Comment. Math. Helvetici 69 (1994) 659–668 0010–2571/94/040659–10$1.50 + 0.20/0

Ahlfors–Weill extensions of conformal mappings and critical points of the Poincaré metric

M. CHUAQUI AND B. OSGOOD

1. Introduction

Nehari showed in [10] that if f is analytic in the unit disk $\mathbf{D}$, and if its Schwarzian derivative $Sf = (f''/f') - (1/2)(f''/f')^2$ satisfies

$$|Sf(z)| \leq \frac{2}{(1-|z|^2)^2}, \tag{1.1}$$

then f is univalent in the disk. Ahlfors and Weill showed in [1] that if the Schwarzian satisfies the stronger inequality

$$|Sf(z)| \leq \frac{2t}{(1-|z|^2)^2} \tag{1.2}$$

for some $0 \leq t < 1$ then, in addition, f has a quasiconformal extension to the sphere. They gave an explicit formula for the extension. The class of analytic functions satisfying either of these conditions is quite large. It was shown by Paatero in [13] that any convex univalent function satisfies (1.1). This was later established in a different way by Nehari in [11], and he went on to prove that a bounded convex function satisfies the Ahlfors–Weill condition.

In [6], Gehring and Pommerenke made a careful study of Nehari's original univalence criterion and showed, among other things, that the condition (1.1) implies that $f(\mathbf{D})$ is a Jordan domain except when f is a Möbius conjugation of the logarithm,

$$F_0(z) = \frac{1}{2}\log\frac{1+z}{1-z}. \tag{1.3}$$

By this we mean that $f = T \circ F_0 \circ \tau$, where T and τ are Möbius transformations and $\tau(\mathbf{D}) = \mathbf{D}$. The function F_0 has $SF_0(z) = 2/(1-z^2)^2$, and $F_0(\mathbf{D})$ is an infinite parallel strip. For topological reasons, it then follows from the Gehring–Pommerenke theorem that other than in the exceptional case f has a homeomorphic extension to the sphere. See also [4]. The main result in this paper is that the same Ahlfors–Weill formula defines a homeomorphic extension of f, though it will not in general be a quasiconformal extension. We discuss this phenomenon via a relationship between the Ahlfors–Weill extension and the Poincaré metric of the image of f. This may be of independent interest.

For economy of notation, though at the risk of sinking a crowded ship, we introduce explicitly several subclasses of univalent functions associated with Nehari type bounds. Thus we let N denote the set of analytic functions in the disk satisfying (1.1), N^* the elements of N other than Möbius conjugations of the function F_0, and N^t those functions satisfying (1.2). We use the notation N_0, N_0^*, N_0^t to indicate that a function f in any of the classes has the normalization $f(0) = 0, f'(0) = 1, f''(0) = 0$. If $f(z) = z + a_2 z^2 + \cdots$ is in any of the classes, then $f/(1 + a_2 f)$ is in the corresponding class of normalized functions, the point being that the normalized function is still analytic, [2]. The function F_0 is normalized in this way. Furthermore, according to [3], Lemma 4, functions in N_0^* are bounded. The family of normalized extremals for the Ahlfors–Weill condition (1.2) is

$$A_t(z) = \frac{1}{\alpha}\frac{(1+z)^\alpha - (1-z)^\alpha}{(1+z)^\alpha + (1-z)^\alpha}, \qquad \alpha = \sqrt{1-t}. \tag{1.4}$$

We thank the referee for his thoughtful and helpful remarks.

2. Preliminary estimates

Several distortion theorems for the classes N_0 and N_0^t were proved in [2] using comparison theorems for the second order, ordinary differential equation associated with the Schwarzian. We continue somewhat in the same vein here for a few basic estimates. We refer to our earlier paper for further background.

LEMMA 1. *If $f \in N_0$ then*

$$\left|\frac{f''}{f'}(z)\right| \le \frac{2|z|}{1-|z|^2}. \tag{2.1}$$

Equality holds at a single $z \neq 0$ if and only if f is a rotation of $F_0(z)$. If $f \in N_0^t$ then

$$\left|\frac{f''}{f'}(z)\right| \le \frac{2t|z|}{1-|z|^2}. \tag{2.2}$$

The inequality (2.2) is not sharp. The proof will show how one may obtain a sharp estimate, but it is not as convenient and explicit as the one given here.

