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Commentarii Mathematici Helvetici

Curvature of curvilinear 4-webs and pencils of one forms: Variation on a theorem of Poincaré, Mayrhofer and Reidemeister

Isao Nakai

Abstract. A curvilinear d-web $W = (F_1, \ldots, F_d)$ is a configuration of d curvilinear foliations F_i on a surface. When $d = 3$, Bott connections of the normal bundles of F_i extend naturally to equal affine connection, which is called *Chern connection.* For $3 < d$, this is the case if and only if the modulus of tangents to the leaves of F_i at a point is constant. A d-web is *associative* if the modulus is constant and *weakly associative* if Chern connections of all 3-subwebs have equal curvature form. We give a geometric interpretation of the curvature form in terms of fake billiard in §2, and prove that a weakly associative d-web is associative if Chern connections of triples of the members are non flat, and then the foliations are defined by members of a pencil (projective linear family of dim 1) of 1-forms. This result completes the classification of weakly associative 4-webs initiated by Poincaré, Mayrhofer and Reidemeister for the flat case. In §4, we generalize the result for $n+2$-webs of $n-$spaces.

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Keywords. Web, Chern connection, Godbillon-Vey class.

A*curvilinear d-web* on a surface S is an d-tuple of foliations of codimension 1, $W = (F_1, \ldots, F_n)$. In this paper we assume that S is real analytic, connected and oriented, and F_i is defined by a real meromorphic 1-form ω_i: of which coefficients are locally fractions of real analytic functions. W is *non singular* at a $p \in S$ if ω_i and $\omega_i \wedge \omega_j$ are analytic and non zero at p for $i \neq j$. $\Sigma(W)$ denotes the set of those p where W is singular. W is *diffeomorphic* to a d-web $W' = (F'_1, \ldots, F'_d)$ on S' if there exists an analytic diffeomorphism of S to S' sending F_i to F'_i for $i = 1, \ldots, d$. An m-subweb of W is an m-tuple of members of W.

First let $d = 3$ and assume W is non singular at p. Since the defining 1-forms $\omega_i, i = 1, 2, 3$ on a surface are linearly dependent, we may assume $\omega_1 + \omega_2 + \omega_3$ vanishes identically on a neighbourhood of p. Then there exists unique 1-form θ on a neighbourhood of p such that $d\omega_i = \theta \wedge \omega_i$ for $i = 1, 2, 3$ [1,2]. The exterior derivative $d\theta$ is independent of the forms ω_i defining F_i as well as the permutation of suffix i. $d\theta$ is called the *web curvature form* of W and denoted $K(W)$. Bott connections of the normal bundles of F_1, F_2, F_3 defined by the transverse dynamics extend to unique affine connection without torsion the so-called *Chern connection* on the complement of $\Sigma(W)$ (see §1 for the definition). And the leaves of F_i are geodesics of the connection. Chern connection has the connection form $\Theta = \begin{pmatrix} -\theta & 0 \\ 0 & -\theta \end{pmatrix}$ with respect to the coframe (ω_1, ω_2) (and ω_3) and the curvature form $d\Theta = \begin{pmatrix} -K(W) & 0 \\ 0 & -K(W) \end{pmatrix}$ [1,2,11]. A 3-web is *hexagonal* (or *flat*) if the web curvature vanishes identically. It is classically known that a hexagonal 3-web is locally diffeomorphic to the 3-web of parallel lines on the plane [1,2] (see Fig.0 and §1).

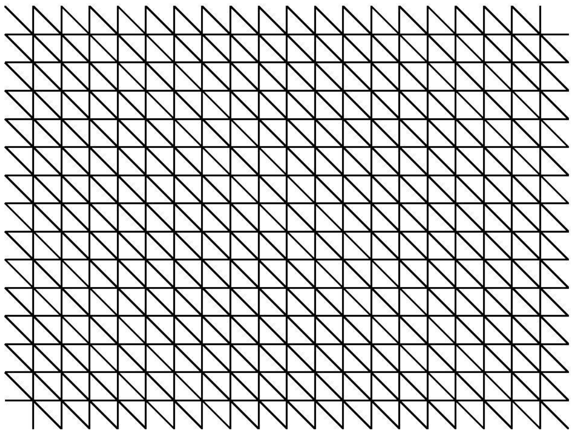

Figure 0. Hexagonal 3-web

A non singular 4-web $W = (F_1, \ldots, F_4)$ possesses the relative and absolute invariants: *web curvature forms of 3-subwebs* and the cross ratio of tangents to the leaves of $F_i, i = 1, \ldots, 4$ passing through a point, which is a special case of the basic affinor in higher dimensional webs (see [12] for the definition). The higher covariant derivatives of the cross ratio generate all other absolute invariants [1,12].

We call d curvilinear foliations as well as a d-web are *associative* if their Bott connections of the normal bundles extend to equal affine connection, in other words, all 3-subwebs have equal Chern connection on the complement of the singular locus. It is easy to see that d foliations $(3 < d)$ are associative if and only if the modulus of tangents to the leaves passing through a point is constant. Clearly if a d-web is associative, it is weakly associative, i.e. Chern connections of 3-subwebs have equal curvature form. But the converse is not always the case. Poincaré [4,16,24], Mayrhofer [20,21] and Reidemeister [25] proved

Theorem 0. Poincaré, Mayrhofer, Reidemeister *Let W be a germ of non singular 4-web on a surface. Assume that all 3-subwebs are hexagonal. Then W*

is diffeomorphic to a germ of 4-web of 4 pencils of lines on the projective plane (Fig.1).

Here a pencil of lines is a family of projective lines passing through a base point.

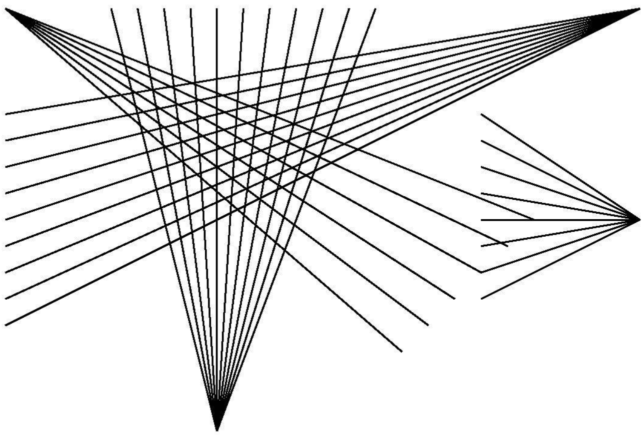

Figure 1. 4-web of pencils of lines

Assume no 3 of base points of the pencils in the theorem are not collinear. Then we may assume, after a projective linear transformation, the base points of 3 pencils in the theorem respectively to $(1:0:0), (0:1:0), (0:0:1)$, and the leaves of those 3 pencils are level lines of the functions (called level functions or defining functions) $u_1 = y/z, u_2 = z/x, u_3 = x/y$ in the homogeneous coordinates x, y, z. These functions enjoy the obvious relation $u_1u_2u_3 = 1$, from which

$$\frac{du_1}{u_1} + \frac{du_2}{u_2} + \frac{du_3}{u_3} = 0.$$

This implies that the connection form is 0 and the 3-web is flat. So 3-subwebs of the 4-web in the theorem are hexagonal (curvature vanishes).

But generically, their Chern connections are not equal since the cross ratio is not constant unless the base points are collinear. Henaut [16] gives a simple proof of Theorem 0, and Goldberg [12] investigated a different approach to this problem. This paper is devoted to generalizing Theorem 0 and finding all weakly associative n-webs.

An *abelian equation* of a germ of plane d-web $W = (F_1, \dots, F_d)$ is a relation of germs of closed one forms ω_i' such that $\omega_i' \wedge \omega_i = 0$ and

$$\omega_1' + \dots + \omega_d' = 0.$$

Denote the $\mathbb{R}$-linear space of Abelian equations by $\mathcal{A}$. The *rank* of W is

$$\text{rank } W = \dim \mathcal{A}.$$

The following bound is classically known (see c.f.[1,2.4]),

$$\text{rank } W \leq \pi(d,2) = \frac{1}{2}(d-1)(d-2),$$

that is Castelnuovo bound: the maximal genus of plane algebraic curves of degree d. One of the most important problems in Web geometry is the following linearization problem ([4]).

Linerarization Problem. Assume a germ of d-web attains the maximal rank $\frac{1}{2}(d-1)(d-2)$. Then is it linearizable?, in other words, does there exist a germ of diffeomorhpism of the plane which sends those leaves to lines?

The importance of the problem is easily understood by the following well known theorem. Let W be a linear $d-$web: the leaves are lines. Then the leaves of F_i define germs of holomorphic curves C_i in the projective dual space. Lie[18,19] and Darboux[6] proved the following theorem.

Algebrization Theorem. Assume W is a linear $d-$web and admits an Abelian equation $\omega'_1 + \cdots + \omega'_d = 0$ such that no ω'_i vanishes identically. Then $C_i, i = 1,\ldots,d$ extend to a (not necessarily irreducible) algebraic curve of degree d.

This was later generalized by Griffiths[13]. By a result in [22] topological structure of linear webs determines the curve C up to projective equivalence except for the case $d = 3$.

