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Autor(en): **Eliashberg, Y. / Polterovich, L.**

Objekttyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **69 (1994)**

PDF erstellt am: **22.07.2024**

Persistenter Link: https://doi.org/10.5169/seals-52274

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Comment. Math. Helvetici 69 (1994) 512–522 0010-2571/94/040512-11$1.50 + 0.20/0
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New applications of Luttinger's surgery

YAKOV ELIASHBERG and LEONID POLTEROVICH

§1. Introduction and main results

Recently Karl Luttinger [L] made a remarkable observation that certain surgeries along a Lagrangian 2-torus in the standard symplectic space $(\mathbb{C}^2, \omega)$ do not change the ambient topology. As a consequence he found restrictions on isotopy classes of embeddings $\mathbb{T}^2 \to \mathbb{C}^2$ which can be represented by Lagrangian ones.

In the present paper, we discuss some new applications of this technique to *linking* of Lagrangian 2-tori in $\mathbb{C}^2$, to *contact geometry* on the 3-torus as well as to study of *complex structures with pseudo-convex boundary* on $\mathbb{T}^2 \times \mathbb{D}^2$.

1.1. *Linking class of totally real tori*

A field of lines on a 2-torus is called *homotopically trivial* if it is homotopic to the kernel of a non-singular closed 1-form. All homotopically trivial line fields are homotopic. A 2-torus in $\mathbb{C}^2$ is called *totally real* if it has no complex tangent lines. From now on we denote by $\ell k(\cdot, \cdot)$ the linking number, and by J the standard complex structure on $\mathbb{C}^2$. All (co)homology groups considered below are integer.

Assume that $L \subset \mathbb{C}^2$ is an embedded oriented totally real 2-torus. Take an arbitrary non-singular tangent vector field, say v on L which generates a homotopically trivial field of lines. For a 1-cycle α on L set

$$\sigma(\alpha) = \ell k(\alpha + \varepsilon Jv, L),$$

where ε is sufficiently small.

One can easily check that σ is a well defined element of $H^1(L)$, in particular σ does not depend on the choice of v. We call σ *the linking class* of a totally real torus L (see [P1], [P2]). Note that this class is closely related to the *Viro quadratic form.*

Supported by United States–Israel Binational Science Foundation, Grant No. 90-00150/2.

As it was shown in [P1] for each cohomology class $\sigma \in H^1(L)$ there exists a totally real embedding $L \to \mathbb{C}^2$ whose linking class is equal to σ. However for Lagrangian submanifolds the situation is quite different. Namely, we prove the following result which was conjectured in [P1], [P2].

THEOREM 1.1.A. *The linking class of every embedded Lagrangian torus in* $\mathbb{C}^2$ *vanishes.*

The theorem is proved below in 3.1.

As a consequence we obtain the following

COROLLARY 1.1.B. (see [P1]). *Let* $M \subset \mathbb{C}^2$ *be an embedded closed* 3*-manifold whose characteristic foliation admits an embedded invariant* 2*-torus L. If L divides M then the restriction of the characteristic foliation to L is homotopically trivial.*

Proof. Notice that L is a Lagrangian torus. Let l be the field of Euclidian normal lines to M along L. Then the field Jl is tangent to the characteristic foliation on L. The needed assertion easily follows now from 1.1.A. □

1.2. *Giroux' theorem*

Homotopically trivial fields of lines on $\mathbb{T}^2$ allow to identify canonically (up to a homotopy) the cotangent bundle $T^*\mathbb{T}^2$ with $\mathbb{T}^2 \times \mathbb{R}^2$ (with this language the zero section is identified with $\mathbb{T}^2 \times \{0\}$).

THEOREM 1.2.A. *Consider an embedded Lagrangian torus in* $T^*\mathbb{T}^2 = \mathbb{T}^2 \times \mathbb{R}^2$ *which does not intersect the zero section and is homologous to it. Then its projection to* $\mathbb{R}^2 - \{0\}$ *is homotopic to a point.*

This result was conjectured by J.-C. Sikorav in [S] who verified it under an additional assumption that the torus is *Lagrangian isotopic* to the zero section. It was proved recently by E. Giroux (see [Gi]) using, in particular, some tools from contact geometry. We give here a different purely symplectic proof (see section 3.2 below).