Proof. Let $y = f''/f'$. Then

$$y' = \frac{1}{2}y^2 + 2p, \qquad y(0) = 0,$$

with $2p(z) = Sf(z)$. We consider the real equation

$$w' = \frac{1}{2}w^2 + \frac{2}{(1-x^2)^2}, \qquad w(0) = 0,$$

on $(-1, 1)$, whose solution is $w(x) = 2x/(1-x^2)$. We want to show that $|y(z)| \leq w(|z|)$.

Fix z_0, $|z_0| = 1$, and let

$$\varphi(\tau) = |y(\tau z_0)|, \qquad 0 \leq \tau < 1.$$

Unless $f(z) = z$ identically the zeros of φ are isolated. Away from these zeros φ is differentiable and $\varphi'(\tau) \leq |y'(\tau z_0)|$. Since $|p(\tau z_0)| \leq 1/(1-\tau^2)^2$ we obtain

$$\begin{aligned}\frac{d}{d\tau}(\varphi(\tau) - w(\tau)) &\leq |y'(\tau z_0)| - w'(\tau) \leq \frac{1}{2}(|y(\tau z_0)|^2 - w^2(\tau)) \\ &= \frac{1}{2}(\varphi(\tau) - w(\tau))(\varphi(\tau) + w(\tau)).\end{aligned}$$

This, together with $\varphi(0) - w(0) = 0$, implies that $\varphi(\tau) - w(\tau)$ can never become positive.

Now suppose that equality holds in (2.1) at $z_1 \neq 0$. Let $z_0 = z_1/|z_1|$ and let $\varphi(\tau)$ be defined as above. Then $\varphi(|z_1|) = w(|z_1|)$ which, by the previous analysis, can happen only if $\varphi(\tau) = w(\tau)$, first on $[0, |z_1|]$, and then for all $\tau \in [0, 1)$ since both functions are analytic. Hence $y(\tau z_0)$ is of the form $e^{i\theta(\tau)}w(\tau)$. Since all inequalities above must be equalities, it follows easily that $\theta(\tau)$ must be constant. From this, it follows in turn that $y(z) = cw(\bar{z}_0 z)$ for all $|z| < 1$, with $|c| = 1$. Integrating this equation and appealing to the normalizations on f shows that $f(z) = e^{-i\theta}F_0(e^{i\theta}z)$. This proves the first part of the lemma.

Next, suppose that $f \in N_0^t$. The proof that $|f''/f'|$ has the bound in (2.2) proceeds exactly as above with the single difference that the comparison equation is

$$w' = \frac{1}{2}w^2 + \frac{2t}{(1-x^2)^2}, \qquad w(0) = 0.$$

The solution is given by

$$w(x) = \frac{2x}{1-x^2} - \frac{2\alpha^2}{1-x^2} A_t(x),$$

where $A_t(x)$ is defined in (1.4). It can be checked that $A_t(x)$ is convex on [0, 1], and hence

$$\max_{0 \le x \le 1} \frac{A_t(x)}{x} = A_t'(0) = 1.$$

Therefore

$$\frac{1-x^2}{2x} w(x) \le 1 - \alpha^2 = t,$$

which proves (2.2).

3. Bounds for the Poincaré metric

The Poincaré metric $\lambda_\Omega |dw|$ of a simply connected domain Ω is defined by

$$\lambda_\Omega(f(z))|f'(z)| = \lambda_{\mathbf{D}}(z) = \frac{1}{1-|z|^2},$$

where $f: \mathbf{D} \to \Omega$ is a conformal mapping of the unit disk onto Ω. From Schwarz's lemma and the Koebe 1/4-theorem one has the sharp inequalities

$$\frac{1}{4}\frac{1}{d(z, \partial\Omega)} \le \lambda_\Omega(z) \le \frac{1}{d(z, \partial\Omega)},$$

where $d(z, \partial\Omega)$ denotes the Euclidean distance from z to the boundary.