In some cases an affirmative answer to the problem is well known. For example

$$\pi(3,2) = 1 \qquad \pi(4,2) = 3 \qquad \textit{and} \qquad \pi(5,2) = 6.$$

In the case $d = 3$, by integrating an Abelian equation

$$\omega_1 + \omega_2 + \omega_3 = 0$$

we obtain

$$\int \omega_1 + \int \omega_2 + \int \omega_3 = 0.$$

The triple of germs $(\omega_1, \int \omega_2, \int \omega_3) : S \to \mathbb{R}^3$ sends F_i to the foliation defined by $i-$th coordinate function on the linear space $\mathbb{C}^2 = \{x_1 + x_2 + x_3 = 0\}$. Therefore, the 3-web W is linearizable and hexagonal.

In the case $d = 4$, an affirmative result was obtained by Wirtinger[28] by using an idea due to Poincaré. So let W be a linear 4-web. By the above theorem, the germs $C_i, i = 1, \ldots, 4$ are contained in an algebraic curve C of degree 4. If W is hexagonal, each 3-subweb admits an Abelian relation hence, by the theorem, each triple of $C_i, i = 1 \ldots, 4$ extends to an algebraic curve of degree 3. Therefore C splits into a union of 4 lines. This tells W consists of 4 pencils of lines. This is the 4-web in Theorem 0. However rank of the 4-web of pencils depends on the configuration of base points (see c.f.[16]).

In the case $d = 5$, the following counter example is known. The *exceptional Bol 5-web* on the plane $E(5)$ consists of the pencils of conics F_i passing through 4 points $p_j, i \neq j$ among five points $p_1, \ldots, p_5$ such that no 3 points are not collinear. By Cremona transformations with 3 of those base points, say p_3, p_4, p_5, those pencils are sent to 2 pencils of lines respectively with base points p_1, p_2 and 3 pencils of conics passing through p_1, p_2 such that the base point sets outside p_1, p_2 intersect mutually with each other (the union of base points off p_1, p_2 consists of 3 points.) Again by Cremona transformation with vertices p_1, p_2 and one of those 3 base points, the 5-web is sent to 4 pencils of lines such that the 4 base points are non degenerate, and one pencil of conics passing through the 4 base points of pencils of lines. By a projective transformation we may assume those 4 base points are $(0,0), (1,0), (1,1), (0,1)$. Therefore $E(5)$ is birationally equivalent to (a germ of) the 5-web defined by the level functions in the coordinates (x, y):

$$u_1 = \frac{x}{y}, \quad u_2 = \frac{x+y-1}{x}, \quad u_3 = \frac{y-x}{1-x}, \quad u_4 = \frac{1-x}{y}, \quad u_5 = \frac{y(1-y)}{x(1-x)}. \tag{1}$$

These functions play the relations

$$u_i u_{i+1} = 1 - u_{i+3}$$

for $i = 1, \ldots, 5$ with the cyclic indices, from which we obtain rather trivial 5 Abelian equations

$$\frac{du_i}{u_i} + \frac{du_{i+1}}{u_{i+1}} + \frac{du_{i+3}}{1-u_{i+3}} = 0.$$

From these equations, we obtain also

$$\sum_{i=1}^{5} \frac{\log\,(1-u_i)}{u_i} + \frac{\log\, u_i}{1-u_i} du_i = 0, \tag{2}$$

and by integrating the equation

$$\sum_{i=1}^{5} \Phi(u_i) = \text{const}, \tag{3}$$

where $\phi(z) = \sum \frac{z^n}{n^2}$ is Euler dilogarithm and $\Phi(z) = \phi(1-z) - \phi(z)$ is Roger dilogarithm.

Relation (3) is nothing but the 5-term relation of 5 cross ratios of quadruples of 5 lines configuration passing through the origin on the plane. In fact, the level function u_i is, after permutation of indices, the cross ratio of the tangent lines of the leaves of $F_j, i \neq j$ passing through a point. This example was generalized by Gelfand-MacPherson[7,9], Damiano[5] and Hain-Hanamura-MacPherson[14,15] to study polylogarithms and also to obtain a combinatorial formula of Pontrjagin classes.

By the above form of $E(5)$ and the symmetry under permutation of the indices in the former presentation, we see that $E(5)$ is hexagonal. The Abelian equations (1), (2) are linearly independent, hence $E(5)$ has the maximal rank $= 6$. One the other hand it is known that $E(5)$ is non linearizable.

A $d-$web $W = (F_1, \dots F_d)$ is *hexagonal* if all 3-subwebs (F_i, F_j, F_k) are hexagonal. Let $H(d)$ denote the set of germs of hexagonal $d-$webs. Bol[1] generalized Theorem 0 as follows.

Theorem 1. (G. Bol) *Let $W \in H(d)$. Assume W is not diffeomorphic to a germ of exceptional Bol 5-web $E(5)$. Then W is diffeomorphic to a $d-$web of d pencils of lines.*

The rank of hexagonal and non exceptional $d-$web depends on the configuration of the base points ([16]).

The equivalent condition of $d-$webs of maximal rank to be linearized is obtained by Henaut[17] in terms of web polynomials. One of crucial open problems in [4] is

Problem. Classify non linearizable $d-$webs of maximal rank, and find all Abelian equations.

The purpose of this paper is to generalize Theorem 0 from a view point of differential geometry. Those webs in Theorems 0 and 1 are all hexagonal hence all 3-subwebs have equal curvature forms 0. We classify all d-webs, for which all 3-subwebs have equal non-zero curvature forms.

Before stating our result we prepare some notions. A *pencil of meromorphic one forms* $P = \{\omega^t\}$ on S is a projective linear family of one forms defined on S, $\omega^t = u\omega^0 + v\omega^1$, $(u, v) \in \mathbb{R}^2 - (0, 0)$, where t stands for the homogeneous coordinate $(u : v)$ as well as the coordinate $\frac{v}{u+v} \in \mathbb{R} \cup \infty$. In the second coordinate, $\omega^t = (1-t)\omega^0 + t\omega^1$ holds for $t \in \mathbb{R} \cup \infty$. P is *non singular* at p if ω^s and $\omega^s \wedge \omega^t$ are analytic and non zero at p for distinct $s, t \in \mathbb{P}^1$. We denote the set of those p where P is singular by $\Sigma(P)$. The *web curvature form $K(P)$ for P* is the 2-form $d\theta$ defined on the complement of $\Sigma(P)$, where θ is unique 1-form such that $d\omega^t = \theta \wedge \omega^t$ for all $t \in \mathbb{P}^1$. ($-\theta$ is called the *connection form of P*.) Clearly all members of P are associative and all triples of the members form 3-webs which have equal web curvature form $K(P)$. Cerveau [3], Ghys [10] and the author [23] applied the web geometry of 3-webs of codimension 1 to classify codimension 2 foliations of 3 manifolds. In the papers by Gelfand-Zakharevich[8] and Regal[26], a generalized structure of linear family of one forms is studied.

Example. Let W be a 3-web defined by level functions x, y and a function f and let

$$\omega_0 = -f_x\ dx, \qquad \omega_1 = -f_y\ dy.$$

Clearly these one forms define the coordinate foliations of x, y. Define

$$\omega^t = (1-t)\ \omega_0 + t\ \omega_1.$$

Then $\omega^{1/2} = -\frac{1}{2}df$ defines the foliation by f. Let

$$\theta = \frac{f_{xy}}{f_y}\ dx\ +\ \frac{f_{xy}}{f_x}\ dy.$$

Then

$$d\omega^t\ =\ \theta \wedge\ \omega^t$$

holds for all $t \in \mathbb{R}$. This tells that all triples $(\omega^{t_1}, \omega^{t_2}, \omega^{t_3})$ have the same connection form $\Theta = \begin{pmatrix} -\theta & 0 \\ 0 & -\theta \end{pmatrix}$ with the coframe $(\omega^{t_1}, \omega^{t_2})$ and the curvature form $d\Theta = \begin{pmatrix} -d\theta & 0 \\ 0 & -d\theta \end{pmatrix}$, where $d\theta = \log\ (\frac{f_x}{f_y})_{xy}\ dx \wedge dy$ is the web curvature form. Corollary 3 asserts that if all 3-subwebs of a $d-$web W have equal non zero curvature form, then W embeds to a one parameter family of foliations defined by an $P = \{\omega^t\}$. A similar result in higher dimensional case is investigated in §4.

Let $W = (F_1, F_2, F_3)$ be a non singular 3-web on an oriented surface S. In this paper *geodesics* mean the leaves of the foliations. A *geodesic triangle* is a smooth triangle $\Delta = \Delta(E_1, E_2, E_3)$ with edges E_i in a leaf $L_{\sigma(i)} \in F_{\sigma(i)}$ for $i = 1, 2, 3$ transversal at the vertices $V_{j,k} = E_j \cap E_k, j \neq k$. Here σ is a permutation of $\{1, 2, 3\}$ and the convention $E_{i+3} = E_i$ is used. The *orientation* of Δ and the edge E_i are given by $\partial\Delta = E_1 + E_2 + E_3$ and $\partial E_i = V_{i,i+1} - V_{i-1,i}$ (see Fig. 2). Define $\sigma(\Delta) = 1$ or -1 alternatively if the orientation is positive or opposite. From later on we assume the permutation σ is trivial.

Let $W = (F_1, F_2, F_3, F_4)$ be a 4-web. A *Schläfli configuration* is a quadruple of geodesic triangles $\Delta_1 = \Delta(E_2, E_3, E_4), \Delta_2 = \Delta(E_1', E_3', E_4'), \Delta_3 = \Delta(E_1'', E_2'', E_4'') \subset \Delta_2$ and $\Delta_4 = \Delta(E_1''', E_2''', E_3''') \subset \Delta_1$ with the following properties (see Fig.3).