1.3. *Contact geometry of the* 3*-torus*

Consider the 3-torus $\mathbb{T}^2 = S^1(\theta) \times \mathbb{T}^2(x, y)$, where (θ, x, y) (mod 1) are angular coordinates. Let $\xi = \operatorname{Ker} \lambda$, where

$$\lambda = \cos 2\pi\theta \, dx + \sin 2\pi\theta \, dy$$

be the standard contact structure.

We identify $H_1(\mathbb{T}^3)$ with $\mathbb{Z} \oplus \mathbb{Z}^2$ and the automorphisms group of $H_1(\mathbb{T}^3)$ with $GL(3, \mathbb{Z})$. Recall [La] that isotopy classes of 3-torus diffeomorphisms are defined by their action on homology. Let $\mathscr{D} \subset SL(3, \mathbb{Z})$ be the stabilizer of the subspace $0 \oplus \mathbb{Z}^2$.

THEOREM 1.3.A. *An element from* $SL(3,\mathbb{Z})$ *can be represented by a contactomorphism of the standard contact structure* ξ *if and only if it belongs to* $\mathscr{D}$.

The proof which is based on 1.2.A is given in Section 3.4 below.

We apply this theorem in order to construct an infinite sequence of pairwise non-isotopic tight contact structures on $\mathbb{T}^3$ with the same Euler class (see Question 8.6.1 in [E2]). Recall that two contact structures are called *isotopic* if there exists a diffeomorphism isotopic to the identity which takes one to another. An immediate consequence of 1.3.A is the following

COROLLARY 1.3.B. *For* $f, g \in SL(3, \mathbb{Z})$, *contact structures* $f_*(\xi)$ *and* $g_*(\xi)$ *are isotopic if and only if* $f^{-1} \circ g$ *belongs to* $\mathscr{D}$.

A theorem by J. Gray states that two contact structures on a compact manifold which are homotopic through contact structures are isotopic. On the other hand the image of the standard contact structure ξ under an arbitrary diffeomorphism of $\mathbb{T}^3$ is homotopic to ξ through plane distributions.

Hence, we have, in particular

COROLLARY 1.3.C. *There exists a sequence* ξ_n, $n \geq 0$, *of contact structures on* $\mathbb{T}^3$ *such that*

(i) ξ_n *is contactomorphic to* ξ *for every* n, *and* $\xi_0 = \xi$;
(ii) *all* ξ_n *are homotopic to* ξ *through two-dimensional distributions*;
(iii) *for* $m \neq n$ *the structures* ξ_m *and* ξ_n *are not homotopic through contact structures on* $\mathbb{T}^3$.

Proof. Take a diffeomorphism f of $\mathbb{T}^3$ such that $[f^n] \notin \mathscr{D}$ for every $n \in \mathbb{Z} - \{0\}$. It follows from 1.3.B and the previous discussion that the structures $\xi_n = f^n_*(\xi)$, $n = 0, \ldots$, are homotopic through plane distributions but not through contact structures. □

REMARK 1.3.D. Giroux in [Gi] used Theorem 1.2.A to construct a tight (see [E2]) contact structure on T^3 which is homotopic (through two-dimensional distributions) but *not isomorphic* to the standard contact structure ξ_0. His structure is symplectically fillable (see [El] for the definition of symplectically and holomorphically fillable structures) while at least some of structures constructed above are holomorphically fillable (see the next section).

1.4. *Complex structures on* $\mathbb{T}^2 \times \mathbb{D}^2$

A contact structure on an oriented 3-manifold is called *positive* if it is (locally) defined by a 1-form, say λ with $\lambda \wedge d\lambda > 0$. A boundary of a complex surface is called *strictly pseudo-convex* if its field of tangent lines is a positive (with respect to the canonical orientation) contact structure.

It was shown in [El] that the manifold $\mathbb{S}^2 \times \mathbb{D}^2$ does not admit a complex structure with strictly pseudo-convex boundary. In the present section we study the space of such structures on $\mathbb{T}^2 \times \mathbb{D}^2$.