Writing $w = f(z)$ and taking the $\partial_z = \partial/\partial z$ derivative of the logarithm of (3.1) gives

$$\frac{\partial_w \lambda_\Omega(f(z))}{\lambda_\Omega(f(z))} f'(z) = \frac{\bar{z}}{1-|z|^2} - \frac{1}{2}\frac{f''}{f'}(z). \tag{3.2}$$

Observe for a normalized function $f \in N_0$ that the Poincaré metric λ_Ω of the image

Ω has a critical point at $w = 0$, and, by Lemma 1, that this must be the unique critical point if f is not a rotation of the logarithm F_0. (In the latter case Ω is a parallel strip and the critical points of λ_Ω are all the points of the axis of symmetry of Ω.) Assuming that $f \in N$ is bounded, we can drop the normalization and reach the same conclusion:

LEMMA 2. *If $f \in N$ is bounded, then λ_Ω has a unique critical point.*

Proof. Since $\Omega = f(\mathbf{D})$ is bounded and $\lambda_\Omega(w) \to \infty$ as $w \to \partial\Omega$, λ_Ω must have at least one critical point. By replacing f by $f \circ T_1$ where T_1 is a Möbius transformation of the disk to itself, and then by $T_2 \circ f \circ T_1$, where T_2 is a complex affine transformation, we may assume that one such critical point is $w = 0 = f(0)$, and furthermore that $f(0) = 0$ and $f'(0) = 1$. The identity (3.2) then forces $f''(0) = 0$, i.e. that $f \in N_0^*$. Hence, as above, $w = 0$ is the unique critical point for λ_Ω since f cannot be a rotation of the log.

REMARK. These are sharp results in the sense that for any $0 < \epsilon < 2$ there is a bounded, univalent function f with $Sf(z) = -2(1+\epsilon)/(1-z^2)^2$ such that λ_Ω, $\Omega = f(\mathbf{D})$, has more than one critical point. In fact, consider the (normalized) functions $A_{-t}(z)$ for $1 \le t < 3$, where $A_t(z)$ is defined in (1.4). For each t the function A_{-t} has $SA_{-t}(z) = -2t/(1-z^2)^2$ and maps $\mathbf{D}$ onto the quasidisk Ω_t consisting of the interior of the union of the circles through the points $1/\alpha$, $-1/\alpha$ and $\pm i(1/\alpha)\tan(\pi\alpha/4)$, where $\alpha = \sqrt{1+t}$. One can check directly that when $t > 1$ the Poincaré metric for Ω_t has exactly three critical points, one at 0 and two on the imaginary axis which are conjugate.

In [7] Kim and Minda showed that $\log \lambda_\Omega$ is a convex function if and only if Ω is a convex domain. See also the papers [9] and [14]. Using the fact that N contains the convex conformal mappings we can add:

COROLLARY 1. *If Ω is a bounded, convex domain, then λ_Ω has a unique critical point.*

In [12] it was shown that

$$|\nabla \log \lambda_\Omega| \le 4\lambda_\Omega \tag{3.3}$$

as a consequence of (3.2) and the classical bound for $|f''/f'|$ that holds for any univalent function in the disk. The inequality (3.3) is equivalent to the coefficient inequality $|a_2| \le 2$. We now give some lower bounds for $|\nabla \log \lambda_\Omega|$.

LEMMA 3. *If* $f \in N_0^*$, *then there exists a constant* $c > 0$ *such that*

$$|\nabla \log \lambda_\Omega(w)| \geq c|w|\lambda_\Omega(w)^{1/2}. \tag{3.4}$$

If $f \in N_0^t$, *then*

$$|\nabla \log \lambda_\Omega(w)| \geq 2(1-t)^{3/2}|w|\lambda_\Omega(w). \tag{3.5}$$

Recall that a function $f \in N_0^*$ is bounded. The constant in (3.4) depends on the bound for f. In an appendix we will give an example to show that the exponent 1/2 is essentially best possible in (3.4).