(1) The edges with suffix i are contained in a common leaf $L_i \in F_i$ for $i = 1, ..., 4$,

(2) Δ_j, Δ_k have common vertex $V_{m,n} = E_m \cap E_n$, where $\{j, k, m, n\} = \{1, 2, 3, 4\}$,

(3) $\Delta_2 + \Delta_4 = \Delta_1 + \Delta_3$, where Δ_i denotes the underlying set of Δ_i.

(4) The 3-subweb $W_i = (F_1, ..., \hat{F_i}, ..., F_4)$ is non singular on a neighbourhood of Δ_i for $i = 1, \ldots, 4$.

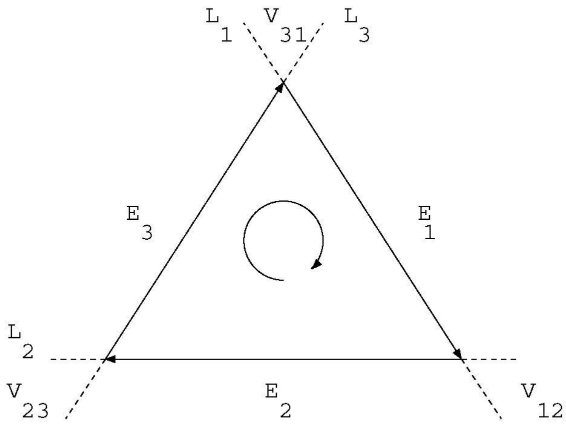

Figure 2. $\Delta(E_1, E_2, E_3)$

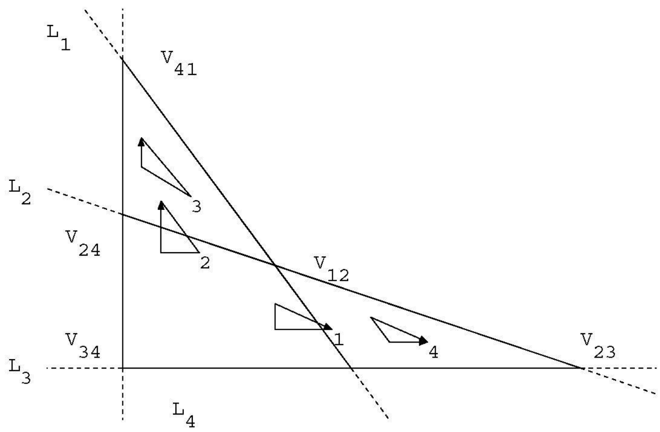

Figure 3. *Schläfli configuration*

In other words a Schläfli configuration is formed by leaves of $F_1, \ldots, F_4$ in general position. The goal of this paper is to prove the following generalization of Theorem 0.

Theorem 2. *Assume 3-subwebs of a 4-web $W = (F_1, F_2, F_3, F_4)$ are non hexagonal. Then the following conditions are equivalent.*

(1) F_i is defined by a 1-form $\omega^{t_i}, t_i \in \mathbb{P}^1$, in a pencil of meromorphic 1-forms $P = \{\omega^t\}$.

(2) The cross ratio $C(F_4, F_3, F_2, F_1)$ of tangents to the leaves of $F_i, i = 1, ..., 4$

passing through a point is constant on the complement of $\Sigma(W)$.

(3) $F_1, \dots, F_4$ *are weakly associative: The web curvature form* $K(F_1, ..., \hat{F}_i, ..., F_4)$ *of the 3-subweb* $W_i = (F_1, ..., \hat{F}_i, ..., F_4)$ *is independent of* i.

(4) For any Schläfli configuration $(\Delta_1, \Delta_2, \Delta_3, \Delta_4)$,

$$\sum_{i=1}^{4} \sigma(\Delta_i) \int_{\Delta_i} K(F_1, ..., \hat{F}_i, ..., F_4) = 0. \tag{$\star$}$$

(5) $F_1, \dots, F_4$ *are associative: Bott connections of the normal bundles of* $F_1, \dots, F_4$ *extend to equal affine connection on the complement of* $\Sigma(W)$.

The following is an immediate consequence of the theorem, and in remarkable contrast to Theorem 1.

Corollary 3. *Let* $W = (F_1, \dots, F_d)$ *be a non hexagonal d-web. The the following conditions are equivalent.*

(1) F_i *is defined by a 1-form* $\omega^{t_i}, t_i \in \mathbb{P}^1$, *in a pencil of meromorphic 1-forms* $P = \{\omega^t\}$.

(2) The modulus of tangents to the leaves of $F_i, i = 1, ..., d$ *passing through a point is constant on the complement of* $\Sigma(W)$.

(3) $F_1, \dots, F_d$ *are weakly associative.*

(5) $F_1, \dots, F_d$ *are associative.*

In §4 we prove a generalization of Theorem 2 for d-webs of an n-manifold, $n + 1 < d$, of codimension one.

All results in this paper remain valid for non singular C^3-smooth webs. The argument is local, so from now on we assume S is a connected domain of $\mathbb{R}^2$.

1. Bott connection and Chern connection

Bott connection of a non singular foliation is defined by the transverse dynamics. To state more precisely in our setting, recall the integrability condition in Frobenius theorem

$$d\omega_i = \theta \wedge \omega_i,$$

where ω_i is a defining one form of F_i such that $\omega_1 + \omega_2 + \omega_3 = 0$. The 1-form θ defines the (partial) connection of the normal bundle of the foliation F_i as follows. Let L be a leaf of F_i, $p, q \in L$, and $C \subset L$ an oriented smooth curve joining p to q. The parallel transport $T(X)$ of a vector X normal to L at p along C is defined by the relation

$$\omega_i(T(X)) = \exp(\int_C \theta) \cdot \omega_i(X).$$

To extend Bott connection to an affine connection of S, consider an (infinitesimally) small geodesic triangle Δ with vertex p. By the transverse dynamics along C, Δ is transported to unique (infinitesimally) small geodesic triangle Δ' with vertex q (Fig.4).

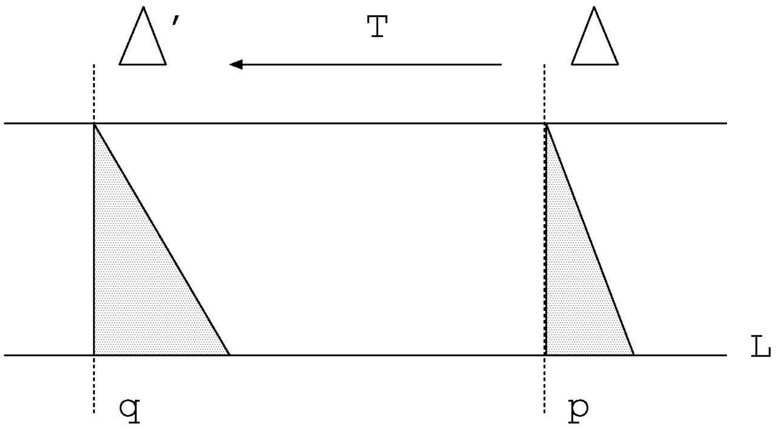

Figure 4. Transverse dynamics and Bott connection

This parallel transport determines a linear map of the tangent spaces T_pS to T_qS. The linear map is defined also for all piecewise "geodesic" curves by composite of those linear maps along geodesic pieces. It is easy to see this transportation determines an affine connection with the structure equation

$$d\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} = \begin{pmatrix}\theta & 0\\ 0 & \theta\end{pmatrix}\wedge\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}$$

with respect to the coframe ω_1, ω_2. The connection form of the affine connection is $\begin{pmatrix}-\theta & 0\\ 0 & -\theta\end{pmatrix}$ and the curvature form is

$$d\begin{pmatrix}-\theta & 0\\ 0 & -\theta\end{pmatrix}+\begin{pmatrix}-\theta & 0\\ 0 & -\theta\end{pmatrix}\wedge\begin{pmatrix}-\theta & 0\\ 0 & -\theta\end{pmatrix} = \begin{pmatrix}-d\theta & 0\\ 0 & -d\theta\end{pmatrix} = \begin{pmatrix}-K(W) & 0\\ 0 & -K(W)\end{pmatrix}.$$

This affine connection is called Chern connection of the 3-web W. Chern connection is in other words unique common extension of Bott connections of F_1, F_2, F_3. The structure group of the connection is $\mathbb{R}^*$: the group of similar transformations, and the holonomy map along a closed cycle C is

$$\exp\,(\int_C \theta)\cdot\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \exp\,(\int_{\text{Area bounded by } C} K(W))\cdot\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

In Web geometry $K(W)$ is called the *Web curvature form.*

Assume that θ is closed, i.e. the web curvature form vanishes identically. Then $\tilde{\omega}_i = \exp(-\int\theta)\cdot\omega_i$ is closed and $\tilde{\omega}_1+\tilde{\omega}_2+\tilde{\omega}_3=0$. By integrating this equation, we obtain the *developing map* $(\int\tilde{\omega}_1, \int\tilde{\omega}_2, \int\tilde{\omega}_3)$ of S into the hyperplane $H=\{(u_1,u_2,u_3)\in\mathbb{R}^3 \mid u_1+u_2+u_3=0\}$, which sends the leaves of the web to the lines defined by $u_i=$ const. in H. Thomsen (c.f.[4,27]) proved that the Hexagonality of 3-webs is equivalent to the closure condition of all piecewise geodesic hexagons as in Fig.5.