THEOREM 1.4.A. *There exists a sequence J_n, $n \geq 0$, of complex structures with strictly pseudo-convex boundary on $\mathbb{T}^2 \times \mathbb{D}^2$ such that*

(i) *any two of them are biholomorphically equivalent and homotopic through complex structures;*

(ii) *for $m \neq n$ the structures J_m and J_n are not homotopic through complex structures with strictly pseudo-convex boundary.*

Proof. We represent $V = \mathbb{T}^2 \times \mathbb{R}^2$ as the quotient space of $\mathbb{C}^2$ by the imaginary lattice $i\mathbb{Z}^2$. We still denote by J the induced complex structure on V. Let (x, y) (mod 1) be angular coordinates on $\mathbb{T}^2$ and $(r, \theta(\text{mod } 1))$ be polar coordinates on $\mathbb{R}^2$. Set

$$N = \mathbb{T}^2 \times \mathbb{D}^2 = \{r \leq 1\}.$$

Denote by $\Sigma = \mathbb{T}^3$ tbe boundary of N. Obviously, Σ is strictly pseudo-convex with respect to J since its field of tangent complex lines is just the standard contact structure ξ defined in 1.3.

Consider a diffeomorphism $F : V \to V$,

$$(r, \theta, x, y) \to (r, \theta + 2x, x, y),$$

and set

$$J_n = DF^n \circ J \circ DF^{-n}.$$

We claim that the sequence $\{J_n\}$ has the desired properties. Indeed, since F preserves Σ we conclude that all $J_n |_N$ are pairwise biholomorphically equivalent and with strictly pseudo-convex boundary. Moreover, for $n \neq 0$ the restriction of F to Σ does not belong to the group $\mathcal{D}$ (see 1.3). Therefore for different values of n the fields of J_n-complex tangent lines to Σ are pairwise non-isotopic through contact structures on $\mathbb{T}^3$ (see 1.3.B) and thus we get (ii).

It remains to check that J_m and J_n are homotopic through complex structures for all m and n. In order to do it we notice that the map $DF : TV \to TV$ is homotopic to the identity through fiberwise linear maps whose restriction to each fiber is an isomorphism (verification of this fact is straightforward and we omit it). Hence the parametric h-principle for immersions of open manifolds (see [H] or [G2, 2.1.2]) implies that F is homotopic to the identity through immersions $V \to V$. Let F_t, $t \in [0; n]$ be such a homotopy with $F_0 = F$ and $F_n = id$. Then

$$J_t(v) = (DF_t^n(v))^{-1} \circ J_n(F_t^n(v)) \circ DF_t^n(v)$$

is the desired homotopy between J_0 and J_n. This completes the proof. □

REMARK 1.4.B. It follows easily from a Bennequin-type inequality proved in [E1, 4.1] that all complex structures with strictly pseudo-convex boundary on $\mathbb{T}^2 \times \mathbb{D}^2$ are homotopic one to another through almost complex structures. Moreover, using additional arguments from [G2] one can show that they are homotopic through complex structures.

REMARK 1.4.C. Let $\mathcal{J}_{conv}$ be the space of complex structures with strictly pseudo-convex boundary on $N = \mathbb{T}^2 \times \mathbb{D}^2$. *How to describe the connected components of $\mathcal{J}_{conv}$?* In order to formulate this question in a more precise way define a diffeomorphism $G_{m,n}$ of N by

$$G_{m,n}(r, \theta, x, y) = (r, \theta + mx + ny, x, y),$$

and consider a complex structure

$$J_{m,n} = DG_{m,n} \circ J \circ DG_{-m,-n}$$

which evidently belongs to $\mathcal{J}_{conv}$. It follows immediately from 1.3.B that for different pairs of integers (m, n) the structures $J_{m,n}$ represent different connected components of $\mathcal{J}_{conv}$. *Is it true that each such a component contains some $J_{m,n}$?*

§2. Surgery along Lagrangian tori

2.1. *The standard model*

Consider cotangent bundle $T^*\mathbb{T}^2$ of the 2-torus $\mathbb{T}^2$ endowed with the standard symplectic structure ω_0. Let $(x, y)\,(\mathrm{mod}\ 1)$ be angular coordinates on the base, and let $(r, \theta(\mathrm{mod}\ 1))$ be polar coordinates on fibers. We identify the hypersurface $\Sigma_0 = \{r = 1\}$ with the 3-torus $\mathbb{T}^3(\theta, x, y)$, and set $N_0 = \{r \le 1\}$.

For $m, n \in \mathbb{Z}$ we define *the Dehn twist* $f_{m,n} : \Sigma_0 \to \Sigma_0$ by

$$(\theta, x, y) \to (\theta, x + m\theta, y + n\theta).$$

Note that $f_{m,n}$ *preserves* the restriction of ω_0 to $T\Sigma_0$.