Proof. The estimate (3.4) is implicit in [6]. We show how it can be deduced, adopting the notation used there. Let h be the inverse of F_0 and let $g = f \circ h$. For $\tau \in \mathbf{R}$ we have $2|g'(\tau)| = (1-|h(\tau)|^2)|f'(h(\tau))| = \lambda_\Omega(g(\tau))^{-1}$. It was shown in [6] that $v = |g'|^{-1/2}$ is convex, with $v(0) = 1$, $v'(0) = 0$. It is not constant when f is not equal to F_0. Now,

$$2\frac{v'}{v}(\tau) = \frac{d}{dt}\log \lambda_\Omega(g(\tau)) \leq |\nabla \log \lambda_\Omega(g(\tau))|\,|g'(\tau)| = |\nabla \log \lambda_\Omega(g(\tau))|v(\tau)^{-2},$$

hence

$$|\nabla \log \lambda_\Omega(g(\tau))| \geq 2v(\tau)v'(\tau) = 2^{3/2}v'(\tau)\lambda_\Omega(g(\tau))^{1/2}.$$

Since v is not constant and f is bounded, it follows that there exists a constant a such that $v'(\tau) \geq a|g(\tau)|$ for $\tau \geq 0$. The estimates can be made uniformly on different rays from the origin by considering $f(e^{i\theta}h)$. This proves (3.4).

Now suppose that $f \in N_0^t$ and write $w = f(z)$. Using (3.2),

$$\frac{1}{2}|\nabla \log \lambda_\Omega(w)| = \frac{|(\partial_w \lambda_\Omega)(f(z))|}{\lambda_\Omega(f(z))} = \frac{\left|\bar{z} - \frac{1}{2}(1-|z|^2)\frac{f''}{f'}(z)\right|}{(1-|z|^2)|f'(z)|}.$$

From Lemma 1, (2.2) we then obtain

$$|\nabla \log \lambda_\Omega(w)| \geq \frac{2(1-t)|z|}{(1-|z|^2)|f'(z)|} = 2(1-t)|z|\lambda_\Omega(w),$$

with $w = f(z)$. But from [2], a function in N_0^t is subject to the sharp bound

$|f(z)| \le A_t(|z|)$, where A_t was defined in (1.4). This can be rearranged to

$$|z| \ge \frac{(1+\alpha|w|)^{1/\alpha} - (1-\alpha|w|)^{1/\alpha}}{(1-\alpha|w|)^{1/\alpha} + (1-\alpha|w|)^{1/\alpha}} = \psi(\alpha|w|), \qquad \alpha = \sqrt{1-t}.$$

The function $\psi(s)$ is concave on [0, 1] with $\psi(0) = 0$ and $\psi(1) = 1$. Hence $\psi(s) \ge s$ and (3.5) follows.

4. Homeomorphic extensions

Let $f \in N$ with $f(z) = z + a_2 z^2 + \cdots$. It was shown in [2] that $-1/a_2 \notin f(\mathbf{D})$, and it follows from Lemma 4 in [3] that unless f is conjugate to F_0 the point $-1/a_2$ will actually lie outside $\overline{f(\mathbf{D})}$. For a fixed $\zeta \in \mathbf{D}$ renormalize in the usual way to

$$g(z) = \frac{f\left(\dfrac{z+\zeta}{1+\bar{\zeta}z}\right) - f(\zeta)}{(1-|\zeta|^2)f'(\zeta)} = z + \left(\frac{1}{2}(1-|\zeta|^2)\frac{f''}{f'}(\zeta) - \bar{\zeta}\right)z^2 + \cdots,$$

which is again in N, and which has $g(0) = 0$ and $g'(0) = 1$. To say that $-2/g''(0) \notin g(\mathbf{D})$ is equivalent to saying that

$$E_f(\zeta) = f(\zeta) + \frac{(1-|\zeta|^2)f'(\zeta)}{\bar{\zeta} - \dfrac{1}{2}(1-|\zeta|^2)\dfrac{f''}{f'}(\zeta)} \notin f(\mathbf{D}), \tag{4.1}$$

and, again, if f is not conjugate to F_0 then

$$E_f(\zeta) \notin \overline{f(\mathbf{D})}. \tag{4.2}$$

In terms of the Poincaré metric, E_f has the expression

$$E_f(z) = f(z) + \frac{1}{\partial_w(\log \lambda_\Omega)(f(z))}, \tag{4.3}$$

by (3.2).

THEOREM 1. *If $f \in N^*$ then*

$$F(z) = \begin{cases} f(z) & \textit{for } |z| \le 1, \\ E_f(1/\bar{z}) & \textit{for } |z| > 1, \end{cases} \tag{4.4}$$

is a homeomorphic extension of f to the sphere.