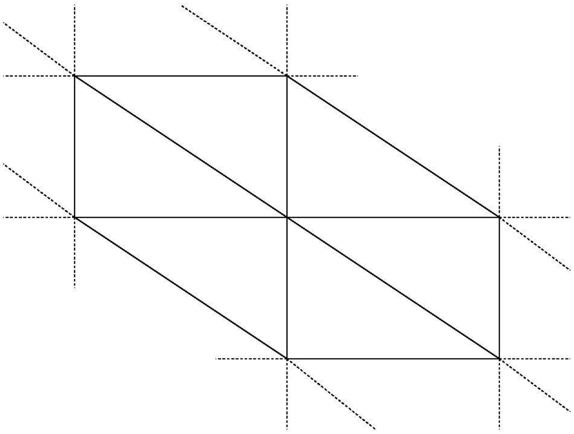

Figure 5. Closed hexagon

In general this "hexagon" is not closed (Fig.6) .

Now assume that $S\subset\mathbb{R}^2$ and the foliations F_1,F_2,F_3 are locally defined by level functions x,y and an $f(x,y)$ on a neighbourhood of the origin such that $f(t,0)=f(0,t)=t$ and $f(t,t)=2t$: $f(x,y)=x+y+k(x-y)xy+\cdots$. Then the web curvature form for the 3-web $W=(F_1,F_2,F_3)$ is

$$K(W)=\frac{\partial^2}{\partial x\partial y}\left(\log\frac{f_x}{f_y}\right)dx\wedge dy=(k+\text{higher terms})dx\wedge dy$$

and the return map $R(x)$ of the x-axis as in Fig.6 is written in the coordinate x on the leaf L centered at p as follows.

$$R(x)=x+kx^3+\cdots .$$

To see Taylor coefficients of R, it suffices to note $R'(x)=1+3kx^2+\cdots$ is the linear term of the holonomy at x anti-clockwisely along the "non-closed hexagon" in Fig.6, while the area of the "hexagon" is proportional to $3x^2$. In the next section we interpret the curvature in terms of "fake billiards", which is composites of transverse dynamics along piecewise geodesic curves.

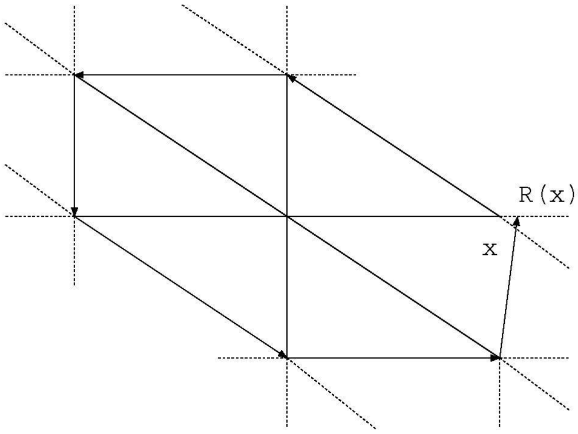

Figure 6. Non-closed hexagon

2. Transverse dynamics and fake billiard

Let $W = (F_1, F_2, F_3)$ be a non singular 3-web on an $S \subset \mathbb{R}^2$ and assume all leaves are connected. Let L_i be a leaf of F_i, $p, q \in L_i$ and $j, k \neq i$. The *transverse dynamics*

$$T_{pq}^{jk} : L_j(p), p \to L_k(q), q$$

is a germ of diffeomorphism of the leaf $L_j(p)$ of F_j passing through p to the leaf $L_k(q)$ of F_k passing through q, which assigns to an $x \in L_j(p)$ close to p unique intersection point $y \in L_i(x) \cap L_k(q)$. *Fake billiard* along boundary of an oriented geodesic triangle $\Delta = \Delta(E_1, E_2, E_3)$ is the return map

$$T_{\partial\Delta}^i = T_{i+2} \circ T_{i+1} \circ T_i : L_{i+1}(V_{i-1,i}), V_{i-1,i} \to L_{i+1}(V_{i-1,i}), V_{i-1,i},$$

where T_j denotes the transverse dynamics along the edge E_j

$$T_{V_{j-1,j}V_{j,j+1}}^{j+1,j+2} : L_{j+1}(V_{j-1,j}), V_{j-1,j} \to L_{j+2}(V_{j,j+1}), V_{j,j+1},$$

(see Fig.7 and Fig.8).

Clearly $T_{\partial\Delta}^i, i = 1, 2, 3$ are conjugate with each other : $T_i^{-1} \circ T_{\partial\Delta}^{i+1} \circ T_i = T_{\partial\Delta}^i$. We denote the derivative of $T_{\partial\Delta}^i$ at $V_{i-1,i}$ by dT_Δ.

Lemma 1. *Fake billiard along an oriented geodesic triangle $\Delta = \Delta(F_1, F_2, F_3)$ has the following derivative at the vertices*

$$dT_\Delta = -exp\left(-\sigma(\Delta) \int_\Delta K(F_1, F_2, F_3)\right),$$

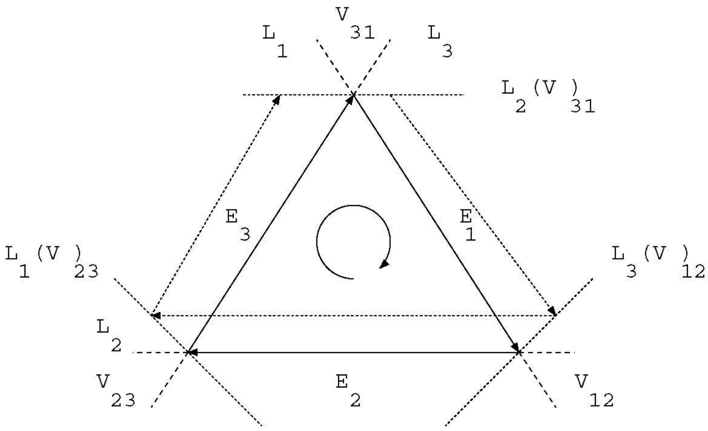

Figure 7. Fake billiard $T^1_{\partial\Delta}$

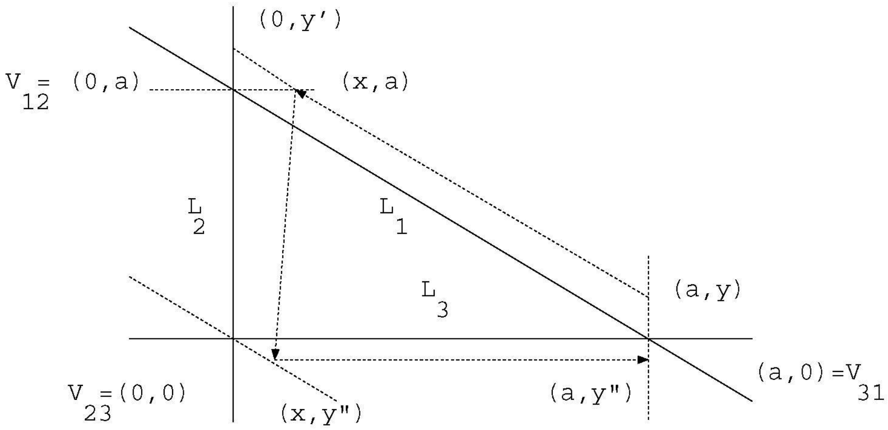

Figure 8. Fake billiard $T^1_{\partial\Delta}$ along another triangle

where $\sigma(\Delta) = 1$ or -1 respectively Δ is the clockwise orientation or anti-clockwise.

Proof. Assume Δ and F_1, F_2, F_3 are defined by level functions f, x and y as in Fig.8.

Then $\sigma(\Delta) = -1$. Let $V_{31} = (a, 0), V_{12} = (0, a)$ and

$$T^{22}_{V_{31}V_{12}}(a, y) = (0, y').$$

Then we obtain

$$\begin{aligned}\log\left(\frac{dy'}{dy}(y=0)\right) &= \int_{V_{31}V_{12}}\left(-\frac{f_x}{f_y}\right)_y dx\\ &= \int_{V_{31}V_{12}}\left(\frac{f_x}{f_y}\right)_y \frac{dx}{dy}\, dy\\ &= \int_{V_{31}V_{12}}\frac{\left(\frac{f_x}{f_y}\right)_y}{\frac{f_x}{f_y}}\, dy\\ &= \int_{V_{31}V_{12}}\left(\log\frac{f_x}{f_y}\right)_y dy.\end{aligned}$$

Let $T^{23}_{V_{12}V_{12}}(0,y') = (x,a)$. Then

$$\log\frac{dx}{dy'}(y'=a) = \log\frac{f_x}{f_y}(0,a). \tag{2}$$

Let $T^{31}_{V_{12}V_{23}}(x,a) = (x,y'')$. Then

$$\log\frac{d''}{dx}(x=0) = \log\frac{f_x}{f_y}(0,0). \tag{3}$$

From (2) and (3), we obtain

$$\begin{aligned}\log\frac{dy''}{dy'} &= \log\frac{dy''}{dx} - \log\frac{dy'}{dx}\\ &= \log\frac{f_x}{f_y}(0,0) - \log\frac{f_x}{f_y}(0,a)\\ &= \int_{V_{12}V_{23}}\left(\log\frac{f_x}{f_y}\right)_y dy.\end{aligned} \tag{4}$$

From (1) and (4)

$$\begin{aligned}\log\left(-\frac{dy''}{dy}\right) &= \int_{V_{31}V_{12}V_{23}}\left(\log\frac{f_x}{f_y}\right)_y dy\\ &= \int_\Delta\left(\log\frac{f_x}{f_y}\right)_{xy} dx\wedge dy\\ &= \int_\Delta K(W).\end{aligned}$$

This completes the proof. □

Let $W = (F_1, F_2, F_3, F_4)$ be a non singular 4-web on $S \subset \mathbb{R}^2$. For the lines $k_i = \{y = a_i x\}, l_i = \{y = b_i x\}, i = 1, 2$ define the cross ratio $C(k_1, k_2, l_1, l_2)$ by

$$C(k_1, k_2, l_1, l_2) = \frac{(a_1 - b_1)(a_2 - b_2)}{(a_1 - b_2)(a_2 - b_1)}.$$

Define the *cross ratio* of tangents to the leaves $L_i(V_{i,j}) \in F_i$ passing through the intersection $V_{i,j} = L_i \cap L_j$ by

$$C(V_{i,j}) = C(T_p L_4(V_{i,j}), T_p L_3(V_{i,j}), T_p L_2(V_{i,j}), T_p L_1(V_{i,j})).$$

From now on we assume $1 < C(k_1, k_2, l_1, l_2) < \infty$ (this holds uniformly on the connected component of $V_{i,j}$ in the non-singular locus).