2.2. *Configurations of marked Lagrangian tori*

Let $L_1, \ldots, L_k \subset \mathbb{C}^2$ be a set of embedded disjoint Lagrangian tori. By *marking* we mean the choice of a basis in $H_1(L_j)$, say α_j, β_j.

Given such a marking, we can identify sufficiently small closed tubular neighbourhood N_j of L_j with N_0 by a *conformally symplectic diffeomorphism* in such a way that L_j goes to the zero section, and the cycles α_j, β_j correspond to the x- and y-coordinate cycles respectively. We assume that all N_j are disjoint. Set $\Sigma_j = \partial N_j \approx \mathbb{T}^3$, and $K = \mathbb{C}^2 - \bigcup_{j=1}^{k} (IntN_j)$. Let $f^{(j)} : \Sigma_j \to \Sigma_j$ be some Dehn twists. Denote by V a manifold obtained as the sum

$$K \cup_{f^{(1)},\Sigma_1} N_1 \cup \cdots \cup_{f^{(k)},\Sigma_k} N_k.$$

The main observation of Luttinger is the following

PROPOSITION 2.2.A. ([L]). *The manifold V associated with an arbitrary configuration $L_1, \ldots, L_k$ of marked Lagrangian tori and an arbitrary sequence $f^{(1)}, \ldots, f^{(k)}$ of Dehn twists is diffeomorphic to $\mathbb{C}^2$. In particular, $H_1(V) = 0$.*

Proof. Note that V admits a symplectic structure which outside a compact set coincides with the standard one on $\mathbb{C}^2$. It follows immediately from well known theorems by M. Gromov and D. McDuff (see [G1], [M]) that V is diffeomorphic to $\mathbb{C}^2$, maybe blown up at finite number of points. On the other hand the signature of V vanishes in view of Novikov's additivity theorem (we thank R. Gompf for this argument), and hence the proposition follows. □

We need below the following corollary of 2.2.A. Set $\Sigma = \amalg \Sigma_j$, $N = \amalg N_j$. Let $\Phi : H_1(\Sigma) \to H_1(K)$ be a homomorphism induced by the inclusion, and let $\Psi : H_1(\Sigma) \to H_1(N)$ be a homomorphism induced by the composition

$$\Sigma \xrightarrow{\amalg f(j)} \Sigma \longrightarrow N,$$

where the last arrow is the inclusion.

COROLLARY 2.2.B. *The homomorphism*

$$\Phi \oplus (-\Psi) : H_1(\Sigma) \to H_1(K) \oplus H_1(N)$$

is an isomorphism.

Proof. Consider the Mayer–Vietoris sequence

$$H_1(\Sigma) \xrightarrow{\Phi \oplus (-\Psi)} H_1(K) \oplus H_1(N) \longrightarrow H_1(V).$$

Since $H_1(V) = 0$ due to 2.2.A, we have that $\Phi \oplus (-\Psi)$ is an epimorphism. But $H_1(\Sigma)$ and $H_1(K) \oplus H_1(N)$ are free $\mathbb{Z}$-modules of the same dimension $3k$. Hence $\Phi \oplus (-\Psi)$ is an isomorphism. □

For our purposes we have to fix a basis in each space $H_1(\Sigma)$, $H_1(K)$, $H_1(N)$. Let $h_1, a_1, b_1, \ldots, h_k, a_k, b_k$ be a basis in $H_1(\Sigma)$ such that for every j the cycles h_j, a_j, b_j correspond to θ-, x- and y-coordinate cycles on $\mathbb{T}^3$ respectively. Let $A_1, B_1, \ldots, A_k, B_k$ be a basis in $H_1(N)$, where for every j the cycles A_j, B_j correspond to x- and y-coordinate cycles on $\mathbb{T}^2$ respectively. Finally, let $H_1, \ldots, H_k$ be the basis in $H_1(K)$ which is defined by relations

$$\ell k(H_i, L_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

(here the orientation of L_j is determined by the marking).

§3. Proof of main theorems

3.1. *Proof of* 1.1.A

Let $L \subset \mathbb{C}^2$ be an embedded Lagrangian torus, and let σ be its linking class. Choose a marking α, β on L and apply the construction of 2.2 with respect to a Dehn twist $f_{m,n}$.