The extension E_f has the important property that it commutes with Möbius transformations of f. If T is a Möbius transformation, then

$$E_{T\circ f} = T(E_f). \tag{4.5}$$

This can be checked directly from the definition, first for complex affine transformations and then, less obviously, for an inversion. It is also true that $E_{f\circ\tau}(z) = E_f(\tau(z))$ for all Möbius transformations τ of the disk onto itself, but we will not make any use of this fact.

Proof of Theorem 1. We first show that F is continuous at all points of the sphere. If $|z| < 1$ this is obvious. Next, using (4.5) we may normalize further and assume that $f \in N_0^*$. It is then clear from

$$E_f(z) = f(z) + \frac{(1-|z|^2)f'(z)}{\bar{z} - \frac{1}{2}(1-|z|^2)\frac{f''}{f'}(z)}$$

that F is continuous outside $\bar{\mathbf{D}}$; from Lemma 1 the denominator vanishes only at $z = 0$, which corresponds to ∞ under the reflection in $|z| = 1$. Finally, recall by the Gehring–Pommerenke theorem that f has a homeomorphic extension to $\bar{\mathbf{D}}$. Thus since $f(\mathbf{D}) = \Omega$ is a Jordan domain, to show that F is continuous on $|z| = 1$ we must see that E_f matches with f there. Because we have normalized to get $f \in N_0^*$ we know that Ω is bounded, and so it suffices to show that $E_f(z) - f(z) \to 0$ as $|z| \to 1$. This is equivalent to $|\nabla \log \lambda_\Omega(w)| \to \infty$ as $w \to \partial\Omega$, which follows from the first part of Lemma 3. We also now conclude that the range of F is all of $\bar{\mathbf{C}}$.

Since f is a homeomorphism of $\bar{\mathbf{D}}$, it remains to show that E_f is injective. Suppose that $E_f(z_1) = E_f(z_2)$. Appealing again to (4.5) we may change f to $T \circ f$ by an appropriate Möbius transformation T and assume that this common value is ∞. But (4.2) now implies that f must be bounded, while on the other hand (4.3) shows that an infinite value of E_f must be a critical point of $\log \lambda_\Omega$. By Lemma 2 such a critical point is unique, hence $z_1 = z_2$ because f is univalent.

We have proved that the mapping F is continuous and injective, and is therefore a homeomorphism onto its range, $\bar{\mathbf{C}}$. This completes the proof of the theorem.

The function E_f is precisely the Ahlfors–Weill extension. For f satisfying $|Sf(z)| \le 2t(1-|z|^2)^{-2}$ the function F defined by (4.4) is a $(1+t/1-t)$-quasiconformal mapping which extends f. In [5] Epstein made an enlightening differential-geometric study of this extension. Independent of the Gehring–Pommerenke results, a function in N_0^t is already $\sqrt{1-t}$-Hölder continuous in $\mathbf{D}$, and so, in particular, it can be extended to $\bar{\mathbf{D}}$; see [2].

The complex dilatation $\mu_F = \partial_{\bar{z}}F/\partial_z F$ of the Ahlfors–Weill extension at a point ζ in the exterior of the disk is

$$\mu_F(\zeta) = -\frac{1}{2}(1-|z|^2)^2 Sf(z),$$

where $z = 1/\bar{\zeta}$. It will therefore not define a quasiconformal mapping at points where $|Sf(z)|$ is at least $2/(1-|z|^2)^2$. There are, however, functions in $N^*\setminus\bigcup_{t<1}N^t$ which do have quasiconformal extensions. For example, take f to be a solution of $Sf \equiv 2$ in $\mathbf{D}$. The function has $|Sf(z)| \le t\pi^2/2$ for $t = 4/\pi^2 < 1$, and so by [10] and [6] its image is a quasidisk. But the formula (4.4) will not provide a global quasiconformal extension. Also, recall from the remark following Lemma 2 that the functions $A_{-t}(z)$, $1 \le t < 3$, with $Sf(z) = -2t/(1-z^2)^2$ (too big to be in N when $t > 1$) all have quasiconformal extensions, but again not via the Ahlfors–Weill extension.