Lemma 5. *Let $\Delta_3 = \Delta(E_1, E_2, E_4)$ be a geodesic triangle of the 3-subweb (F_1, F_2, F_4) (Fig.9). Then the following composite of transverse dynamics along $\partial\Delta_3$,*

$$\tilde{T}_{\Delta_3} = T^{34}_{V_{41}V_{12}} \circ T^{33}_{V_{24}V_{41}} \circ T^{43}_{V_{12}V_{24}} : L_4(V_{12}), V_{12} \to L_4(V_{12}), V_{12}$$

has the derivative at V_{12}

$$d\tilde{T}_{\Delta_3} = \frac{C(V_{41})}{C(V_{41}) - 1}(1 - C(V_{24}))\; dT_{\Delta_3}.$$

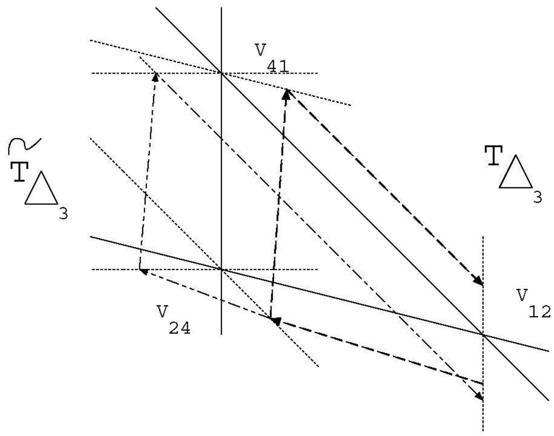

Figure 9.

Proof. Let f_i be a level function of F_i defined on a neighbourhood of Δ_3. Let

$$T^{32}_{V_{41}} : \ L_3(V_{41}), V_{41} \to L_2(V_{41}), V_{41}$$
$$T^{13}_{V_{24}} : \ L_1(V_{24}), V_{24} \to L_3(V_{24}), V_{24}$$

denote the transverse dynamics respectively along the leaves of F_1, F_2 such that

$$f_1 \circ T^{32}_{V_{41}} = f_1 \ , \quad f_2 \circ T^{13}_{V_{24}} = f_2.$$

It is easy to see

$$d\left(f_4 \circ T^{13}_{V_{24}}\right) = (1 - C(V_{24}))\ df_4,$$
$$d\left(f_4 \circ T^{32}_{V_{41}}\right) = \frac{(C(V_{41})}{(C(V_{41}) - 1)}\ df_4,$$

from which

$$d(f_4 \circ \ T^{32}_{V_{41}} \circ T^{33}_{V_{24}V_{41}} \circ T^{13}_{V_{24}}) = \frac{(C(V_{41})}{(C(V_{41}) - 1)}(1 - C(V_{24}))\ df_4.$$

By definition we obtain

$$\tilde{T}_{\Delta_3} = T^{24}_{V_{41}V_{12}} \circ T^{32}_{V_{41}} \circ T^{33}_{V_{24}V_{41}} \circ T^{13}_{V_{24}} \circ T^{41}_{V_{12}V_{24}},$$

from which we obtain the statement. □

Lemma 6. *Let $\Delta_1, \Delta_2, \Delta_3$ be as in Fig.10. Then*

$$d\tilde{T}_{\Delta_3} = dT_{\Delta_2} \cdot dT_{-\Delta_1} \cdot C(V_{34}),$$

where $-\Delta_1$ denotes the triangle Δ_1 with reverted orientation.

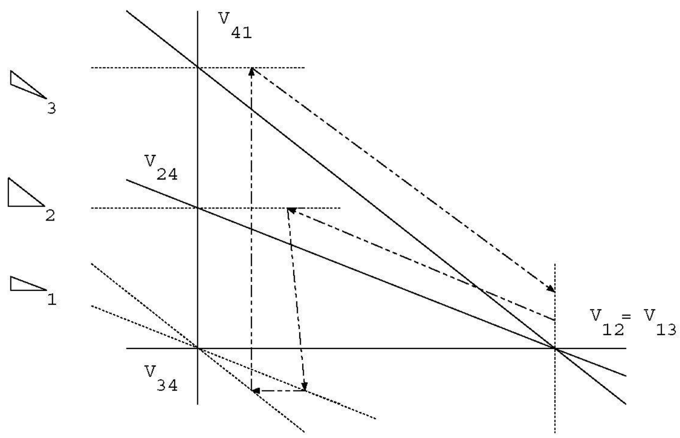

Figure 10.

Proof. Let $T_{\Delta_2 V_{13}}, T_{-\Delta_1 V_{13}}, T_{\Delta_3 V_{24}}$ denote fake billiards along $\partial\Delta_2, -\partial\Delta_1, \partial\Delta_3$ starting at V_{13}, V_{24}. It is easy to see fake billiard $T_{\Delta_2 V_{13}} \circ T_{-\Delta_1 V_{13}}$ is conjugate with

$$\tilde{T}_{\Delta_3 V_{24}} \circ T^{13}_{V_{34}V_{24}} \circ T^{41}_{V_{13}V_{34}} \circ T^{24}_{V_{34}V_{13}} \circ T^{32}_{V_{24}V_{34}} = \tilde{T}_{\Delta_3 V_{24}} \circ T^{13}_{V_{34}V_{24}} \circ T^{21}_{V_{34}} \circ T^{32}_{V_{24}V_{34}},$$

where $T^{21}_{V_{34}} : L_2(V_{34}), V_{34} \to L_1(V_{34}), V_{34}$ is defined by $f_3 \circ T^{21}_{V_{34}} = f_3$, f_3 being the defining function of F_3. Differentiating the equality we obtain the statement using the equality

$$C(V_{34}) \cdot d\left(f_4 \circ T^{21}_{V_{34}}\right) = df_4.$$

□

Lemma 7. *Let $\Delta_1, \Delta_2, \Delta_3$ be as in Fig.10. Then*

$$dT_{\Delta_1} \cdot dT_{-\Delta_2} \cdot dT_{\Delta_3} = -\frac{C(V_{41}) - 1}{C(V_{41})}\, C(V_{34}) \frac{1}{C(V_{24}) - 1}$$

Proof. The statement follows from Lemmas 5 and 6.

Proposition 8. *Let $(\Delta_1, \Delta_2, \Delta_3, \Delta_4)$ be Schläfli configuration as in Fig.11. Then*

$$\sum (-1)^i \int_{\Delta_i} K(F_1, ..., \hat{F}_i, ..., F_4)$$
$$= log\, \frac{C(V_{41}) - 1}{C(V_{41})} \cdot \frac{C(V_{23}) - 1}{C(V_{23})} \cdot \frac{1}{C(V_{24}) - 1} \cdot \frac{1}{C(V_{13}) - 1} \cdot C(V_{34}) \cdot C(V_{12})$$

Proof. We may assume $V_{34} = (0, 0)$, F_3, F_4 are defined by the coordinate functions y, x and F_1, F_2 by functions f, g respectively. Let $V_{12} = (a, b)$ and $P_1 = (0, b), P_2 = (a, 0)$. Let $\square$ denote the geodesic rectangle with the vertices V_{12}, P_1, V_{34}, P_2, and Δ'_1 (resp. Δ'_2) the geodesic triangle with the vertices V_{12}, V_{24}, P_1 (resp. V_{12}, V_{13}, P_2) and let $\Delta''_1 = \Delta_4 + \Delta'_2, \Delta''_2 = \Delta_3 + \Delta'_1$.

Then

$$\int_{\square} K(F_1, F_3, F_4) - \int_{\square} K(F_2, F_3, F_4) = \log \frac{k\, C(V_{12})}{C(P_1) C(P_2)}$$

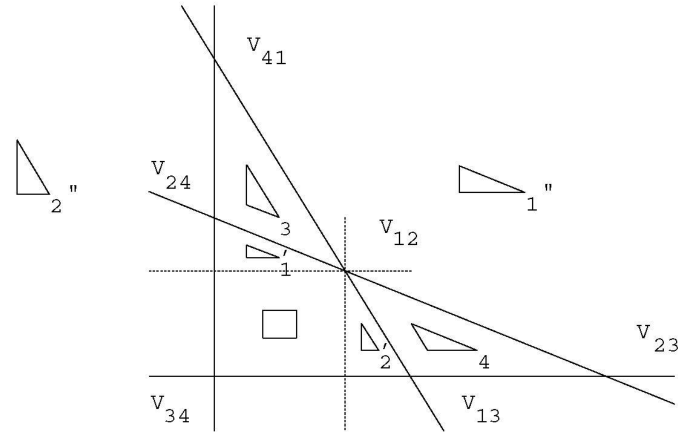

Figure 11.