Recall that using homotopically trivial fields of lines one can define the canonical trivialisation of the (co)tangent bundle to a 2-torus. Consider a trivialisation of the normal bundle to L which is obtained from the canonical one of TL by the multiplication by J. It is easy to see that after the identification of a tubular neighbourhood of L with N_0 (see 2.2) this trivialisation coincides with the canonical one of $T^*\mathbb{T}^2$.

In view of this we have that the maps $\Phi : H_1(\Sigma) \to H_1(K)$ and $\Psi : H_1(\Sigma) \to H_1(N)$ act as follows:

$$\Phi(h) = H, \qquad \Phi(a) = \sigma(\alpha)H, \qquad \Phi(b) = \sigma(\beta)H;$$

$$\Psi(h) = mA + nB, \qquad \Psi(a) = A, \qquad \Psi(b) = B.$$

(The numeration of the basis elements is omitted since we work with one torus). Hence in the bases (h, a, b) and (H, A, B) the map $\Phi \oplus (-\Psi)$ is given by the matrix

$$\begin{pmatrix} 1 & \sigma(\alpha) & \sigma(\beta) \\ -m & -1 & 0 \\ -n & 0 & -1 \end{pmatrix}.$$

Its determinant equals to $1 - \sigma(\alpha)m - \sigma(\beta)n$. On the other hand 2.2.B implies that this determinant equals to ± 1 for all m and n. Hence $\sigma(\alpha) = \sigma(\beta) = 0$. This completes the proof. □

3.2. *Proof of* 1.2.A

Let us represent a neighbourhood of the zero section in $T^*\mathbb{T}^2$ as a tubular neighbourhood $\mathcal{U}$ of the standard Lagrangian torus $L_1 = \mathbb{S}^1 \times \mathbb{S}^1 \subset \mathbb{C}^2$. Let L_2 be an embedded Lagrangian torus in $\mathcal{U}$ which is disjoint from L_1 and homologous to L_1 *inside* $\mathcal{U}$. The assertion we have to prove can be reformulated as follows: every cycle $e \in H_1(L_2)$ is unlinked with L_1:

$$\ell k(e, L_1) = 0.$$

Denote by $\tau : \mathcal{U} \to L_1$ the natural projection and by $\tau_* : H_1(L_2) \to H_1(L_1)$ the induced *isomorphism*. We need the following simple topological

LEMMA 3.2.A. *For every $e \in H_1(L_2)$ the following equality holds*:

$$\ell k(e, L_1) = \ell k(\tau_* e, L_2),$$

where we assume that τ preserves orientations of L_1 and L_2.

The proof is given in 3.3 below.

Let α_2, β_2 be a marking of L_2, and let $\alpha_1 = \tau_* \alpha_2$, $\beta_1 = \tau_* \beta_2$ be the "coherent" marking of L_1. Set $u = \ell k(\alpha_1, L_2) = \ell k(\alpha_2, L_1)$, $v = \ell k(\beta_1, L_2) = \ell k(\beta_2, L_1)$. Choose disjoint tubular neighbourhoods N_1, N_2 of L_1, L_2 respectively *inside* $\mathcal{U}$, and apply the surgery procedure 2.2 associated with Dehn twists $f^{(1)} = f_{m,n}$ and $f^{(2)} = f_{m,n}$ for some integer m, n. Now consider the action of Φ and Ψ in corresponding bases $(h_1, a_1, b_1, h_2, a_2, b_2)$ and $(H_1, A_1, B_1, H_2, A_2, B_2)$. A straightforward computation (which uses also 1.1.A) shows that $\Phi \oplus (-\Psi)$ is given by the matrix

	h_1	a_1	b_1	h_2	a_2	b_2
H_1	1	0	0	0	u	v
A_1	$-m$	-1	0	0	0	0
B_1	$-n$	0	-1	0	0	0
H_2	0	u	v	1	0	0
A_2	0	0	0	$-m$	-1	0
B_2	0	0	0	$-n$	0	-1

whose determinant is equal to $1 - (um + vn)^2$. On the other hand, this determinant equals to ± 1 for each choice of m and n due to 2.2.B. Hence $u = v = 0$, and the desired assertion follows. □

