi_1(X) = G$, and if $\bar{\beta}_1(G) = 0$, then

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$\chi(X) = \bar{\beta}_2(X \text{ rel. } G)$. Thus all arguments of Sections 2 and 3 concerning 4-manifolds can be carried through. Moreover, via the l_2-signature theorem, one can obtain statements concerning the signature of X. We plan to return to these aspects in a separate paper.

6.2. Here we only note as an immediate consequence of Proposition 1.1 that finitely presented groups G with the Kazhdan (T) property have $\bar{\beta}_1(G) = 0$. Indeed, (T) implies $H^1(G; l_2G) = 0$; but $H^1(G; l_2G) = H^1(X; l_2G) = H^1_{(2)}(\tilde{X})$, and since $H^1_{(2)}(\tilde{X})$ maps onto $\bar{H}^1(\tilde{X})$ it follows that $\bar{\beta}_1(\tilde{X} \text{ rel. } G) = \bar{\beta}_1(G) = 0$.

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