The alternative sum in the equality of the proposition is

$$
\begin{aligned}
&-\{\int_{\Delta_1'} + \int_{\Box} + \int_{\Delta_1''} K(F_2,F_3,F_4)\} + \{\int_{\Delta_2'} + \int_{\Box} + \int_{\Delta_2''} K(F_1,F_3,F_4)\} \\
&\quad - \int_{\Delta_3} K(F_1,F_2,F_4) + \int_{\Delta_4} K(F_1,F_2,F_3) \\
&\quad = \{-\int_{\Delta_1'} K(F_2,F_3,F_4) + \int_{\Delta_2''} K(F_1,F_3,F_4) - \int_{\Delta_3} K(F_1,F_2,F_4)\} \\
&\quad + \{-\int_{\Box} K(F_2,F_3,F_4) + \int_{\Box} K(F_1,F_3,F_4)\} \\
&\quad\quad - \{-\int_{\Delta_1'} K(F_1,F_3,F_4) + \int_{\Delta_1''} K(F_2,F_3,F_4) - \int_{\Delta_4} K(F_1,F_2,F_3)
\end{aligned}
$$

By Lemma 4 and Lemma 7

$$
\begin{aligned}
&= \log \{-\frac{C(V_{41})(C(V_{41})-1)}{(C(V_{24})-1)C(V_{41})}\}\{+\frac{C(V_{12})C(V_{34})}{C(P_1)C(P_2)}\}\{-\frac{C(V_{41})(C(V_{23})-1)}{(C(V_{13})-1)C(V_{23})}\} \\
&= \log \{\frac{C(V_{41})-1}{C(V_{41})} \cdot \frac{C(V_{23})-1}{C(V_{23})} \cdot \frac{1}{C(V_{24})-1} \cdot \frac{1}{C(V_{13})-1} \cdot C(V_{34}) \cdot C(V_{12})\}.
\end{aligned}
$$

□

3. Proof of Theorem 0 and Theorem 2

First we prove Theorem 2. The implications $(1) \to (2), (3) \to (4), (1) \to (5)$ are clear. The implication $(1) \to (3)$ follows from the uniqueness of the 1-form $\theta(P)$.

Proof of $(5) \to (3)$. For each 3-subweb $W' = (F_i, F_j, F_k)$ if Bott connections of F_i, F_j, F_k extend to equal affine connection, it is Chern connection of W'. Therefore common extension of all Bott connections is Chern connection of 3-subwebs. □

Proof of $(2) \to (1)$. Assume W is non singular at p. Let ω_i be a meromorphic 1-form defining F_i on a neighbourhood of p. Then ω_3 is presented as $\omega_3 = \lambda\omega_1 + \mu\omega_2$ with meromorphic functions λ, μ on a neighbourhood of p. Now we may assume $\lambda = -1, \mu = 2$ replacing ω_1, ω_2 and F_4 is defined by $\omega_4 = \lambda'\omega_1 + \mu'\omega_2$. The ratio $C = \lambda'/\mu'$ is non zero constant by assumption. Therefore we may assume $\omega_4 = C\omega_1 + \omega_2$. Define $\omega^t = (2-t)\omega_1 + (t-1)\omega_2$. Then $\omega_1 = \omega^1$, $\omega_2 = \omega^2$, $\omega_3 = \omega^3$ and $\omega_4 = \omega^{2+c/1+c}$. By analyticity this relation holds on S. □

Proof of $(4) \to (3)$. Consider the Schläfli configuration such that $V_{12} = V_{24} = V_{41}$ as in Fig.12.

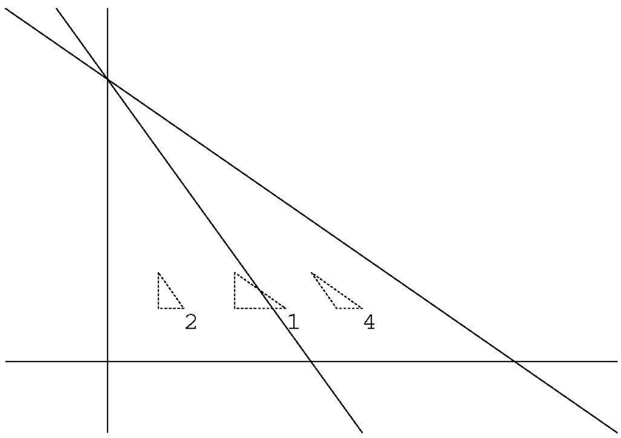

Figure 12.

By Hypothesis (4)

$$\int_{\Delta_1} K(W_1) = \int_{\Delta_2} K(W_2) + \int_{\Delta_4} K(W_4),$$

where $K(W_i)$ denotes the web curvature form of the 3-subweb of W forgetting the i-th foliation. This tells that the integral of the curvature form over a geodesic triangle can be calculated by decomposing into small geodesic triangles. Now decompose the Δ_1 into infinitely many geodesic triangles of the same type as Δ_2 as in Fig.13.

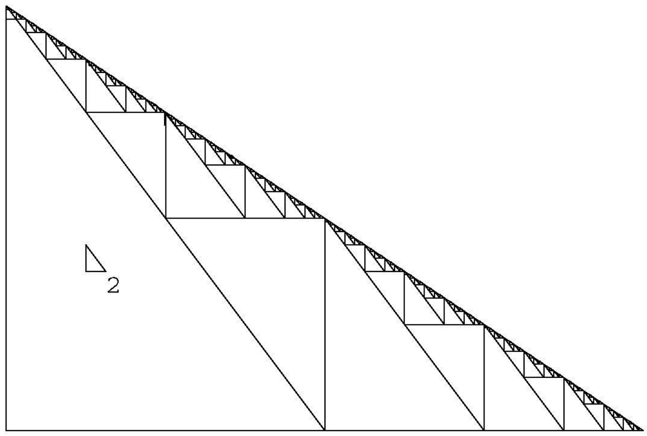

Figure 13.

Then it follows that

$$\int_{\Delta_1} K(W_1) = \int_{\Delta_1} K(W_2).$$

This holds for all geodesic triangles Δ_1 in the non singular locus of W. Therefore $K(W_1) = K(W_2)$ holds on S. This argument applies to show all 3-subwebs have equal curvature form. □

Proof of (3), (4) $\longrightarrow$ (2). By analyticity, if the cross ratio is locally constant, it is constant on the domain of definition S. So we may assume W is non singular and F_1, F_2, F_3, F_4 are defined by level functions f, g and the coordinate functions y, x of $\mathbb{R}^2$ respectively. Then the cross ratio is

$$C(F_4, F_3, F_2, F_1) = \frac{f_x\ g_y}{f_y\ g_x}.$$

By suitable coordinate transformation of x, y we may assume $f(t,0) = f(0,t) = g(t,0) = t$, and applying the linearlization theorem to the dynamics $t \longrightarrow g(0,t)$ we may assume $g(0,t) = kt$. Here k is the cross ratio at the origin, and f, g are of the form

$$\begin{aligned} f(x,y) &= x + y + mxy + O, \\ g(x,y) &= x + ky + nxy + O', \end{aligned} \tag{*}$$

O, O' being the remainder terms of x, y of order 3 which vanish identically on the x and y axes. It is easy to see that only similar transformations $(x, y) \to (cx, cy)$, $c \neq 0$, respect the dynamics $t \to g(0,t) = kt$. Therefore the ratio $(m : n)$ as well as k gives rise to an absolute invariant of 4-webs. By definition, the cross ratio at (x, y) is

$$C(x,y) = C(F_4, F_3, F_2, F_1)(x,y) = \frac{(k+nx)(1+my)}{(1+ny)(1+mx)} + O''.$$

Denote $C(x, 0) = A(x)$ and $C(0, y) = B(y)$ for simplicity. On the x-axis $C(x, y)$ restricts to

$$A(x) = \frac{k+nx}{1+mx} + \cdots = k + (n-km)x + \cdots .$$

By Proposition 5 applied to Schläfli configuration as in Fig.12 we obtain

$$\frac{A(kx)}{A(x)} = \frac{A(kx)-1}{A(x)-1}.$$

This relation admits unique solution by Taylor expansion with given initial value $C(0)$. With the initial condition $C(0) = k$ the solution is

$$C(x,0) = A(x) = k \; + \; \frac{(n-km)x}{1 - \frac{(n-km)x}{k-1}}.$$

Similarly we obtain

$$C(0,y) = B(y) = k \; + \; \frac{k(m-n)y}{1 - \frac{k(m-n)y}{k-1}}.$$

By the hypothesis (3) $K(W_1) = K(W_2)$, so

$$\frac{\partial^2}{\partial x \partial y} \log\left(\frac{f_x}{f_y}\right) = \frac{\partial^2}{\partial x \partial y} \log\left(\frac{g_x}{g_y}\right),$$

from which

$$\frac{\partial^2}{\partial x \partial y} \log C(x,y) = \frac{\partial^2}{\partial x \partial y}\left(\log\left(\frac{f_x}{f_y}\right) - \log\left(\frac{g_x}{g_y}\right)\right) = 0.$$

Therefore

$$C(x,y) = A(x)B(y)/k.$$

Now assume a non singular 4-web $W' = (F_1', \ldots, F_4')$ (not necessarily of the above normal form) is defined by the level functions f', g', y, x, and assume the cross ratio function $C'(x, y)$ is a product of two linear fractions $A'(x), B'(y)$ of x

and y. Let ϕ and ψ be the diffeomorphisms of the x-axis and the y-axis, which normalize W' to the above normal form (*). Since the cross ratio is an absolute invariant,

$$C'(x,y) = C(\phi(x), \psi(y)).$$

First assume $n \neq m, km$. This is equivalent that $A(x), B(y)$ are not constant. Then it follows from the above equality that

ϕ, ψ x

Since the transverse dynamics of F_1, F_2 in the normal form sending the x-axis to the y-axis respecting the origin are linear maps, those dynamics for W' are also linear fractions of x. This argument applies to germs of the normal form W at all points (x, y) on a neighbourhood of the origin to show that

the transverse dynamics sending horizontal lines to vertical lines are all linear fractions of x.