Appendix: An example

We return to the first part of Lemma 3. We want to construct a function $f \in N_0^*$ showing that the exponent 1/2 in the bound $|\nabla \log \lambda_\Omega(w)| \ge a|w|\lambda_\Omega(w)^{1/2}$, $\Omega = f(\mathbf{D})$, is, in general, best possible. As the proof of Lemma 3 shows, this will be the case provided the convex function v, introduced in the proof, has bounded derivative.

The extremal F_0 maps the disk onto the strip $-\pi/4 < \operatorname{Im} w < \pi/4$. We want to construct g, analytic in this strip, so that $f = g \circ F_0$ will be in N_0^*, and $v(\tau) = |g'(\tau)|^{-1/2}$ will be convex with bounded derivative for τ on the real axis.

Let $a > 0$, to be chosen, and let

$$g'(\zeta) = \frac{a}{a+\zeta^2}\cdot$$

If $a > \sqrt{\pi}/2$ then g' will be regular in the strip and $v(\tau)$, $\tau \in \mathbf{R}$, will be a convex function with bounded derivative. We compute the Schwarzian of g to be

$$Sg(\zeta) = \frac{-2a}{(a+\zeta^2)^2}\cdot$$

Then $f = g \circ F_0$ is normalized and

$$Sf(z) = Sg(F_0(z))(F_0'(z))^2 + SF_0(z) = \frac{2}{(1-z^2)^2}\left\{1 - \frac{4a}{(a+\zeta^2)^2}\right\}, \qquad \zeta = F_0(z).$$

It is not hard to show that if a is sufficiently large then

$$\left|1-\frac{4a}{(a+\zeta^2)^2}\right| \leq 1,$$

so that $f \in N_0^*$.

REFERENCES

[1] AHLFORS, L. and WEILL, G., *A uniqueness theorem for beltrami equations*. Proc. Amer. Math. Soc., *13*: 975–978, 1962.

[2] CHUAQUI, M. and OSGOOD, B., *Sharp distortion theorems associated with the Schwarzian derivative*. Jour. London Math. Soc. (2), *48*: 289–298, 1993.

[3] CHUAQUI, M. and OSGOOD, B., *An extension of a theorem of Gehring and Pommerenke*. Israel Jour. Math., to appear.

[4] DUREN, P. L. and LEHTO, O., *Schwarzian derivatives and homeomorphic extensions*. Ann. Acad. Sci. Fenn. Ser. A 1 Math., *477*: 3–11, 1970.

[5] EPSTEIN, C., *The hyperbolic Gauss map and quasiconformal reflections*. J. reine u. angew. Math., *372*: 96–135, 1986.

[6] GEHRING, F. W. and POMMERENKE, CH., *On the Nehari univalence criterion and quasicircles*. Comment. Math. Helv., *59*: 226–242, 1984.

[7] KIM, S.-A. and MINDA, D., *The hyperbolic and quasihyperbolic metrics in convex regions*. J. Analysis, *1*: 109–118, 1993.

[8] MINDA, D. and OVERHOLT, M., *The minimum points of the Poincaré metric*. Complex Variables, *21*: 265–277, 1993.

[9] NEHARI, Z., *The Schwarzian derivative and schlicht functions*. Bull. Amer. Math. Soc., *55*: 545–551, 1949.

[10] NEHARI, Z., *A property of convex conformal maps*. Jour. d'Analyse Math., *30*: 390–393, 1976.

[11] OSGOOD, B., *Some properties of f''/f' and the Poincaré metric*. Indiana Univ. Math. Jour., *31*: 449–461, 1982.

[12] PAATERO, V., *Über die konforme Abbildung von Gebieten daren Ränder von beschränkter Drehung sind*. Ann. Acad. Sci. Fenn. Ser. A XXXIII, *9*: 1–79, 1931.

[13] YAMASHITA, S., *The Schwarzian derivative and local maxima of the Bloch derivative*. Math. Japonica, *37*: 1117–1128, 1992.

P. Universidad Católica de Chile,
Chile

and

Department of Mathematics,
Stanford University,
Stanford,
CA 94305-2125,
USA

Received March 21, 1994