3.3. *Proof of* 3.2.A

Let $v_1 \in H_1(L_1)$ (respectively, $v_2 \in H_1(L_2)$) be a class Poincare dual to $\ell k(\cdot, L_2)$ (respectively, to $\ell k(\cdot, L_1)$). We have to show that $\tau_* v_2 = v_1$, in other words that 1-cycles representing these classes are homologous inside $\mathcal{U}$. Let $\mathcal{R}$ be a smooth embedded 3-chain which spans L_1 in $\mathbb{C}^2$ and has the following properties:

- $\mathcal{R}$ is transversal to $\partial\mathcal{U}$ and to L_2;
- $\mathcal{R} \cap \mathcal{U} \approx \mathbb{T}^2 \times [0; 1]$, where $\mathbb{T}^2 \times \{0\} = L_1$ and $\mathbb{T}^2 \times \{1\} \subset \partial\mathcal{U}$.

Let $\mathcal{R}'$ be a small shift of $\mathcal{R}$ along the field of normals, such that $\mathcal{R} \cap \mathcal{R}' = \varnothing$ and $\mathcal{R}'$ intersects $\partial\mathcal{U}$ transversally along a torus L. Note that L and L_2 are homologous inside $\mathscr{U}$. Let Q be a 3-chain such that $Q \subset \mathscr{U}$ and $\partial Q = L \cup L_2$. We shall assume that Q is an immersed 3-manifold transversal to $\mathscr{R}$ and to L_1. Finally, set $S = Q \cup (\mathscr{R}' - \mathscr{U})$. Note that S is a 3-chain with the following properties:

- S spans L_2 in $\mathbb{C}^2$;
- S is transversal to $\mathscr{R}$ and to L_1 and intersects $\mathscr{R}$ *inside* $\mathscr{U}$.

Set $W = S \cap \mathscr{R}$. Obviously, W is a 2-chain in $\mathscr{U}$ whose boundary components are $S \cap L_1$ and $\mathscr{R} \cap L_2$. Moreover, 1-cycles $S \cap L_1$ on L_1 and $\mathscr{R} \cap L_2$ on L_2 represent classes v_1 and v_2 respectively. Hence $\tau_* v_2 = v_1$, and the proof is complete. □

3.4. *Proof of* 1.3.A

Assume that f is a linear automorphism of $\mathbb{T}^3$ with $[f] \in \mathscr{D}$. One can easily check that the form $f^*\lambda$ is isotopic to λ through contact forms, and hence f is isotopic to a contactomorphism.

The proof of the inverse assertion is divided into several steps.

(1) We represent $\mathbb{T}^3$ as the hypersurface $\Sigma_0 = \{r = 1\}$ in $T^*\mathbb{T}^2$ (see 2.1). Then λ is just the restriction of the standard Liouville form

$$r \cos 2\pi\theta \, dx + r \sin 2\pi\theta \, dy$$

on $T^*\mathbb{T}^2$. Let $f: \mathbb{T}^3 \to \mathbb{T}^3$ be a contactomorphism, that is $f^*\lambda = \varphi\lambda$ for some non-vanishing function $\varphi(\theta, x, y)$. Since α and $-\alpha$ are isotopic through contact forms, we can assume that φ is *positive*.

(2) We claim that the map $F: \Sigma_0 \to T^*\mathbb{T}^2$, given in coordinates (r, θ, x, y) on $T^*\mathbb{T}^2$ by

$$(\theta, x, y) \mapsto \left(\frac{1}{\varphi(\theta, x, y)}, f(\theta, x, y)\right)$$

is symplectic, that is $F^*\omega_0 = \omega_0 |_{T\Sigma}$. Indeed,

$$\begin{aligned} F^*\omega_0 &= F^* \, d(r \cdot (\cos 2\pi\theta \, dx + \sin 2\pi\theta \, dy)) \\ &= d\left(\frac{1}{\varphi} \cdot f^*\lambda\right) = d\lambda = \omega_0. \end{aligned}$$

(3) Take a Lagrangian torus $L = \{\theta = \text{const}\} \subset \Sigma_0$. Due to the previous step, its image $F(L)$ is a Lagrangian torus in $T^*\mathbb{T}^2$ disjoint from the zero section. Obviously, the projection $F(L) \to \mathbb{R}^2 - \{0\}$ (see 1.2) is homotopic to a point if and only if $[f] \in \mathcal{D}$. The desired assertion follows now from 1.2.A. □

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Received April 26, 1993; May 1993