It is easy to see that this implies also

F_1, F_2

x y

(see Fig.14.)

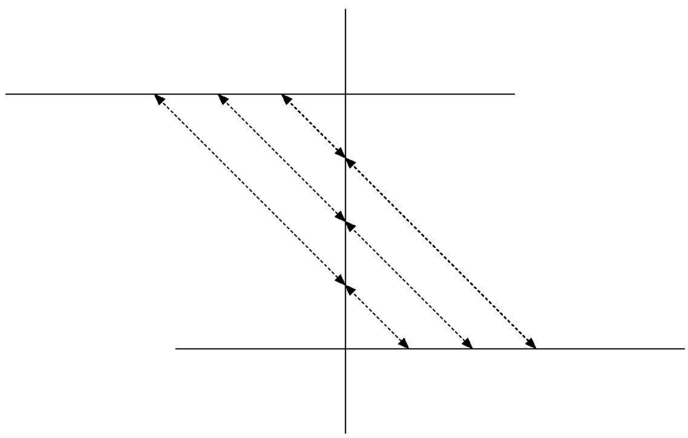

Figure 14.

Therefore we may assume that the restrictions of the level functions f, g to the horizontal and the vertical lines are linear fractions in x, y. We may write as

$$f = \frac{a(x)y + b(x)}{c(x)y + d(x)}$$

with functions a, b, c, d of x.

Claim.

$$f(x,y) = \frac{c_1xy + c_2x + c_3y + c_4}{c'_1xy + c'_2x + c'_3y + c'_4}. \qquad (\ **)$$

Proof of the claim. $u(x) = f(x, y_1), v(x) = f(x, y_2), w(x) = f(x, y_3)$ are linear fraction of x for all small distinct y_1, y_2, y_3. The coefficients $a(x), b(x), c(x), d(x)$ are solved as rational functions of u, v, w hence f is a rational function of x, y. The numerator and denominator are of degree 1 in x and y by the above result. Therefore f is in the above form. □

Now we will see that the curvature form $K(F_1, F_3, F_4)$ of the 3-subweb (F_1, F_3, F_4) vanishes at the origin. (This can be also seen by straight forward calculation.) Recall the return map R defined as in §1, Fig.6. Since R is defined by composing the various transverse dynamics respecting the origin, all of which are linear fractions by the above form of f, R is also a linear fraction of x. On the other hand the rotation map has the expansion $R(x) = x + k(0,0)x^3 + \cdots$, where the second order term is missing. Therefore R is the identity and in particular the web curvature vanishes at the origin. This argument applies at all point in the domain of definition S to show that the curvature form vanishes identically. This contradicts the hypothesis of the theorem.

Next assume $n = m$ and $n \neq km$. Then by the same argument as the above case $B(y)$ is constant and

$$C(x,y) = C(x,0) = A(x) = k \ + \ \frac{(n-km)x}{1 - \frac{(n-km)x}{k-1}}$$

is not constant. In this case exchange the roles of F_4 and F_1 (or F_2) in the above argument. Then it reduces to the first case $n \neq m, km$, since the cross ratio C is not constant on the leaves of F_1, F_2, F_3 passing through the origin.

Similar argument applies to the case $n = km$ and $n \neq m$. The rest is the case $m = n = 0$. Clearly this implies that the cross ratio function is locally constant. This completes the proof of Theorem 2. □

Proof of Theorem 0. Assume that the 4-web W is weakly associative, hexagonal and the cross ratio is not constant along the x and y axes. By the above result F_1, F_2, F_3, F_4 are respectively defined by the level functions f, g, y, x such that f, g are in the form (**). So the web extends naturally to a 4-web on the projective plane: F_3, F_4 extend to the pencils of lines with base points $p_1 = (1:0:0), p_2 = (0:1:0)$, and F_1, F_2 extend to pencils lines if $c_1 = c'_1$ and pencils of conics with 4 base points p_1, p_2, q_1, q_2 and p_1, p_2, q_3, q_4, respectively, if $c_1 \neq c'_1$. First assume F_1, F_2 are pencils of conics. If those base points degenerate, i.e three of 4 base points are collinear, then the pencil of conics degenerates, i.e. it splits as a union of the line passing through the collinear 3 points and a pencil of lines passing through the remaining point. This is equivalent that the level function f (or g) is a linear fraction and defines a pencil of lines. So assume F_1, F_2 are non degenerate and consider the 3-subweb (F_1, F_2, F_3) on the projective plane. By Cremona transformation with vertices p_1, p_2 and q_1, the foliations F_1, F_3, F_4 are transformed to the pencils of lines $\bar{F}_1, \bar{F}_3, \bar{F}_4$ respectively with base points $\bar{q}_2, \bar{p}_1, \bar{p}_2$ and F_2 is transformed to pencil of conics $\bar{F}_2$ with 4 base points $\bar{p}_1, \bar{p}_2, \bar{q}_3, \bar{q}_4$, where $\bar{p}_1, \bar{p}_2$ are the images of lines p_2q_1, p_1q_1 and $\bar{q}_2, \bar{q}_3, \bar{q}_4$ are the image of q_2, q_3, q_4 respectively. If these 4 base points degenerate, $\bar{F}_2$ is a pencil of lines and the statement of the theorem is proved. So assume $\bar{F}_2$ is non degenerate as in Fig.15.

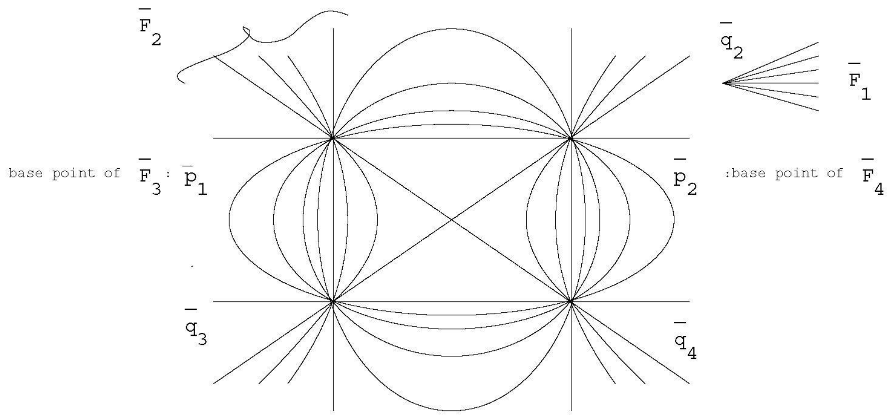

Figure 15.

Claim. *3-web $(\bar{F}_1, \bar{F}_2, \bar{F}_3)$ is hexagonal if and only if $\bar{q}_2$ coincides with one of $\bar{q}_3, \bar{q}_4$ if and only if q_2 coincides with one of q_3, q_4.*

Proof of the claim. First consider the 3-web of the plane defined by the coordinate functions x, y and a rational function f. The web curvature form is

$$\frac{\partial^2}{\partial x \partial y} \log \frac{f_x}{f_y}\, dx \wedge dy.$$

If the web is hexagonal, this vanishes identically, hence

$$\frac{f_x}{f_y} = A(x) \cdot B(y)$$

with rational functions A, B. In particular the set T of those $p \in \mathbb{R}^2$ where the foliations are non singular but the web is singular is a finite union of leaves defined by $x, y =$ const.. We apply this fact to the 3-web $(\bar{F}_1, \bar{F}_2, \bar{F}_3)$, assuming the base points $\bar{p}_1, \bar{q}_2$ are situated on the line at infinity by a projective transformation. The singular locus $T(W')$ of our web $W' = (\bar{F}_1, \bar{F}_2, \bar{F}_3)$ is the set of those $p \in \mathbb{P}^2 - \bar{p}_1\bar{q}_2$, where the leaf of $\bar{F}_2$ is non singular and tangent to the leaf of either $\bar{F}_1$ or $\bar{F}_3$. By the above argument $T(W')$ is a union of some lines passing through $\bar{p}_1$ or $\bar{q}_2$. Since all leaves of $\bar{F}_2$ contain $\bar{p}_1$, the lines passing through $\bar{p}_1$ and the leaves of $\bar{F}_2$ have no tangent point off $\bar{p}_1$. So the singular locus $T(W')$ is the set of those $p \in \mathbb{P}^2 - \bar{p}_1\bar{q}_2$ where $\bar{F}_2$ is non singular and tangent to $\bar{F}_1$. Assume $T(W')$ contains a line ℓ passing through $\bar{p}_1$. Then ℓ ought to be the line spanned by $\bar{p}_1, \bar{q}_2$, because the tangent line to the leaves of $\bar{F}_2$ at $p \in \ell$ is asymptotic to ℓ as p tends to $\bar{p}_1$ and all those tangent lines contain $\bar{q}_2$ hence ℓ also contains $\bar{q}_2$. Therefore the singular locus $T(W')$ consists of lines passing through $\bar{q}_2$ but $\bar{p}_1$. Let $\ell' \in \bar{F}_1$ be a line in the singular locus passing through $\bar{q}_2$. Then ℓ' is tangent to $\bar{F}_2$ on $\ell' - \bar{q}_2$, hence ℓ' is contained in a conic in $\bar{F}_2$. This tells that $\bar{q}_2$ is in the union X of singular leaves of $\bar{F}_2$, which is the union of projective lines spanned by two base points of $\bar{F}_2$, and ℓ' is one of those lines. It is easily seen by Fig.16 that if $\bar{q}_2$ is not one of the base points of $\bar{F}_2$, the singular locus has a component which is not contained in X. Therefore $\bar{q}_2$ is one of $\bar{p}_1, \bar{p}_2, \bar{q}_3, \bar{q}_4$. If $\bar{q}_2$ is either $\bar{p}_1$ or $\bar{p}_2$, then F_1 coincides with either F_3 or F_4 and W is a hexagonal 3-web. Therefore the only possibility is the case $\bar{q}_2$ is either $\bar{q}_3$ or $\bar{q}_4$. □

By the claim, $\bar{q}_2$ coincides with $\bar{q}_3$ or $\bar{q}_4$. Cremona transformation with vertices p_1, p_2 and q_2 transforms the foliations of our web to pencils of lines.

Next assume F_1 is non degenerate and F_2 is a pencil of lines with a base point q_3. By the same argument as above, 3-subwebs of W are hexagonal if and only if q_3 coincides with one of q_1, q_2. Then Cremona transformation with the vertices p_1, p_2, q_3 transforms F_1 to the pencil of lines with base point $\bar{q}$, where $\{q_1, q_2\} = \{q_3, q\}$ and $\bar{q}$ is the image of q under the transformation.

If F_1 is degenerate and F_2 is non degenerate, a similar argument applies.

The rest is the case where F_1, F_2, F_3, F_4 are pencils of lines. This completes the proof of Theorem 0. □

4. Associative webs and weakly associative webs of codimension 1 in higher dimension

Let $W = (F_1, \ldots, F_{n+1})$ be a non singular (i.e. F_i's are non singular and in general position) $(n+1)$-web of codimension 1 on an open subset of $\mathbb{R}^n$. Chern connection γ_i of W is an extension of Bott connection of the normal bundle of F_i (see [5] for the definition). In the case $n = 3$ twice the average of the curvature forms of $\gamma_1, \ldots, \gamma_4$ had been already defined by Blaschke [1], so we call it *Blaschke curvature form.*

To define Blaschke curvature form for $n+1$-web of codimension 1 on an n-manifold M, assume that F_i is locally defined by a level function u_i. A relation $W(u_1,\ldots,u_{n+1})=0$ of those u_i is called *web function.*Blaschke curvature form is defined by

$$d\Gamma = d(-\sum_{i=1,\ldots,n+1}(\log F_{u_i})_{u_i}du_i) = \frac{1}{2}\sum_{i,j=1,\ldots,n}(\log\frac{F_{u_i}}{F_{u_j}})_{u_iu_j}du_i\wedge du_j.$$

It is easy to see that $d\Gamma$ is independent of the choice of defining level functions. When $(u_1,\ldots,u_n)$ is a local coordinate $(x_1,\ldots,x_n)$ of M and $u_{n+1}=f(x_1,\ldots,x_n)$, a web function is given by

$$F = u_{n+1} - f(u_1,\ldots,u_n)$$

and

$$d\Gamma = d(-\sum_{i=1,\ldots,n}(\log f_{x_i})_{x_i}dx_i) = \frac{1}{2}\sum_{i,j=1,\ldots,n}(\log \frac{f_{x_i}}{f_{x_j}})_{x_ix_j}dx_i\wedge dx_j.$$

By definition we obtain

Proposition 9. *Blaschke curvature form restricts to that of the $i+1$-web on the intersections of leaves of $n-i$ foliations cut out by the other $i+1$ foliations for $i=2,3,\ldots,n-1$. Conversely Blaschke curvature form is determined by the web curvature forms of the 3-webs on $n-2$-intersections.*

We call $n+2$ foliations of codimension 1 of an n-manifold M are *associative* if the modulus of tangent hyperplanes to the leaves passing through a point is constant, and we say those foliations are *weakly associative* if all $n+1$-subwebs have equal Blaschke curvature form. If the modulus of the tangent hyperplanes is constant, the cross ratios of 4-webs on $n-2$-intersections of the leaves are constant. In particular 3-subwebs of those 4-webs have equal web curvature forms by Theorem 2 and by the second half of Proposition 9, $n+2$ foliations are weakly associative. In the following we discuss the converse.

For simplicity assume that M is Euclidean n space, F_i is defined by the i-th coordinate x_i and F_{n+1},F_{n+2} are defined by level functions f,g respectively on a neighbourhood of the origin. Assume that Blaschke curvature forms of all $n+1$-subwebs are equal. Then the 4-webs on $n-2$-intersections are weakly associative by Proposition 9. Assume that those curvilinear 4-webs are not hexagonal, i.e. all 3-subwebs are not hexagonal. Then the cross ratio of tangents to the leaves of those curvilinear 4-webs is constant on each $(n-2)$-intersection of the leaves by Theorem 2.

Lemma 10. *Let $H_i, i=1,\ldots,n+2$ be codimension one subspaces in Euclidean n-space in general position. The modulus of those H_i is determined by the modulus* *of 4-lines in intersections of $n-2$ of those $H_i, i \neq 1, n+1, n+2$ cut out by the remaining 4 hyperplanes.*

Proof. Assume H_i is defined by $x_i = 0$ for $i = 1, \ldots, n$, H_{n+1} is defined by $x_1 + \cdots + x_n = 0$ and H_{n+2} is defined by $x_1 + a_2x_2 + \cdots + a_nx_n = 0$. The x_1x_i-plane is the $n-2$-intersection of $H_j, j \neq 1, i, n+1, n+2$. On this plane the other 4 subspaces cut out 4 lines $x_1 = 0, x_i = 0, x_1 + x_i = 0$ and $x_1 + a_ix_i = 0$. The modulus of these 4 lines determines the coefficient a_i. □

By the lemma the modulus of tangent hyperplanes of the leaves of $F_i, i = 1, \ldots, n+2$ is constant along $n-1$-intersections of $F_i, i \neq j, k, \ell$ for distinct j, k, ℓ. Therefore it is constant on the domain of definition. We can state the result in the following general form.

Theorem 11. *Assume d foliations $F_1, \ldots, F_d$, $n+2 \leq d$ of codimension 1 on an n-manifold are non singular, in general position and also the curvilinear 3-webs on the intersection of leaves of $n-2$ foliations cut out by 3 of the remaining 4 foliations are not hexagonal. Then the following conditions are equivalent.*

(1) $F_1, \ldots, F_d$ are associative: the modulus of tangent planes to the leaves of $F_1, \ldots, F_d$ passing through a point is constant.

(2) $F_1, \ldots, F_d$ are weakly associative: Blaschke curvature forms of $(n+1)$-subwebs are equal.

The statement remains valid for webs with generic singularity. In the hexagonal case the statement is not true. In fact d pencils of hyperplanes on $\mathbb{R}^n$ satisfies (2) but (1). The author does not know if there exist other such examples. It seems interesting to classify all such webs, generalizing Theorem 0 and Theorem 1.

Clearly a non singular associative d-web of codimension 1 on an n-manifold, $n < d$, can be defined by a d-tuple of members of n-dimensional non singular linear family of one forms $L = \{\omega_v\}_{v \in \mathbb{R}^n}$. We say such d foliations are generic if the m points in the projectivization $PL = P^{n-1}$ are non degenerate and not contained in a quadric hypersurface. Then all members of L are integrable by Frobenius theorem. The following proposition cited in [23] is a special case of diagonalizable $n+1$-webs (see [11]).

Proposition 12. *If all members of a non singular linear family of one forms L on an open subset of $\mathbb{R}^n$ of dimension n are integrable and dim $L = n \geq 3$, there exists unique closed one form θ such that*

$$d\omega_v = \theta \wedge \omega_v.$$

for all $\omega_v \in L$.

By the proposition we obtain

Proposition 13. *If linearly independent integrable one forms $F_1, \dots, F_d$ (possibly singular) on a n-manifold are weakly associative and generic and d is sufficiently large, then the d-web $(F_1, \dots, F_d)$ is parallelizable at non singular points.*

Here a non singular d-web $(F_1, \dots, F_d)$ is *parallelizable* if it is locally diffeomorphic to the d-web by d foliations by parallel hyperplanes in $\mathbb{R}^n$. In the paper [23] a more detailed structure of such parallelizable webs is investigated.

Non generic case is studied by Gelfand- Zakharevich [8] and M.H.Rigal [26] from different view points.

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